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Bifurcations for a predator–prey system with two delays ☆

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Abstract

In this paper, a predator–prey system with two delays is investigated. By choosing the sum τ of two delays as a bifurcation parameter, we show that Hopf bifurcations can occur as τ crosses some critical values. By deriving the equation describing the flow on the center manifold, we can determine the direction of the Hopf bifurcations and the stability of the bifurcating periodic solutions. In addition, special attention is paid to the global continuation of local Hopf bifurcations. Using a global Hopf bifurcation result of [J. Wu, Symmetric functional differential equations and neural networks with memory, Trans. Amer. Math. Soc. 350 (1998) 4799–4838], we may show the global existence of periodic solutions.

Keywords: Predator–prey system; Delay; Hopf bifurcation; Normal form; Periodic solution

1. Introduction

Since the work of Volterra, time delays were already incorporated into the mathematical models of population dynamics. For a long time, it has been recognized that delays can have very complicated impact on the dynamics of a system (see, for example, monographes by Hale and Lunel [9], Kuang [11] and Wu [22]). For example, delays can cause the loss of stability and can induce various oscillations and periodic solutions. It is well known that periodic solutions can arise through the Hopf bifurcation in delay differential equations. However, theses bifurcating periodic solutions are generally local, i.e., they exist in a small neighborhood of the critical value. Therefore, we want to known if these nonconstant periodic solutions obtained through local Hopf bifurcations exist globally. Over the last two decades, a great deal of research has been devoted to the global existence of nonconstant periodic solutions in delayed scalar equations (see, for example, [7,8,13,14,16,20]). Recently, the existence of nonconstant periodic solutions of the planar delay differential system have been investigated by Taboas [18], Ruan and Wei [15], Wei and Li [21] for neural network systems and by Leung [12], Zhao et al. [24] and Song and Wei [17] for predator–prey systems.

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In the present paper, we again devote our attention to bifurcating phenomenons and the global existence of periodic solutions of a predator–prey system with two delays described by

$$\begin{cases} \dot{x}(t) = x(t)\big[r_1 - a_{11}x(t) - a_{12}y(t-\nu_2)\big], \\ \dot{y}(t) = y(t)\big[-r_2 + a_{21}x(t-\nu_1) - a_{22}y(t)\big], \end{cases} \tag{1.1}$$

where r_1, r_2, a_{12}, a_{21} are positive constants and $\nu_1, \nu_2, a_{11}, a_{22}$ are nonnegative constants. The variables $x(t)$ and $y(t)$ denote the population of the prey and the predator, respectively. a_{11}, a_{22} are self-limitation constants. In the absence of predators, the prey species is governed by the logistic equation $\dot{x}(t) = x(t)[r_1 - a_{11}x(t)]$. In the presence of predators, there is a hunting term, $a_{12}y(t-\nu_2)$, with a certain delay ν_2 , called the hunting delay. In the absence of prey species, the predators decrease. The positive feedback $a_{21}x(t-\nu_1)$ has a nonnegative delay which is the time of the predator maturation.

The stability, local Hopf bifurcation and other dynamics of system (1.1) have been extensively studied (see, for example, [1,3,6,10,19], and references therein). In [6], taking the single delay, ν_1, as a parameter and assuming the ratio ν_2/ν_1 is constant, the author studied the properties of the local Hopf bifurcation. However, the global existence of periodic solutions have not been studied. In the present paper, choosing the sum τ of two delays as a bifurcation parameter, we investigate not only the properties of the local Hopf bifurcation but also the global continuation of the local Hopf bifurcation. In addition, we would like to mention the work of Zhao et al. [24]. By proving the existence of nontrivial fixed points of an appropriate map, Zhao et al. [24] studied the global existence of periodic solutions of the following delayed Gause-type predator–prey system:

$$\begin{cases} \dot{x}(t) = x(t)\big[g\big(x(t)\big) - p\big(x(t)\big)y(t)\big], \\ \dot{y}(t) = y(t)\big[-d + h\big(x(t-\tau)\big)\big]. \end{cases}$$

Unfortunately, their results are not applied to system (1.1). However, we may employ degree theory methods developed by Wu [23] to show the global existence of periodic solutions of (1.1).

This paper is organized as follows. In Section 2, using the sum of two delays as a parameter, the local stability of the positive equilibrium and existence of local Hopf bifurcations are addressed. In Section 3, by using the normal form theory of retarded functional differential equations developed by Faria and Magalháes, the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions are determined. Finally, the global existence of these bifurcating periodic solutions is investigated in Section 4.

2. Stability of the positive equilibrium and existence of local Hopf bifurcations

Clearly, if

$$r_1a_{21} - r_2a_{11} > 0,$$

then system (1.1) has a positive equilibrium $E_* = (x_*, y_*)$, where

$$x_* = \frac{r_1a_{22} + r_2a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \qquad y_* = \frac{r_1a_{21} - r_2a_{11}}{a_{11}a_{22} + a_{12}a_{21}}.$$

For convenience, let us introduce new variables $u_1(t) = x(t-\nu_1)$, $u_2(t) = y(t)$, $\tau = \nu_1 + \nu_2$ so that system (1.1) can be written as the following equivalent system with a single delay:

$$\begin{cases} \dot{u}_1(t) = u_1(t)\big[r_1 - a_{11}u_1(t) - a_{12}u_2(t-\tau)\big], \\ \dot{u}_2(t) = u_2(t)\big[-r_2 + a_{21}u_1(t) - a_{22}u_2(t)\big]. \end{cases} \tag{2.1}$$

In what follows, we will investigate the effect of the delay τ on the dynamics of (2.1). It is well known that the stability of an equilibrium and local Hopf bifurcations involve the distribution of roots of the corresponding characteristic equation. It is easy to see that the associated characteristic equation of system (2.1) at the positive equilibrium (x_*, y_*) has the following form:

$$\lambda^2 + p\lambda + r + qe^{-\lambda\tau} = 0, \tag{2.2}$$

where $p = a_{11}x_* + a_{22}y_* \geqslant 0$, $r = a_{11}a_{22}x_*y_* \geqslant 0$, $q = a_{12}a_{21}x_*y_* > 0$ and $\tau = \nu_1 + \nu_2$. In the sequel, we will investigate the distribution of roots of Eq. (2.2). Obviously, $\lambda = 0$ is not a root of (2.2). For $\tau = 0$ the characteristic equation becomes

$$\lambda^2 + p\lambda + r + q = 0,$$

which has two roots with negative real parts if $a_{11} + a_{22} \neq 0$ and have a pair of purely imaginary roots if $a_{11} = a_{22} = 0$.

Now for $\tau \neq 0$, if $i\omega(\omega > 0)$ is a root of (2.2), then

$$-\omega^2 + p\omega i + r + q\cos\omega\tau - iq\sin\omega\tau = 0.$$

Separating the real and imaginary parts, we have

$$\begin{cases} \omega^2 - r = q\cos\omega\tau, \\ p\omega = q\sin\omega\tau, \end{cases} \tag{2.3}$$

which leads to

$$\omega^4 + \left(p^2 - 2r\right)\omega^2 + r^2 - q^2 = 0. \tag{2.4}$$

By a simple analysis, one immediately obtain the following results: (i) if $a_{11}a_{22} - a_{12}a_{21} \geqslant 0$, then Eq. (2.4) has no positive root; (ii) if $a_{11}a_{22} - a_{12}a_{21} < 0$, then Eq. (2.4) has only one positive root ω_0 defined by

$$\omega_0 = \frac{\sqrt{2}}{2}\left\{-\left(a_{11}^2x_*^2 + a_{22}^2y_*^2\right) + \left[\left(a_{11}^2x_*^2 - a_{22}^2y_*^2\right)^2 + 4x_*^2y_*^2a_{12}^2a_{21}^2\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}. \tag{2.5}$$

Let

$$\tau_j = \begin{cases} \frac{2(j+1)\pi}{\omega_0}, & a_{11} = a_{22} = 0, \\ \frac{1}{\omega_0}\arccos\{\frac{\omega_0^2 - a_{11}a_{22}x_*y_*}{a_{12}a_{21}x_*y_*}\} + \frac{2j\pi}{\omega_0}, & a_{11} + a_{22} \neq 0, \end{cases} \tag{2.6}$$

where $j = 0, 1, 2, \ldots$. Then when $\tau = \tau_j$, Eq. (2.2) has a pair of purely imaginary roots $\pm i\omega_0$. Denote by

$$\lambda(\tau) = \alpha(\tau) + i\omega(\tau)$$

the root of Eq. (2.2) such that

$$\alpha(\tau_j) = 0, \qquad \omega(\tau_j) = \omega_0.$$

Substituting $\lambda(\tau)$ into (2.2) and taking the derivative with respect to τ, we have

$$\left[\frac{d\lambda}{d\tau}\right]^{-1} = \frac{(2\lambda + p)e^{\lambda\tau}}{\lambda q} - \frac{\tau}{\lambda},$$

which, together with (2.3) and (2.5), leads to

$$\begin{aligned} \operatorname{sign}\left\{\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_j}\right\} &= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{(2\lambda + p)e^{\lambda\tau}}{\lambda q}\right]_{\tau=\tau_j}\right\} \\ &= \operatorname{sign}\left\{\operatorname{Re}\left[\frac{p\cos\omega_0\tau_j - 2\omega_0\sin\omega_0\tau_j + i[2\omega_0\cos\omega_0\tau_j + p\sin\omega_0\tau_j]}{iq\omega_0}\right]\right\} \\ &= \operatorname{sign}\left\{\frac{1}{q^2\omega_0}[2\omega_0 q\cos\omega_0\tau_j + pq\sin\omega_0\tau_j]\right\} \\ &= \operatorname{sign}\left\{\frac{1}{q^2}\left[2\omega_0^2 - 2r + p^2\right]\right\} \\ &= \operatorname{sign}\left\{\frac{\sqrt{(a_{11}^2x_*^2 - a_{22}^2y_*^2)^2 + 4x_*^2y_*^2a_{12}^2a_{21}^2}}{a_{12}^2a_{21}^2x_*^2y_*^2}\right\} > 0. \end{aligned} \tag{2.7}$$

Therefore, we can obtain the following results about the distribution of roots of Eq. (2.2).

Lemma 2.1. *Let ω_0, τ_j $(j = 0, 1, \ldots)$ be defined by* (2.5) *and* (2.6)*, respectively.*

(I) *If $a_{11}a_{22} - a_{12}a_{21} \geqslant 0$, then all roots of Eq.* (2.2) *have negative real parts for all $\tau \geqslant 0$.*
(II) *If $a_{11}a_{22} - a_{12}a_{21} < 0$ and $a_{11} + a_{22} \neq 0$, then Eq.* (2.2) *has a pair of simple imaginary roots $\pm i\omega_0$ at $\tau = \tau_j$. Furthermore, if $\tau \in [0, \tau_0)$, then all roots of Eq.* (2.2) *have negative real parts*; *if $\tau = \tau_0$, then all roots of* (2.2) *except $\pm i\omega_0$ have negative real parts*; *and if $\tau \in (\tau_j, \tau_{j+1})$, Eq.* (2.2) *has $2(j+1)$ roots with positive real parts.*
(III) *If $a_{11} = a_{22} = 0$, then Eq.* (2.2) *has a pair of simple imaginary roots $\pm i\omega_0$ at $\tau = \tau_j$ and has at least two roots with positive real parts for all $\tau > 0$.*

By the transversality condition (2.7), Lemma 2.1 and the Hopf bifurcation theorem for functional differential equations [9], we can state the following theorem.

Theorem 2.1. *Suppose $r_1a_{21} - r_2a_{11} > 0$ and let τ_j $(j = 0, 1, \ldots)$ be defined by* (2.6)*. Then we have the following.*

(i) *If $a_{11}a_{22} - a_{12}a_{21} \geqslant 0$, then the positive equilibrium E_* of* (2.1) *is asymptotically for all $\tau \geqslant 0$.*
(ii) *If $a_{11}a_{22} - a_{12}a_{21} < 0$ and $a_{11} + a_{22} \neq 0$, then the positive equilibrium E_* of* (2.1) *is asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. If $a_{11} = a_{22} = 0$, the positive equilibrium E_* of* (2.1) *is unstable for all $\tau > 0$.*
(iii) *If $a_{11}a_{22} - a_{12}a_{21} < 0$, then system* (2.1) *undergoes a Hopf bifurcation at the positive equilibrium E_* for $\tau = \tau_j$.*

3. Direction of Hopf bifurcations and stability of the bifurcating periodic orbits

In the previous section, we shown that system (2.1) undergoes the Hopf bifurcation at the critical values τ_j, $j = 0, 1, \ldots$. In this section, we will employ the algorithm of Faria and Magelháes [4,5] to compute explicitly the normal forms of system (2.1) on the center manifold. After that we will analyze the direction of the Hopf bifurcation and stability of the nontrivial periodic orbits of system (2.1) at $\tau = \tau_j$. Throughout this section, we refer to [4,5] for explanations of notations involved.

By the translation $z(t) = (u_1(t), u_2(t)) - E_*$ and rescaling the time by $t \mapsto t/\tau$, (2.1) can be written as

$$\dot{z}(t) = L(\tau)z_t + F(z_t, \tau), \tag{3.1}$$

in the phase space $C = C([-1, 0]; R^2)$, where, for $\varphi = (\varphi_1, \varphi_2)^T \in C$,

$$L(\tau)(\varphi) = \tau \begin{pmatrix} -x_*(a_{11}\varphi_1(0) + a_{12}\varphi_2(-1)) \\ y_*(a_{21}\varphi_1(0) - a_{22}\varphi_2(0)) \end{pmatrix}$$

and

$$F(\varphi, \tau) = \tau \begin{pmatrix} -a_{11}\phi_1^2(0) - a_{12}\phi_1(0)\phi_2(-1) \\ a_{21}\phi_1(0)\phi_2(0) - a_{22}\phi_2^2(0) \end{pmatrix}. \tag{3.2}$$

The linear map $L(\tau)$ may be expressed in integral form as

$$L(\tau)\phi = \int_{-1}^{0} \left[d\eta_\tau(\theta)\right]\phi(\theta),$$

where $\eta : [-1, 0] \to R^n$ is a function of bounded variation.

Introducing the new parameter $\mu = \tau - \tau_j$ so that $\mu = 0$ is a Hopf bifurcation value of (3.1), we can rewrite (3.1) as

$$\dot{z}(t) = L(\tau_j)z_t + \widetilde{F}(z_t, \mu), \tag{3.3}$$

where $\widetilde{F}(z_t, \mu) = L(\mu)z_t + F(z_t, \tau_j + \mu)$.

Let $\omega_j = \omega_0\tau_j$ and $\Lambda_0 = \{i\omega_j, -i\omega_j\}$. From the discussion in Section 2, we know that the characteristic equation of $\dot{z}(t) = L(\tau_j)z_t$ has a pair of simple imaginary roots $\pm i\omega_j$ and no other roots in the imaginary axis which are multiples of $\pm i\omega_j$. Thus, the nonresonance conditions relative to Λ_0 are satisfied. Let $\Phi = (\phi_1, \phi_2)$ be a matrix whose columns form a basis of the center space P of $\dot{z}(t) = L(\tau_j)z_t$ with $\phi_1(\theta) = e^{i\omega_j\theta}v$, $\phi_2(\theta) = e^{-i\omega_j\theta}\bar{v}$, where the bar means complex conjugation, and v is a vector in $\mathbb{C}^2$ that satisfies

$$L(\tau_j)\phi_1 = i\omega_j v. \tag{3.4}$$

With $C^* = C([0,1]; R^{2*})$, where R^{2*} is the 2-dimensional space of row vectors, we consider the adjoint bilinear form on $C^* \times C$ defined by

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-1}^{0}\int_{0}^{\theta} \psi(\xi - \theta)\, d\eta_{\tau_j}(\theta)\, \phi(\xi)\, d\xi, \quad \text{for } \psi \in C^*,\ \phi \in C.$$

Suppose $\Psi = col(\psi_1, \psi_2)$ is a matrix whose rows form a basis for the dual space P^* in C^* with $\langle \Psi, \Phi \rangle = I$ (the 2×2 identity matrix). Then we have $\Psi(s) = col(\psi_1(s), \psi_2(s)) = col(u^T e^{-i\omega_j s}, \bar{u}^T e^{i\omega_j s})$, $s \in [0,1]$, for $u \in \mathbb{C}^2$ such that

$$\langle \psi_1, \phi_1 \rangle = 1, \qquad \langle \psi_1, \phi_2 \rangle = 0.$$

By directly computing, we can choose

$$v = \begin{pmatrix} \frac{i\omega_0 + a_{22}y_*}{a_{21}y_*} \\ 1 \end{pmatrix}, \qquad u = d\begin{pmatrix} 1 \\ \frac{i\omega_0 + a_{11}x_*}{a_{21}y_*} \end{pmatrix}, \tag{3.5}$$

where $d = \frac{a_{21}y_*}{2i\omega_0 + a_{11}x_* + a_{22}y_* + \tau_j(i\omega_0 + a_{11}x_*)(i\omega_0 + a_{22}y_*)}$ such that $\langle \Psi(s), \Phi(\theta) \rangle = I$.

Faria and Magalháes [4,5] show that (3.3) can be written as an infinite dimensional ordinary differential equation on the Banach space BC of functions from $[-1,0]$ into R which are uniformly continuous on $[-1,0)$ and with a jump discontinuity at 0. Elements of BC are of the form $\phi + X_0\alpha$ where $\phi \in C$, $\alpha \in R^n$ and $X_0(\theta)$ is defined by

$$X_0(\theta) = \begin{cases} 1, & \theta = 0, \\ 0, & -r \leqslant \theta < 0. \end{cases}$$

Let $\pi : BC \to P$ denote the projection $\pi(\varphi + X_0\alpha) = \Phi[\langle \Psi, \varphi \rangle + \Psi(0)\alpha]$. We can write $BC = P \oplus \ker\pi$ with the property that $Q \subset \ker\pi$, where Q is an infinite dimensional complementary subspace of P in C. Let A be the infinitesimal generator for the flow of the linear system $\dot{z}(t) = L(\tau_j)z_t$. Decompose z_t in (3.3) according to the decomposition of BC, in the form $z_t = \Phi x(t) + y_t$, with $x \in R^m$ and $y_t \in \ker\pi \cap D(A) = Q \cap C^1 \stackrel{\text{def}}{=} Q^1$ where $D(A)$ is the domain of A. Define the 2×2 diagonal matrix

$$B = \begin{pmatrix} i\omega_j & 0 \\ 0 & -i\omega_j \end{pmatrix}.$$

Then Eq. (3.1) is equivalent to the following system:

$$\begin{aligned} \dot{x} &= Bx + \Psi(0)\widetilde{F}(\Phi x + y, \mu), \\ \frac{d}{dt}y &= A_{Q^1}y + (I - \pi)X_0\widetilde{F}(\Phi x + y, \mu), \end{aligned} \tag{3.6}$$

where $A_{Q^1} : Q^1 \to \ker\pi$ is such that $A_{Q^1}\phi = \dot{\phi} + X_0[L(\tau_j)(\phi) - \dot{\phi}(0)]$.

Let $\widetilde{F}_j$ be the jth Fréchet derivative of $\widetilde{F}$, and denote $f_j = (f_j^1, f_j^2)$, where

$$\begin{aligned} \frac{1}{j!}f_j^1(x,y,\mu) &= \Psi(0)\widetilde{F}_j(\Phi x + y, \mu), \\ \frac{1}{j!}f_j^2(x,y,\mu) &= (I - \pi)X_0\widetilde{F}_j(\Phi x + y, \mu), \end{aligned} \tag{3.7}$$

then (3.6) can be written as

$$\dot{x} = Bx + \sum_{j\geqslant 2} \frac{1}{j!} f_j^1(x, y, \mu),$$
$$\frac{d}{dt} y = A_{Q^1} y + \sum_{j\geqslant 2} \frac{1}{j!} f_j^2(x, y, \mu),$$

where $f_j^1(x, y, \mu)$ and $f_j^2(x, y, \mu)$ are homogeneous polynomials in (x, y, μ) of degree j with coefficients in $\mathbb{C}^2$, $\ker\pi$, respectively. Since the nonresonance conditions relative to Λ_0 are satisfied, the normal form theory due to Faria and Magalháes [4,5] implies that the center manifold is locally given by $y = 0$ and a normal form of (3.3) on this center manifold of the origin at $\mu = 0$ is given by

$$\dot{x} = Bx + \frac{1}{2} g_2^1(x, 0, \mu) + \frac{1}{3!} g_3^1(x, 0, \mu) + \text{h.o.t.}, \tag{3.8}$$

where g_2^1, g_3^1 are the second- and third-order terms in (x, μ), respectively, and h.o.t. stands for higher-order terms.

Let $V_j^{m+p}(X)$ be the homogeneous polynomials of degree j in $m + p$ real variables, $x = (x_1, \ldots, x_m)$, $\mu = (\mu_1, \ldots, \mu_p)$ with coefficients in X,

$$V_j^{m+p}(X) = \left\{ \sum_{|(q,l)|=j} c_{(q,l)} x^q \mu^l \colon c_{(q,l)} \in X \right\}, \quad \text{for } (q, l) = (q_1, \ldots, q_m, l_1, \ldots, l_p) \in N_0^{m+p},$$

with the norm given by the sum of the norms of the coefficients. Define

$$\big(M_j^1 p\big)(x, \mu) = D_x p(x, \mu) Bx - Bp(x, \mu),$$

where $p \in V_j^{2+1}(\mathbb{C}^2)$. It follows from [5] that

$$V_j^3\big(\mathbb{C}^2\big) = \operatorname{Im}\big(M_j^1\big) \oplus \operatorname{Ker}\big(M_j^1\big), \qquad g_j^1(x, 0, \mu) \in \operatorname{Ker}\big(M_j^1\big)$$

and

$$\operatorname{Ker}\big(M_j^1\big) = \operatorname{span}\big\{x^q \mu^l e_k \colon q_1\lambda_1 + q_2\lambda_2 = \lambda_k,\ k = 1, 2,\ q \in N_0^2,\ l \in N_0,\ q_1 + q_2 + l = j\big\},$$

where $(\lambda_1, \lambda_2) = (i\omega_j, -i\omega_j)$ and $\{e_1, e_2\}$ is the canonical basis of $\mathbb{C}^2$. Hence,

$$\operatorname{Ker}\big(M_2^1\big) = \operatorname{span}\left\{ \begin{pmatrix} x_1\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu \end{pmatrix} \right\},$$
$$\operatorname{Ker}\big(M_3^1\big) = \operatorname{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\mu^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\mu^2 \end{pmatrix} \right\}. \tag{3.9}$$

From (3.2), (3.3) and (3.7), we get

$$f_2^1(x, 0, \mu) = \Psi(0)\big[2L(\mu)(\Phi x) + F_2(\Phi x, \tau_j)\big]. \tag{3.10}$$

Thus, the second-order terms as described in (3.8) of the normal form on the center manifold are given by

$$\begin{aligned} \frac{1}{2} g_2^1(x, 0, \mu) &= \frac{1}{2} \operatorname{Proj}_{\operatorname{Ker}(M_2^1)} f_2^1(x, 0, \mu) \\ &= \operatorname{Proj}_{\operatorname{Ker}(M_2^1)} \Psi(0)\big(L(\mu)(\phi_1)x_1 + L(\mu)(\phi_2)x_2\big), \end{aligned}$$

where we have use (3.9) and (3.10). Note that $L(\mu) = (\mu/\tau_j)L(\tau_j)$. This, together with (3.4), yields

$$\frac{1}{2} g_2^1(x, 0, \mu) = \begin{pmatrix} A_1 x_1 \mu \\ \overline{A_1} x_2 \mu \end{pmatrix}, \tag{3.11}$$

with $A_1 = i\omega_0 u^T v$, for $u = \binom{u_1}{u_2}$ and $v = \binom{v_1}{v_2}$ defined as in (3.5).

Next we will compute the third-order terms $g_3^1(x,0,\mu)$ as described in (3.8). We first note that

$$g_3^1(x,0,\mu) = \mathrm{Proj}_{\mathrm{Ker}(M_3^1)} \tilde{f}_3^1(x,0,\mu) = \mathrm{Proj}_S \tilde{f}_3^1(x,0,0) + O\big(|x|\mu^2\big),$$

for

$$S = \mathrm{span}\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \right\}.$$

It is sufficient to compute only $\mathrm{Proj}_S \tilde{f}_3^1(x,0,0)$ for the purpose of determining the generic Hopf bifurcation. In fact $\tilde{f}_3$ denotes the third terms after the computation of the normal form up to the second terms, which is given by

$$\tilde{f}_3^1(x,0,0) = f_3^1(x,0,0) + \frac{3}{2}\big[\big(D_x f_2^1\big)U_2^1 - \big(D_x U_2^1\big)g_2^1\big](x,0,0) + \frac{3}{2}\big[\big(D_y f_2^1\big)h\big](x,0,0), \tag{3.12}$$

where $U_2^1(x,0) = (M_2^1)^{-1} f_2^1(x,0,0)$ and $h = h(x)(\theta)$ is evaluated by the following system

$$\begin{aligned} &\dot{h}(x) - D_x h(x) Bx = \Phi\Psi(0)\big[2L(0)(\Phi x) + F_2(\Phi x, \tau_j)\big], \\ &\dot{h}(x)(0) - L(\tau_j)\big(h(x)\big) = 2L(0)(\Phi x) + F_2(\Phi x, \tau_j), \end{aligned} \tag{3.13}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ with respect to θ.

From (3.11), it is clear to see

$$\big(D_x U_2^1\big) g_2^1(x,0,0) = 0. \tag{3.14}$$

In addition, it follows from (3.2), (3.3) and (3.7) that $f_3^1 = 0$. Thus, to compute $\mathrm{Proj}_S \tilde{f}_3^1(x,0,0)$, it is sufficient to determine $\mathrm{Proj}_S[(D_x f_2^1)U_2^1](x,0,0)$ and $\mathrm{Proj}_S[(D_y f_2^1)h](x,0,0)$.

We first determine $\mathrm{Proj}_S[(D_x f_2^1)U_2^1](x,0,0)$. From (2.2), (3.2) and (3.7), we have

$$f_2^1(x,y,0) = 2\tau_j \begin{pmatrix} u_1 & u_2 \\ \bar{u}_1 & \bar{u}_2 \end{pmatrix} \begin{pmatrix} -a_{11} l_1^2 - a_{12} l_1 l_3 \\ a_{21} l_1 l_2 - a_{22} l_2^2 \end{pmatrix}, \tag{3.15}$$

where

$$\begin{aligned} l_1 &= \varphi_1(0) = x_1 v_1 + x_2 \bar{v}_1 + y^{(1)}(0), \\ l_2 &= \varphi_2(0) = x_1 v_2 + x_2 \bar{v}_2 + y^{(2)}(0), \\ l_3 &= \varphi_2(-1) = x_1 v_2 e^{-i\omega_j} + x_2 \bar{v}_2 e^{i\omega_j} + y^{(2)}(-1). \end{aligned}$$

Furthermore,

$$f_2^1(x,0,0) = \begin{pmatrix} b_{20} x_1^2 + 2b_{11} x_1 x_2 + b_{02} x_2^2 \\ \bar{b}_{02} x_1^2 + 2\bar{b}_{11} x_1 x_2 + \bar{b}_{20} x_2^2 \end{pmatrix}, \tag{3.16}$$

where

$$b_{20} = 2i\omega_j u^T \begin{pmatrix} v_1^2/x_* \\ v_2^2/y_* \end{pmatrix}, \qquad b_{11} = 0, \qquad b_{02} = -2i\omega_j u^T \begin{pmatrix} \bar{v}_1^2/x_* \\ \bar{v}_2^2/y_* \end{pmatrix}. \tag{3.17}$$

By a direct but tedious computation, we can choose

$$U_2^1(x,0) = \big(M_2^1\big)^{-1} f_2^1(x,0,0) = \frac{1}{i\omega_j} \begin{pmatrix} b_{20} x_1^2 - 2b_{11} x_1 x_2 - \frac{1}{3} b_{02} x_2^2 \\ \frac{1}{3}\bar{b}_{02} x_1^2 + 2\bar{b}_{11} x_1 x_2 - \bar{b}_{20} x_2^2 \end{pmatrix},$$

which, together with (3.18), leads to

$$\mathrm{Proj}_S\big[\big(D_x f_2^1\big)U_2^1\big](x,0,0) = \begin{pmatrix} \frac{2i}{\omega_j}(b_{20} b_{11} - 2|b_{11}|^2 - \frac{1}{3}|b_{02}|^2) x_1^2 x_2 \\ -\frac{2i}{\omega_j}(\bar{b}_{20}\bar{b}_{11} - 2|b_{11}|^2 - \frac{1}{3}|b_{02}|^2) x_1 x_2^2 \end{pmatrix}.$$

This, together with (3.17), means

$$\mathrm{Proj}_S\big[\big(D_x f_2^1\big)U_2^1\big](x,0,0)=\begin{pmatrix}-\frac{2i}{3\omega_j}|b_{02}|^2x_1^2x_2\\ \frac{2i}{3\omega_j}|b_{02}|^2x_1x_2^2\end{pmatrix}. \tag{3.18}$$

Next we compute $\mathrm{Proj}_S[(D_y f_2^1)h](x,0,0)$, where $h(x)$, given by (3.13), is a second-order homogeneous polynomial in $(x_1,x_2)\in\mathbb{C}^2$ and coefficients in $\mathrm{Ker}\,\pi$, as

$$h=\big(h^{(1)}(\theta),h^{(2)}(\theta)\big)^T=h_{20}(\theta)x_1^2+h_{11}(\theta)x_1x_2+h_{02}(\theta)x_2^2.$$

This, together with (3.13) and (3.17), immediately leads to $h_{02}=\bar h_{20}$ and $h_{11}=0$. From (3.15), after computing, we get

$$\mathrm{Proj}_S\big[\big(D_y f_2^1\big)h\big](x,0,0)=\begin{pmatrix}2b_{21}^*x_1^2x_2\\ 2\bar b_{21}^*x_1x_2^2\end{pmatrix}, \tag{3.19}$$

where

$$b_{21}^*=\tau_j u^T\begin{pmatrix}[-(a_{11}+\frac{i\omega_0}{x_*})h_{20}^{(1)}(0)-a_{12}h_{20}^{(2)}(-1)]\bar v_1\\ [a_{21}h_{20}^{(1)}(0)-(a_{22}+\frac{i\omega_0}{y_*})h_{20}^{(2)}(0)]\bar v_2\end{pmatrix}. \tag{3.20}$$

Clearly, we still need to compute $h_{20}(\theta)$. By (3.13), (3.16) and (3.17), one can obtain the following equation:

$$\dot h_{20}-2i\omega_j h_{20}=(\Phi_1,\Phi_2)\begin{pmatrix}b_{20}\\ \bar b_{02}\end{pmatrix}$$

with the boundary condition

$$\dot h_{20}(0)-L(\tau_j)(h_{20})=2i\omega_j\begin{pmatrix}v_1^2/x_*\\ v_2^2/y_*\end{pmatrix}.$$

It follows that

$$h_{20}(\theta)=-\frac{1}{i\omega_j}\left(b_{20}e^{i\omega_j\theta}v+\frac{1}{3}\bar b_{02}e^{-i\omega_j\theta}\bar v\right)+e^{2i\omega_j\theta}c,$$

where $c=(c_1,c_2)^T$ with

$$c_1=\frac{2i\omega_0[v_1^2(2i\omega_0+a_{22}y_*)/x_*-v_2^2a_{12}x_*e^{-2i\omega_j}/y_*]}{(2i\omega_0+a_{11}x_*)(2i\omega_0+a_{22}y_*)+a_{12}a_{21}x_*y_*e^{-2i\omega_j}},$$
$$c_2=\frac{2i\omega_0[(2i\omega_0+a_{11}x_*)v_2^2/y_*+a_{21}y_*v_1^2/x_*]}{(2i\omega_0+a_{11}x_*)(2i\omega_0+a_{22}y_*)+a_{12}a_{21}x_*y_*e^{-2i\omega_j}}.$$

Combining these, we have

$$h_{20}^{(1)}(0)=-\frac{2}{3}\big(\mathrm{Re}\{u_1v_1\}+2u_1v_1,\mathrm{Re}\{u_2v_1\}+2u_2v_1\big)\begin{pmatrix}v_1^2/x_*\\ v_1^2/y_*\end{pmatrix}+c_1,$$
$$h_{20}^{(2)}(0)=-\frac{2}{3}\big(\mathrm{Re}\{u_1v_2\}+2u_1v_2,\mathrm{Re}\{u_2v_2\}+2u_2v_2\big)\begin{pmatrix}v_1^2/x_*\\ v_1^2/y_*\end{pmatrix}+c_2,$$
$$h_{20}^{(2)}(0)=-\frac{2}{3}\big(\mathrm{Re}\big\{u_1v_2e^{-i\omega_j}\big\}+2u_1v_2e^{-i\omega_j},\mathrm{Re}\big\{u_2v_2e^{-i\omega_j}\big\}+2u_2v_2e^{-i\omega_j}\big)\begin{pmatrix}v_1^2/x_*\\ v_1^2/y_*\end{pmatrix}+c_2e^{-2i\omega_j}.$$

It follows from (3.5), (3.17) and (3.20) that, after expatiatory computation,

$$a_{21}^*=-\frac{2}{3}\tau_j u^T\begin{pmatrix}\frac{\omega_0 i}{x_*}(|v_1|^2\bar u_1/x_*,\bar u_2\bar v_1^2v_2^2/y_*)\\ \frac{\omega_0 i}{y_*}(\bar u_1v_1^2\bar v_2^2/x_*,|v_2|^2\bar u_2/y_*)\end{pmatrix}-i\omega_j u^T\begin{pmatrix}[2v_1^2/x_*-c_1]\bar v_1/x_*\\ [2v_2^2/y_*-c_2]\bar v_2/y_*\end{pmatrix}. \tag{3.21}$$

At the moment, we can compute $\mathrm{Proj}_S\tilde f_3^1(x,0,0)$. From the above discussion and combining (3.12) with (3.18)–(3.19), one can obtain

$$\frac{1}{3!}g_3^1(x,0,0) = \frac{1}{3!}\operatorname{Proj}_S \tilde{f}_3^1(x,0,0) = \begin{pmatrix} A_2 x_1 x_2^2 \\ \overline{A_2} x_1 x_2^2 \end{pmatrix}$$

with $A_2 = -\frac{i}{6\omega_j}|b_{02}|^2 + \frac{1}{2}a_{21}^*$. Thus, the normal form (3.8) on the center manifold has the form

$$\dot{x} = Bx + \begin{pmatrix} A_1 x_1 \mu \\ \overline{A_1} x_2 \mu \end{pmatrix} + \begin{pmatrix} A_2 x_1 x_2^2 \\ \overline{A_2} x_1 x_2^2 \end{pmatrix} + O\big(|x|\mu^2 + |x^4|\big),$$

which can be written in real coordinates w through the change of variables $x_1 = w_1 - ix_2$, $x_2 = w_1 + iw_2$. Transforming to polar coordinates $w_1 = \rho\cos\xi$, $w_2 = \rho\sin\xi$, this normal form becomes

$$\begin{aligned} \dot{\rho} &= K_1\mu\rho + K_2\rho^3 + O\big(\mu^2\rho + |(\rho,\mu)|^4\big), \\ \dot{\xi} &= -\omega_j + O\big(|(\rho,\mu)|\big) \end{aligned} \tag{3.22}$$

with $K_1 = \operatorname{Re} A_1 = \operatorname{Re}(i\omega_0 u^T v)$, $K_2 = \operatorname{Re} A_2 = \frac{1}{2}\operatorname{Re}(b_{21}^*)$. From (3.5), one can obtain

$$\begin{aligned} \operatorname{sign}\{\operatorname{Im}(u^T v)\} &= \operatorname{sign}\{-\operatorname{Im}((u^T v)^{-1})\} \\ &= \operatorname{sign}\left\{-\operatorname{Im}\left(1 + \frac{\tau_j(i\omega_0 + a_{11}x_*)(i\omega_0 + a_{22}y_*)}{2i\omega_0 + a_{11}x_* + a_{22}y_*}\right)\right\} \\ &= \operatorname{sign}\left\{-\frac{\tau_j\omega_0(2\omega_0^2 + a_{11}^2x_*^2 + a_{22}^2y_*^2)}{\Upsilon}\right\} < 0, \end{aligned}$$

where $\Upsilon = (4\omega_0)^2 + (a_{11}x_* + a_{22}y_*)^2$. So,

$$K_1 = \operatorname{Re}\big(i\omega_0 u^T v\big) > 0.$$

By (3.21), one can get

$$\begin{aligned} K_2 &= \frac{1}{2}\operatorname{Re}\big(b_{21}^*\big) \\ &= \frac{1}{2}\operatorname{Re}\left\{-i\omega_j u^T \begin{pmatrix} [2v_1^2/x_* - c_1]\bar{v}_1/x_* \\ [2v_2^2/y_* - c_2]\bar{v}_2/y_* \end{pmatrix}\right\} \\ &= \frac{1}{2}\omega_j \operatorname{Im}\left\{\frac{2|v_1|^2}{x_*^2}u_1v_1 + \frac{2|v_2|^2}{y_*^2}u_2v_2 - \left(\frac{1}{x_*}u_1\bar{v}_1c_1 + \frac{1}{y_*}u_2\bar{v}_2c_2\right)\right\}. \end{aligned} \tag{3.23}$$

It is well know [2] that the sign of K_1K_2 determines the direction of the bifurcation (supercritical if $K_1K_2 < 0$, subcritical if $K_1K_2 > 0$), and the sign of K_2 determines the stability of the nontrivial periodic orbits (stable if $K_2 < 0$, unstable if $K_2 > 0$). Thus, if system (2.1) is given, then we will analyze the direction of the Hopf bifurcation and stability of the bifurcating periodic orbits at $\tau = \tau_j$.Thus, we have the following statement.

Theorem 3.1. *Let K_2 be defined by* (3.23). *If $r_1a_{21} - r_2a_{11} > 0$ and $a_{11}a_{22} - a_{12}a_{21} < 0$, then the flow on the center manifold of the origin for* (3.3) *at $\mu = 0$ is given in polar coordinates by* (3.22). *Moreover, if $K_2 < 0$, there exists a unique nontrivial periodic orbit in the neighborhood of $\rho = 0$ for $\mu > 0$ and the bifurcating periodic orbits are stable, while if $K_2 > 0$, there exists a unique nontrivial periodic orbit in the neighborhood of $\rho = 0$ for $\mu < 0$ and the bifurcating orbits are unstable.*

As an example, we consider the following system:

$$\begin{cases} \dot{x}(t) = x(t)\big[r_1 - a_{12}y(t - \nu_2)\big], \\ \dot{y}(t) = y(t)\big[-r_2 + a_{21}x(t - \nu_1)\big]. \end{cases} \tag{3.24}$$

Following the previous procedure, we can obtain

$$E_* = (x_*, y_*) = \left(\frac{r_2}{a_{21}}, \frac{r_1}{a_{12}}\right), \qquad \omega_0 = \sqrt{r_1 r_2}, \qquad \tau_j = \frac{2(j+1)\pi}{\omega_0}, \quad j = 0, 1, \ldots,$$

and

$$v = \begin{pmatrix} \frac{i\omega_0}{a_{21} y_*} \\ 1 \end{pmatrix}, \qquad u = \begin{pmatrix} \frac{a_{21} y_*}{2i\omega_0 - \tau_j \omega_0^2} \\ \frac{i\omega_0}{2i\omega_0 - \tau_j \omega_0^2} \end{pmatrix}.$$

Clearly,

$$u_1 v_1 = u_2 v_2 = u_1 \bar{v}_1 = u_2 \bar{v}_2 = u_2 = \frac{2 - i\omega_j}{4 + \omega_j^2}$$

and

$$c_1 = \frac{2a_{12}^2(-2r_1 + i\omega_0)}{3r_1^2 a_{21}}, \qquad c_2 = \frac{2a_{12}(2r_2 + i\omega_0)}{3r_1 r_2}.$$

Substituting these into (3.23), one can obtain

$$\begin{aligned} K_2 &= \frac{1}{2}\omega_j \operatorname{Im}\left\{ \frac{2|v_1|^2}{x_*^2} u_1 v_1 + \frac{2|v_2|^2}{y_*^2} u_2 v_2 - \left(\frac{1}{x_*} u_1 \bar{v}_1 c_1 + \frac{1}{y_*} u_2 \bar{v}_2 c_2 \right) \right\} \\ &= \frac{1}{2}\omega_j \operatorname{Im}\left\{ \left(\frac{2|v_1|^2}{x_*^2} + \frac{2}{y_*^2} + \frac{1}{x_*} c_1 - \frac{1}{y_*} c_2 \right) u_2 \right\} \\ &= \frac{1}{2}\omega_j \operatorname{Im}\left\{ \frac{2a_{12}^2 (r_1 + r_2)(2 - i\omega_j)}{3r_1^2 r_2 (4 + \omega_j^2)} \right\} \\ &= -\frac{2a_{12}^2 (r_1 + r_2)\omega_j^2}{3r_1^2 r_2 (4 + \omega_j^2)} < 0. \end{aligned}$$

Using Theorems 2.1 and 3.1, one immediately obtain the following.

Corollary 3.1. *Assume that the condition $r_1 a_{21} - r_2 a_{11} > 0$ holds. Then system* (3.24) *undergoes a Hopf bifurcation at the positive equilibrium E_* for $\tau = \tau_j$ and the bifurcating periodic orbits are stable on the center manifold.*

Remark 3.1. Although system (3.24) has been studied in [6], the author obtained the same result only for $r_1 = r_2$ and $\nu_1 = \nu_2$. Clearly, these limits have not been required in the present paper.

4. Global continuation of local Hopf bifurcations

In this section, we study the global continuation of periodic solutions bifurcating from the positive equilibrium E_*. Throughout this section, we follow closely the notations in [23]. For simplification of notations, setting $z(t) = (z_1(t), z_2(t))^T = (u_1(t), u_2(t))^T$, we may rewrite system (2.1) as the following functional differential equation:

$$\dot{z}(t) = F(z_t, \tau, p), \tag{4.1}$$

where $z_t(\theta) = (z_{1t}(\theta), z_{2t}(\theta))^T = (z_1(t+\theta), z_2(t+\theta))^T \in C([-\tau, 0], R^2)$. It is obvious that if

$$r_1 a_{21} - r_2 a_{11} > 0,$$

then (4.1) has three boundary equilibria $E_1 = (0, 0)$, $E_2 = (0, -\frac{r_2}{a_{22}})$, $E_3 = (\frac{r_1}{a_{11}}, 0)$ and a positive equilibrium $E_* = (x_*, y_*)$. Following the work of [23], we need to define

$$\begin{aligned} &\mathbb{X} = C\big([-\tau, 0], R^2\big), \\ &\Gamma = Cl\big\{(z, \tau, p) \in \mathbb{X} \times R \times R^+;\ z \text{ is a nonconstant periodic solution of (4.1)}\big\}, \\ &\mathcal{N} = \big\{(\bar{z}, \bar{\tau}, \bar{p});\ F(\bar{z}, \bar{\tau}, \bar{p}) = 0\big\}. \end{aligned}$$

Let $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ denote the connected component passing through $(E_*,\tau_j,\frac{2\pi}{\omega_0})$ in Γ, where τ_j is defined by (2.6). From Theorem 2.1, we know that $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ is nonempty.

For the benefit of readers, we first state the global Hopf bifurcation theory due to Wu [23] for functional differential equations.

Lemma 4.1. *Assume that (z_*,τ,p) is an isolated center satisfying the hypotheses (A_1)–(A_4) in [23]. Denote by $\ell_{(z_*,\tau,p)}$ the connected component of (z_*,τ,p) in Γ. Then either*

(i) *$\ell_{(z_*,\tau,p)}$ is unbounded, or*
(ii) *$\ell_{(z_*,\tau,p)}$ is bounded, $\ell_{(z_*,\tau,p)}\cap\Gamma$ is finite and*

$$\sum_{(z,\tau,p)\in\ell_{(z_*,\tau,p)}\cap\mathcal{N}}\gamma_m(z_*,\tau,p)=0,$$

for all $m=1,2,\dots,$ where $\gamma_m(z_*,\tau,p)$ is the mth crossing number of (z_*,τ,p) if $m\in J(z_*,\tau,p)$, or it is zero if otherwise.

Clearly, if (ii) in Lemma 4.1 is not true, then $\ell_{(z_*,\tau,p)}$ is unbounded. Thus, if the projections of $\ell_{(z_*,\tau,p)}$ onto z-space and onto p-space are bounded, then the projection of $\ell_{(z_*,\tau,p)}$ onto τ-space is unbounded. Further, if we can show that the projection of $\ell_{(z_*,\tau,p)}$ onto τ-space is away from zero, then the projection of $\ell_{(z_*,\tau,p)}$ onto τ-space must include interval $[\tau,\infty)$. Following this ideal, we can prove our results on the global continuation of local Hopf bifurcation.

Lemma 4.2. *If the condition $r_1a_{21}-r_2a_{11}>0$ holds, then all periodic solutions to* (2.1) *are uniformly bounded.*

Proof. Let $(u_1(t),u_2(t))$ be a solution of (2.1). Then it follows from (2.1) that

$$\begin{cases} u_1(t)=u_1(0)\exp\left\{\displaystyle\int_0^t\big[r_1-a_{11}u_1(s)-a_{12}u_2(s-\tau)\big]\,ds\right\},\\ u_2(t)=u_2(0)\exp\left\{\displaystyle\int_0^t\big[-r_2+a_{21}u_1(s)-a_{22}u_2(s)\big]\,ds\right\},\end{cases}$$

which implies that the solution of (2.1) cannot cross the x-axis and y-axis. Thus, the nonconstant periodic orbits must be located in the interior of each quadrant.

In the following, suppose that $(u_1(t),u_2(t))$ is a nonconstant periodic solution of (2.1). Then we immediately have $u_1(t)>0$. Otherwise, $u_1(t)<0$, and either $u_2(t)>0$ or $u_2(t)>0$. If $u_1(t)<0$ and $u_2(t)>0$, then the first equation of (2.1) implies

$$\dot u_2(t)=u_2(t)\big[-r_2+a_{21}u_1(t)-a_{22}u_2(t)\big]<0,$$

which contradicts the fact that $u_2(t)$ is periodic, while if $u_1(t)<0$ and $u_2(t)<0$, then the second equation of (2.1) implies

$$\dot u_1(t)=u_1(t)\big[r_1-a_{11}u_1(t)-a_{12}u_2(t-\tau)\big]>0,$$

which contradicts the fact that $u_1(t)$ is periodic. Thus, we need only to consider the following two cases.

For periodic functions $u_1(t)$ and $y(t)$, define

$$\begin{aligned} u_1(\xi_1)&=\min\{u_1(t)\}, &\quad u_1(\eta_1)&=\max\{u_1(t)\},\\ u_2(\xi_2)&=\min\{u_2(t)\}, &\quad u_2(\eta_2)&=\max\{u_2(t)\}. \end{aligned} \tag{4.2}$$

Case 1. Suppose that $u_1(t)>0$ and $u_2(t)>0$. From (2.1), we get

$$0=r_1-a_{11}u_1(\eta_1)-a_{12}u_2(\eta_1-\tau), \tag{4.3}$$

$$0=-r_2+a_{21}u_1(\eta_2)-a_{22}u_2(\eta_2). \tag{4.4}$$

Since $u_2(t) > 0$, it follows from (4.3) that

$$0 < u_1(\eta_1) \leqslant \frac{r_1}{a_{11}}. \tag{4.5}$$

On the other hand, by (4.4)–(4.5), we have

$$0 < u_2(\eta_2) \leqslant \frac{r_1 a_{21} - r_2 a_{11}}{a_{11} a_{22}}. \tag{4.6}$$

Case 2. Suppose that $u_1(t) > 0$ and $u_2(t) < 0$. Similarly, from (2.1), we have

$$0 = r_1 - a_{11} u_1(\eta_1) - a_{12} u_2(\eta_1 - \tau), \tag{4.7}$$

$$0 = -r_2 + a_{21} u_1(\xi_2) - a_{22} u_2(\xi_2). \tag{4.8}$$

Since $u_1(t) > 0$, it follows from (4.8) that

$$0 > u_2(\xi_2) > -\frac{r_2}{a_{22}}, \tag{4.9}$$

which, together with (4.7), implies

$$r_1 - a_{11} x(\eta_1) + \frac{r_2 a_{12}}{a_{22}} > 0.$$

Thus, we get

$$0 < u_1(\eta_1) < \frac{r_1 a_{22} + r_2 a_{12}}{a_{11} a_{22}}. \tag{4.10}$$

From the above discussion, the lemma follows immediately. □

Lemma 4.3. *If $r_1 a_{21} - r_2 a_{11} > 0$, then system* (2.1) *has no nonconstant periodic solution with period τ.*

Proof. Suppose for a contradiction that system (2.1) has nonconstant periodic solution with period τ. Then the following system (4.11) of ordinary differential equations has nonconstant periodic solution:

$$\begin{cases} \dot{u}(t) = u_1(t)\big[r_1 - a_{11} u_1(t) - a_{12} u_2(t)\big], \\ \dot{u}(t) = u_2(t)\big[-r_2 + a_{21} u_1(t) - a_{22} u_2(t)\big], \end{cases} \tag{4.11}$$

which has the same equilibria to system (2.1), i.e., $E_1 = (0,0)$, $E_2 = (0, -\frac{r_2}{a_{22}})$, $E_3 = (\frac{r_1}{a_{11}}, 0)$ and a unique positive equilibrium $E_* = (x_*, y_*)$. Note that x-axis and y-axis are the invariable manifold of system (4.11) and the orbits of system (4.11) do not intersect each other. Thus, there are no solutions crossing the coordinate axes. On the other hand, note the fact that if system (4.11) has a periodic solution, then there must be the equilibrium in its interior, and that E_1, E_2 and E_3 are located on the coordinate axis. Thus, we conclude that the periodic orbit of system (4.11) must lie in the first quadrant. It is well known that the positive equilibrium E_* is global asymptotically stable to the first quadrant (see, for example, [3,10]). Thus, there is not periodic orbit in the first quadrant, too. Lemma 4.3 is complete. □

Theorem 4.1. *Suppose that $r_1 a_{21} - r_2 a_{11} > 0$. Let ω_0 and τ_j $(j = 0, 1, \ldots)$ be defined by* (2.5) *and* (2.6)*, respectively.*

(i) *If $a_{11} a_{22} - a_{12} a_{21} < 0$ and $a_{11} + a_{22} \neq 0$, then, for each $\tau > \tau_j$ $(j \geqslant 1)$, system* (1.1) *has at least j periodic solutions.*
(ii) *If $a_{11} = a_{22} = 0$, then, for each $\tau > \tau_j$ $(j = 0, 1, \ldots)$, system* (1.1) *has at least $j + 1$ periodic solutions.*

Proof. It is sufficient to prove that the projection of $\ell_{(E_*, \tau_j, \frac{2\pi}{\omega_0})}$ onto τ-space is $[\bar{\tau}, \infty)$ for each $j > 0$, where $\bar{\tau} \leqslant \tau_j$.

The characteristic matrix of (4.1) at an equilibrium $\bar{z} = (\bar{z}^{(1)}, \bar{z}^{(2)}) \in R^2$ takes the following form:

$$\Delta(\bar{z}, \tau, p)(\lambda) = \lambda \,\mathrm{Id} - DF(\bar{z}, \bar{\tau}, \bar{p})\big(e^{\lambda \cdot} \mathrm{Id}\big),$$

i.e.,

$$\Delta(\bar{z},\tau,p)(\lambda)=\begin{pmatrix}\lambda-r_1+2a_{11}\bar{z}^{(1)}+a_{12}\bar{z}^{(2)} & a_{12}\bar{z}^{(1)}e^{-\lambda\tau_2}\\ -a_{21}\bar{z}^{(2)}e^{-\lambda\tau_1} & \lambda+r_2-a_{21}\bar{z}^{(1)}+2a_{22}\bar{z}^{(2)}\end{pmatrix}. \tag{4.12}$$

$(\bar{z},\bar{\tau},\bar{p})$ is called a center if $F(\bar{z},\bar{\tau},\bar{p})=0$ and $\det(\Delta(\bar{z},\bar{\tau},\bar{p})(\frac{2\pi}{p}i))=0$. A center $(\bar{z},\bar{\tau},\bar{p})$ is said to be isolated if it is the only center in some neighborhood of $(\bar{z},\bar{\tau},\bar{p})$. It follows from (4.13) that

$$\det\bigl(\Delta(E_1,\tau,p)(\lambda)\bigr)=\lambda^2+(r_2-r_1)\lambda-r_1r_2=0, \tag{4.13}$$

$$\det\bigl(\Delta(E_2,\tau,p)(\lambda)\bigr)=\lambda^2-\left(r_1+r_2+\frac{r_2a_{12}}{a_{22}}\right)\lambda+\frac{r_2(r_1a_{22}+r_2a_{12})}{a_{22}}=0 \tag{4.14}$$

and

$$\det\bigl(\Delta(E_3,\tau,p)(\lambda)\bigr)=\lambda^2+\left(r_1+r_2-\frac{r_1a_{21}}{a_{11}}\right)\lambda+\frac{r_1(r_2a_{11}-r_1a_{21})}{a_{11}}=0. \tag{4.15}$$

Obviously, each of Eqs. (4.13), (4.14) and (4.15) has no purely imaginary roots provided that $r_1a_{21}-r_2a_{11}>0$. Thus, we conclude that (4.1) has no the center of the form as (E_i,τ,p) $(i=1,2,3)$. On the other hand, from the discussion in Section 2 about the local Hopf bifurcation, it is easy to verify that $(E_*,\tau_j,\frac{2\pi}{\omega_0})$ is a isolated center, and there exist $\epsilon>0,\delta>0$ and a smooth curve $\lambda:(\tau_j-\delta,\tau_j+\delta)\to C$ such that $\det(\Delta(\lambda(\tau)))=0$, $|\lambda(\tau)-\omega_0|<\epsilon$ for all $\tau\in[\tau_j-\delta,\tau_j+\delta]$ and

$$\lambda(\tau_j)=\omega_0 i,\qquad \frac{d\,\mathrm{Re}\,\lambda(\tau)}{d\tau}\bigg|_{\tau=\tau_j}>0.$$

Let

$$\Omega_{\epsilon,\frac{2\pi}{\omega_0}}=\left\{(\eta,p);\ 0<\eta<\epsilon,\ \left|p-\frac{2\pi}{\omega_0}\right|<\epsilon\right\}.$$

It is easy to verify that on $[\tau_j-\delta,\tau_j+\delta]\times\partial\Omega_{\epsilon,\frac{2\pi}{\omega_0}}$,

$$\det\left(\Delta(E_*,\tau,p)\left(\eta+\frac{2\pi}{p}i\right)\right)=0\quad\text{if and only if }\eta=0,\ \tau=\tau_j,\ p=\frac{2\pi}{\omega_0}.$$

Therefore, the hypotheses (A_1)–(A_4) in [23] are satisfied. Moreover, if we define

$$H^{\pm}\left(E_*,\tau_j,\frac{2\pi}{\omega_0}\right)(\eta,p)=\det\left(\Delta(E_*,\tau_j\pm\delta,p)\left(\eta+i\frac{2\pi}{p}\right)\right),$$

then we have the crossing number of isolated center $(E_*,\tau_j,\frac{2\pi}{\omega_0})$ as follows

$$\begin{aligned}\gamma\left(E_*,\tau_j,\frac{2\pi}{\omega_0}\right)&=\deg_B\left(H^-\left(E_*,\tau_j,\frac{2\pi}{\omega_0}\right),\Omega_{\epsilon,\frac{2\pi}{\omega_0}}\right)\\&\quad-\deg_B\left(H^+\left(E_*,\tau_j,\frac{2\pi}{\omega_0}\right),\Omega_{\epsilon,\frac{2\pi}{\omega_0}}\right)\\&=-1.\end{aligned}$$

Thus, we have

$$\sum_{(\bar{z},\bar{\tau},\bar{p})\in C_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}}\gamma(\bar{z},\bar{\tau},\bar{p})<0,$$

where $(\bar{z},\bar{\tau},\bar{p})$, in fact, has all or parts of the form $(E_*,\tau_k,\frac{2\pi}{\omega_0})$ $(k=0,1,\ldots)$. It follows from Lemma 4.1 that the connected component $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ through $(E_*,\tau_j,\frac{2\pi}{\omega_0})$ in Γ is unbounded. From (2.6), one can show that if $a_{11}+a_{22}\neq 0$ (respectively $a_{11}=a_{22}=0$), then, for $j\geqslant 1$ (respectively $j\geqslant 0$),

$$\tau_j=\frac{1}{\omega_0}\arccos\left\{\frac{\omega_0^2-a_{11}a_{22}x_*y_*}{a_{12}a_{21}x_*y_*}\right\}+\frac{2j\pi}{\omega_0}\geqslant\frac{2\pi}{\omega_0},$$

which leads to

$$\frac{2\pi}{\omega_0} < \tau_j. \tag{4.16}$$

Now we prove that the projection of $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ onto τ-space is $[\bar{\tau},\infty)$, where $\bar{\tau} \leqslant \tau_j$. Clearly, it follows from the proof of Lemma 4.3 that system (1.1) with $\tau = 0$ has no nontrivial periodic solution. Hence, the projection of $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ onto τ-space is away from zero.

For a contradiction, we suppose that the projection of $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ onto τ-space is bounded. This means that the projection of $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ onto τ-space is included in a interval $(0,\tau^*)$. Noticing $\frac{2\pi}{\omega_0} < \tau_j$ and applying Lemma 4.3 we have $p < \tau^*$ for $(z(t),\tau,p)$ belonging to $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$. This implies that the projection of $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ onto p-space is bounded. Then, applying Lemma 4.2 we get that the connected component $\ell_{(E_*,\tau_j,\frac{2\pi}{\omega_0})}$ is bounded. This contradiction completes the proof. □

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