Cancellation and periodicity properties of iterated morphisms

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Abstract

In this note we prove two cancellation properties of iterated morphisms and use these properties to give a simple method for deciding whether or not a given infinite D0L word is ultimately periodic.

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1. Introduction and definitions

In this note we prove two cancellation properties of iterated morphisms. As an application of these properties we give a new solution to the D0L periodicity problem solved in [3,5].

We assume that the reader is familiar with the basics concerning iterated morphisms (see [1,6,7]). For all unexplained notation and terminology we refer to these references.

As usual, the free monoid generated by a finite nonempty alphabet X is denoted by X∗. The set of right-infinite words over X is denoted by Xω and X∞ = X∗ ∪ Xω. If u is a finite word, uω is the word uu · · ·. An infinite word w is called ultimately periodic if there exist finite words u, v such that w = uvω.

Let h : X∗ → X∗ be a morphism and let a ∈ X be a letter. If a is a prefix of h(a) and the language {h(n)(a) | n ≥ 0} is infinite, then hω(a) is the unique infinite word having prefix h(n)(a) for all n ≥ 0.

A morphism h : X∗ → Y∗ is called elementary if there do not exist a set Z smaller than X and two morphisms f : X∗ → Z∗, g : Z∗ → Y∗ such that h = gf. Elementary morphisms are injective on finite and infinite words. More precisely, if h : X∗ → Y∗ is elementary and u, v ∈ X∞, then h(u) = h(v) implies that u = v (see [6, Theorem III 1.6]).

Suppose h : X∗ → X∗ is an elementary morphism. Define

F(h) = {a ∈ X | h(n)(a) ∈ X for all n ≥ 1}.

Clearly, if a ∈ F(h) then h(a) ∈ F(h). Hence h induces a permutation of F(h). It follows that

h(card(X)!)(a) = a if a ∈ F(h).

Here card(X)! can be replaced by an integer at most G(card(X)), where G(k) is Landau’s function specifying the maximum order of an element of the symmetric group on k symbols (see [4]).

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2. Two cancellation properties of iterated morphisms

In this section we state and prove two cancellation properties. The first of these is a direct generalization of a result of Ehrenfeucht and Rozenberg [2].

Theorem 1. Let \( h : X^* \to X^* \) be a morphism and let \( u, v \in X^\omega \). Suppose that there is a positive integer \( n \) such that

\[
h^n(u) = h^n(v).
\]

Then

\[
h^{\text{card}(X)}(u) = h^{\text{card}(X)}(v).
\]

Proof. We use induction on \( \text{card}(X) \). If \( \text{card}(X) = 1 \), the claim holds. Assume inductively that the claim holds if \( \text{card}(X) < k \) where \( k \geq 2 \) and consider an alphabet \( X \) having \( k \) letters.

If \( h \) is elementary, \( h^n(u) = h^n(v) \) implies that \( u = v \). Hence the claim holds.

If \( h \) is not elementary, there exist an alphabet \( Y \) smaller than \( X \) and morphisms \( f : X^* \to Y^* \), \( g : Y^* \to X^* \) such that \( h = gf \). Hence \( (gf)^n(u) = (gf)^n(v) \). Therefore \( (fg)^n f(u) = (fg)^n f(v) \). By the inductive assumption we get

\[
(fg)^{\text{card}(Y)} f(u) = (fg)^{\text{card}(Y)} f(v).
\]

Hence

\[
(gf)^{\text{card}(Y)+1} (u) = (gf)^{\text{card}(Y)+1} (v),
\]

which implies the claim. \( \square \)

Theorem 2. Let \( h : X^* \to X^* \) be a morphism and let \( u \in X^\omega \). Suppose that there is a positive integer \( n \) such that

\[
u = h^n(u).
\]

Then

\[
u = h^{\text{card}(X)!}(u).
\]

Proof. We use induction on \( \text{card}(X) \). If \( \text{card}(X) = 1 \), the claim holds. Assume inductively that the claim holds if \( \text{card}(X) < k \) where \( k \geq 2 \) and consider an alphabet \( X \) having \( k \) letters.

Suppose first that \( h \) is elementary. The claim holds if all letters of \( u \) belong to \( F(h) \). To proceed, assume that

\[
u = a_1 \cdots a_i b v,
\]

where \( a_1, \ldots, a_i \in F(h), b \in X - F(h) \) and \( v \in X^\omega \). Then \( h^n(bv) = bv \). Because \( b \notin F(h) \), this implies that \( (h^n)^\omega(b) = bv \). Hence there exists a positive integer \( m \leq \text{card}(X) \) such that \( (h^m)^\omega(b) = bv \). Therefore \( h^m(bv) = bv \) and

\[
u = a_1 \cdots a_i b v = h^{\text{card}(X)!}(a_1 \cdots a_i b v) = h^{\text{card}(X)!}(u).
\]

Suppose then that \( h \) is not elementary. Then there exist an alphabet \( Y \) smaller than \( X \) and morphisms \( f : X^* \to Y^* \), \( g : Y^* \to X^* \) such that \( h = gf \). Because \( u = (gf)^n(u) \) we have \( f(u) = (fg)^n f(u) \) where \( f(u) \) is an infinite word. Now the inductive hypothesis implies that

\[
f(u) = (fg)^{\text{card}(Y)!} f(u).
\]

Hence

\[
h(u) = h^{\text{card}(Y)!+1}(u).
\]

By assumption, there exists a positive integer \( p \geq \text{card}(X) \) such that \( u = h^p(u) \). Because

\[
h^{p+1}(u) = h^{p+1}(h^{\text{card}(Y)!}(u)),
\]

we have

\[
h(u) = h^{\text{card}(Y)!+1}(u).
\]

Hence

\[
h(u) = h^{\text{card}(Y)!+1}(u) = h^{\text{card}(Y)!+1}(u).
\]

...
Theorem 1 implies that
\[ h^p(u) = h^p(h^{\text{card}(Y)\!}(u)). \]

Hence
\[ u = h^{\text{card}(Y)!}(u), \]

which implies the claim. \(\square\)

3. The periodicity problem for infinite DOL words

The decidability of the DOL periodicity problem is an immediate consequence of the following theorem.

**Theorem 3.** Let \( X \) be an alphabet having \( k \) letters. Let \( h : X^* \rightarrow X^* \) be a morphism and let \( a \in X \) be a letter such that \( h^\omega(a) \) exists. Let \( h(a) = au \) where \( u \in X^*. \) Let
\[ w_1 = auh(u) \cdots h^{k-1}(u) \]

and let \( w_2 \) be a primitive word and \( q \) be a positive integer such that
\[ h^k(u)h^{k+1}(u) \cdots h^{k+q-1}(u) = w_2^q. \]

(a) If \( h^\omega(a) \) is ultimately periodic, then \( h^\omega(a) = w_1w_2^\alpha. \)

(b) \( h^\omega(a) \) is ultimately periodic if and only if there exist integers \( \alpha \geq 0, \beta \geq 1 \) and words \( w_3, w_4 \in X^* \) such that
\[ h(w_1) = w_1w_2^\beta w_3, \quad h(w_2) = (w_4w_3)^\beta, \quad w_2 = w_3w_4. \] (1)

**Proof.** Suppose first that \( h^\omega(a) \) is ultimately periodic. Then there exist positive integers \( i \) and \( j, i < j, \) such that
\[ h^i(u)h^{i+1}(u) \cdots = h^j(u)h^{j+1}(u) \cdots. \]

In other words we have
\[ h^i(uh(u) \cdots) = h^j(uh^{j-i}(u)h^{j-i+1}(u) \cdots). \]

By Theorem 1 we have
\[ h^k(uh(u) \cdots) = h^k(uh^{j-i}(u)h^{j-i+1}(u) \cdots) \]
or, equivalently,
\[ h^k(u)h^{k+1}(u) \cdots = h^{k+j-i}(u)h^{k+j-i+1}(u) \cdots = h^{j-i}(h^k(u)h^{k+1}(u) \cdots). \]

This implies by Theorem 2 that
\[ h^k(u)h^{k+1}(u) \cdots = h^{k+1}(h^k(u)h^{k+1}(u) \cdots). \]

Because
\[ h^k(u)h^{k+1}(u) \cdots = h^k(u)h^{k+1}(u) \cdots h^{k+k!-1}(u)h^{k+k!}(u)h^{k+k!+1}(u) \cdots = w_2^q h^{k+1}(h^k(u)h^{k+1}(u) \cdots), \]

we get
\[ w_2^q h^{k+1}(h^k(u)h^{k+1}(u) \cdots) = h^{k+1}(h^k(u)h^{k+1}(u) \cdots). \]

Hence
\[ h^{k+k!}(u)h^{k+k!+1}(u) \cdots = w_2^\omega, \]

and
\[ h^\omega(a) = w_1w_2^\omega. \]
Because $h(w_1 w_2^\omega) = w_1 w_2^\omega$, we have $h(w_1)h(w_2)^\omega = w_1 w_2^\omega$. Hence there exists a positive integer $\beta$ such that $h(w_2)$ is a conjugate of $w_2^\beta$. In other words, there exist words $w_3, w_4 \in X^*$ such that $h(w_2) = (w_4 w_3)^\beta$ and $w_2 = w_3 w_4$. Because $h(w_1)$ is longer than $w_1$, the equation $h(w_1)(w_4 w_3)^\omega = w_1 (w_3 w_4)^\omega$ implies that there is a nonnegative integer $\alpha$ such that the first equation of (1) holds.

Finally, if there exist integers $\alpha \geq 0$, $\beta \geq 1$ and words $w_3, w_4 \in X^*$ such that (1) holds, then

$$h^\omega(\alpha) = h^\omega(w_1) = w_1 w_2^\omega$$

is ultimately periodic. □

References