Journal of King Saud University – Science (2016) 28, 359–367

King Saud University

Journal of King Saud University – Science

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ORIGINAL ARTICLE

Matric variate Pearson type II-Riesz distribution

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Received 16 June 2015; accepted 22 August 2015
Available online 12 September 2015

KEYWORDS

Matric variate;
Elliptical distribution;
Real;
Complex;
Quaternion and octonion random matrices;
Beta-Riesz type I distributions

Abstract The Pearson type II distribution is well known and is used in the general framework of real normed division algebras and Riesz distribution theory. Also, the so called Pearson type II-Riesz distribution, based on the Kotz–Riesz distribution, is presented in a unified way valid in the context of real, complex, quaternion and octonion random matrices. Specifically, the central nonsingular matric variate generalised Pearson type II-Riesz distribution and beta-Riesz type I distributions are derived in the addressed multiple numerical field settings.

1. Introduction

Matrix distribution theory has transformed the vision of statistics applications in the last century; the usual real and univariate setting was generalised for large random objects in the standard numerical fields, constituting powerful techniques used in several branches of knowledge. That tendency allowed that any imaginable approach and application of univariate statistics could be taught in a greater framework. As usual, matrix generalisations based on real Gaussian models appeared in numerous papers over the past 50 years; verbatim copies of those classical results were translated separately into the complex and quaternion cases, without showing the underlying fact explained by certain abstract theories of mathematics.

Extensions to matrix variate non-Gaussian models opened an interesting perspective in the context of generalised invariant statistics and propitiate some strong results which are now widely applied in recent areas such as statistical shape theory and MANOVA. For example, transition to unified studies, of special families of distributions such as Pearson type II, took several years and required strong mathematical theories, which were usually out of the scope of statistical papers. In this case, the addressed distribution emerges in the following context: let $\mathbf{X}$ and $\mathbf{U}_1$ be random matrices independently distributed as matrix multivariate normal distribution and a Wishart distribution, respectively; then the random matrix $\mathbf{R} = \mathbf{L}^{-1}\mathbf{X}$, where $\mathbf{L}$ is any square root of $\mathbf{U} = \mathbf{L}^*\mathbf{L} = \mathbf{U}_1 + \mathbf{X}^*\mathbf{X}$, has a matric variate Pearson type II distribution. In the real case under normality, the *matric variate Pearson type II distribution* (also known as *matric variate inverted T distribution*) was studied separately by Khatri (1959), Dickey (1967) and Press (1982). Recently, in a general and unified setting, Díaz-García and Gutiérrez-Jáimez (2012) studied the *real, complex, quaternion and octonion* versions of this distribution.

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Peer review under responsibility of King Saud University.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.jksus.2015.08.006
1018-3647

Within the context of Bayesian inference, the posterior mean and generalised maximum likelihood estimators were found by Fang and Li (1999), assuming a matric variate Pearson type II distribution as the sampling model, and considering the posterior and marginal laws as the corresponding noninformative prior distributions. Meanwhile, with the frequentist approach, Díaz-García and Gutiérrez-Jáimez (2006) and Kotz and Nadarajah (2004) studied the normal regression based on Studentised errors.

In multivariate analysis the matric variate Pearson type II distribution is a source of interesting potential studies, for example, let $\mathbf{R}$ be a matric variate Pearson type II random matrix, then $\mathbf{R}^*\mathbf{R}$ follows a matrix multivariate beta type I distribution, a law which plays a fundamental role in MANOVA theory, see Khatri (1959, 1970) and Muirhead (1982).

A family of distributions on symmetric cones, termed the *matrix multivariate Riesz distributions*, was first introduced by Hassairi and Lajmi (2001) under the name of Riesz natural exponential family (Riesz NEF); it was based on a special case of the so-termed Riesz measure from Faraut and Korányi (1994, p. 137), going back to Riesz (1949). This Riesz distribution generalises the matrix multivariate gamma and Wishart distributions, containing them as particular cases. Subsequently, Díaz-García (2015c,a) proposed two versions of the Riesz distribution and two generalisations of a class of Kotz type distributions. The addressed general laws are termed *matrix multivariate Kotz–Riesz distribution* and contains the matrix multivariate normal distribution as a particular case.

With a similar philosophy, we can search a generalisation of the matric variate Pearson type II distribution, in the following way: let $\mathbf{R} = \mathbf{X}\mathbf{L}^{-1}$, where $\mathbf{L}$ is a upper triangular matrix such that $\mathbf{U} = \mathbf{L}^*\mathbf{L} = \mathbf{U}_1 + \mathbf{X}^*\mathbf{X}$; if we assume that $\mathbf{X}$ and $\mathbf{U}_1$ are independently distributed matrix multivariate Kotz–Riesz distribution and matrix multivariate Riesz distribution, then we can derive the required distribution of $\mathbf{R}$, which will be called the *matric variate Pearson type II-Riesz distribution*.

In the last 30 years, the theory of random matrix distributions has reached a substantial development involving certain special areas of mathematics. Essentially, these advances have been archived through two approaches based on the *theory of Jordan algebras* and the *theory of real normed division algebras*. A basic source of the mathematical tools of theory of random matrices distributions under Jordan algebras can be found in Faraut and Korányi (1994); and specifically, some works in the context of theory of random matrix distributions based on Jordan algebras are provided in Massam (1994), Casalis and Letac (1996), Hassairi and Lajmi (2001) and Hassairi et al. (2005), and the references therein. Parallel results on theory of random matrix distributions based on real normed division algebras have been also developed in random matrix theory and statistics, see Gross and Richards (1987), Dumitriu (2002), Forrester (2005) and Díaz-García and Gutiérrez-Jáimez (2011, 2013), among others. Instead of using Jordan algebras, Ishi (2000) and Boutouria and Hassiri (2009) studied several basic properties of the matrix multivariate Riesz distribution under *the structure theory of normal j-algebras* and *theory of Vinberg algebras*, respectively.

Finally, the application of some particular fields as *the octonions* seems to be unclear at present. An excellent review of the history, construction and properties of octonions can be found in Baez (2002); moreover, that author comments:

> *"Their relevance to geometry was quite obscure until 1925, when Élie Cartan described 'triality' – the symmetry between vector and spinors in 8-dimensional Euclidian space. Their potential relevance to physics was noticed in a 1934 paper by Jordan, von Neumann and Wigner on the foundations of quantum mechanics...Work along these lines continued quite slowly until the 1980s, when it was realised that the octionions explain some curious features of string theory...* ***However, there is still no*** **proof** ***that the octonions are useful for understanding the real world****. We can only hope that eventually this question will be settled one way or another."*

For the sake of completeness, the octonions will be considered in this work, but we must recognise that the application of the associated results can only be conjectured. Even so, some expectations are emerging, for example, Forrester (2005, Section 1.4.5, pp. 22-24) proved that the bi-dimensional eigenvalue density function of a 2×2 octonionic matrix Gaussian ensemble is obtained from the eigenvalue general joint density function of a Gaussian ensemble with $m = 2$ and $\beta = 8$, see notation in Section 2. Moreover, according to Faraut and Korányi (1994) and Sawyer (1997), it is easy to check that the results of this work are valid for the *algebra of Albert*, i.e., when the involved hermitian matrices or certain products of hermitian matrices are 3×3 octonionic matrices.

The present paper is organised as follows: basic concepts and notations of abstract algebra and Jacobians are summarised in Section 2; and, definitions and properties of the nonsingular central matric variate Pearson type II-Riesz and beta type I distributions are studied in Section 3. We emphasise that the results are derived in the context of real normed division algebras, a useful integrated and unified approach recently implemented in matrix distribution theory.

2. Preliminary results

A detailed discussion of real normed division algebras can be found in Baez (2002) and Neukirch et al. (1990). For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{A} \times \mathfrak{A} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}, x, y, z \in \mathfrak{A}$,

$$m(x, \lambda y + z) = \lambda m(x; y) + m(x; z),$$
$$m(\lambda x + y; z) = \lambda m(x; z) + m(y; z).$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

1. *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
2. *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,

3. *commutative* if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
4. *unital* if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{A}$ is nonzero and $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$ or $y = 0_{\mathfrak{A}}$ for all $x, y \in \mathfrak{A}$.

The term "division algebra", comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_{\mathfrak{A}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel we assume $\mathbb{F} = \mathfrak{R}$ and consider classes of division algebras over $\mathfrak{R}$ or "*real division algebras*" for short.

We introduce the algebras of *real numbers* $\mathfrak{R}$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, then $\mathfrak{A}$ is isomorphic to $\mathfrak{R}$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{A}$ be a real division algebra with identity 1. Then $\mathfrak{A}$ is said to be *normed* if there is an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that

$$(xy, xy) = (x, x)(y, y) \quad \text{forall } x, y \in \mathfrak{A}.$$

Let $\mathfrak{A}$ be a division algebra over the real numbers. Then $\mathfrak{A}$ has dimension either 1, 2, 4 or 8. In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see Edelman and Rao (2005) and Khatri (1984), respectively.

Finally, observe that

$\mathfrak{R}$ is a real commutative associative normed division algebra,
$\mathfrak{C}$ is a commutative associative normed division algebra,
$\mathfrak{H}$ is an associative normed division algebra,
$\mathfrak{O}$ is an alternative normed division algebra.

Let $\mathcal{L}^{\beta}_{n,m}$ be the set of all $n \times m$ matrices of rank $m \leqslant n$ over $\mathfrak{A}$ with m distinct positive singular values, where $\mathfrak{A}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{A}^{n\times m}$ be the set of all $n \times m$ matrices over $\mathfrak{A}$. The dimension of $\mathfrak{A}^{n\times m}$ over $\mathfrak{R}$ is βmn. Let $\mathbf{A} \in \mathfrak{A}^{n\times m}$, then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose.

Table 1 sets out the equivalence among the same concepts in the four normed division algebras.

We denote by $\mathfrak{S}^{\beta}_m$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m\times m}$ such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m\times m}$. Thus, $\mathfrak{P}^{\beta}_m$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^{\beta}_{n,m}$; then $\mathfrak{P}^{\beta}_m$ is an open subset of $\mathfrak{S}^{\beta}_m$.

Let $\mathfrak{D}^{\beta}_m$ consisting of all $\mathbf{D} \in \mathfrak{A}^{m\times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$ and let $\mathfrak{T}^{\beta}_U(m)$ be the subgroup of all *upper triangular* matrices $\mathbf{T} \in \mathfrak{A}^{m\times m}$ such that $t_{ij} = 0$ for $1 < i < j \leqslant m$.

For any matrix $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $d\mathbf{X}$ denotes the*matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $\mathfrak{S}^{\beta}_m$, $\mathfrak{D}^{\beta}_m$ or $\mathcal{V}^{\beta}_{m,n}$, see Díaz-García and Gutiérrez-Jáimez (2011, 2013).

If $\mathbf{X} \in \mathfrak{A}^{n\times m}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{A}^{n\times m}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{n}\bigwedge_{j=1}^{m} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx^{(k)}_{ij}.$$

If $\mathbf{S} \in \mathfrak{S}^{\beta}_m$ (or $\mathbf{S} \in \mathfrak{T}^{\beta}_U(m)$ with $t_{ii} > 0, i = 1, \dots, m$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}^{\beta}_m$ or in $\mathfrak{T}^{\beta}_U(m)$) denotes the exterior product of the $m(m-1)\beta/2 + m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i>j}^{m} \bigwedge_{k=1}^{\beta} ds^{(k)}_{ij}.$$

Observe that the Lebesgue measure $(d\mathbf{S})$ requires that $\mathbf{S} \in \mathfrak{P}^{\beta}_m$, i.e., $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}^{\beta}_m$ then $(d\mathbf{\Lambda})$ (the Lebesgue measure in $\mathfrak{D}^{\beta}_m$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda^{(k)}_{i}.$$

If $\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}$ then

$$(\mathbf{H}^*_1 d\mathbf{H}_1) = \bigwedge_{i=1}^{m} \bigwedge_{j=i+1}^{n} \mathbf{h}^*_j d\mathbf{h}_i.$$

where $\mathbf{H} = (\mathbf{H}^*_1 | \mathbf{H}^*_2)^* = (\mathbf{h}_1, \dots, \mathbf{h}_m | \mathbf{h}_{m+1}, \dots, \mathbf{h}_n)^* \in \mathfrak{U}^{\beta}(n)$. It can be proved that this differential form does not depend on the choice of the $\mathbf{H}_2$ matrix. When $n = 1$; $\mathcal{V}^{\beta}_{m,1}$ defines the unit sphere in $\mathfrak{A}^m$, an $(m-1)\beta$-dimensional surface in $\mathfrak{A}^m$. When $n = m$ and denoting $\mathbf{H}_1$ by $\mathbf{H}$, $(\mathbf{H}d\mathbf{H}^*)$ is termed the *Haar measure* on $\mathfrak{U}^{\beta}(m)$.

The surface area or volume of the Stiefel manifold $\mathcal{V}^{\beta}_{m,n}$ is

$$\operatorname{Vol}(\mathcal{V}^{\beta}_{m,n}) = \int_{\mathbf{H}_1 \in \mathcal{V}^{\beta}_{m,n}} (\mathbf{H}_1 d\mathbf{H}^*_1) = \frac{2^m \pi^{mn\beta/2}}{\Gamma^{\beta}_m[n\beta/2]}, \tag{1}$$

where $\Gamma^{\beta}_m[a]$ denotes the multivariate *Gamma function* for the space $\mathfrak{S}^{\beta}_m$. This can be obtained as a particular case of the *generalised gamma function of weight* κ for the space $\mathfrak{S}^{\beta}_m$ with $\kappa = (k_1, k_2, \dots, k_m) \in \mathfrak{R}^m$, taking $\kappa = (0, 0, \dots, 0) \in \mathfrak{R}^m$. In

Table 1 Notation.

Real	Complex	Quaternion	Octonion	Generic notation
Semi-orthogonal	Semi-unitary	Semi-symplectic	Semi-exceptional type	$\mathcal{V}^{\beta}_{m,n}$
Orthogonal	Unitary	Symplectic	Exceptional type	$\mathfrak{U}^{\beta}(m)$
Symmetric	Hermitian	Quaternion hermitian	Octonion hermitian	$\mathfrak{S}^{\beta}_m$

general, for $\text{Re}(a) \geqslant (m-1)\beta/2 - k_m$, Gross and Richards (1987) and Faraut and Korányi (1994) have defined,

$$\Gamma_m^\beta[a,\kappa] = \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A})(d\mathbf{A})$$

$$= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a+k_i-(i-1)\beta/2] \tag{2}$$

$$= [a]_\kappa^\beta \Gamma_m^\beta[a], \tag{3}$$

where $\text{etr}(\cdot) = \exp(\text{tr}(\cdot))$, $|\cdot|$ denotes the determinant, and for $\mathbf{A}\in\mathfrak{S}_m^\beta$

$$q_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m}\prod_{i=1}^{m-1}|\mathbf{A}_i|^{k_i-k_{i+1}} \tag{4}$$

with $\mathbf{A}_p = (a_{rs})$, $r,s = 1,2,\dots,p$, $p = 1,2,\dots,m$ is termed the *highest weight vector*, see Gross and Richards (1987). Also,

$$\Gamma_m^\beta[a] = \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1}(d\mathbf{A})$$

$$= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a-(i-1)\beta/2],$$

and $\text{Re}(a) > (m-1)\beta/2$.

In other branches of mathematics the *highest weight vector* $q_\kappa(\mathbf{A})$ is also termed the *generalised power* of $\mathbf{A}$ and is denoted as $\Delta_\kappa(\mathbf{A})$, see Faraut and Korányi (1994) and Hassairi and Lajmi (2001).

Several properties of $q_\kappa(\mathbf{A})$ can be easily obtained, a list of them is given next:

1. Let $\mathbf{A} = \mathbf{L}^*\mathbf{D}\mathbf{L}$ be the L'DL decomposition of $\mathbf{A}\in\mathfrak{P}_m^\beta$, where $\mathbf{L}\in\mathfrak{T}_U^\beta(m)$ with $l_{ii} = 1$, $i = 1,2,\dots,m$ and $\mathbf{D} = \text{diag}(\lambda_1,\dots,\lambda_m)$, $\lambda_i \geqslant 0$, $i = 1,2,\dots,m$. Then

$$q_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_i}. \tag{5}$$

2.

$$q_\kappa(\mathbf{A}^{-1}) = q^*_{-\kappa^*}(\mathbf{A}), \tag{6}$$

where $\kappa^* = (k_m,k_{m-1},\dots,k_1)$, $-\kappa^* = (-k_m,-k_{m-1},\dots,-k_1)$,

$$q^*_\kappa(\mathbf{A}) = |\mathbf{A}_m|^{k_m}\prod_{i=1}^{m-1}|\mathbf{A}_i|^{k_i-k_{i+1}} \tag{7}$$

and

$$q^*_\kappa(\mathbf{A}) = \prod_{i=1}^m \lambda_i^{k_{m-i+1}}, \tag{8}$$

see Faraut and Korányi (1994, pp. 126–127 and Proposition VII.1.5).

Alternatively, let $\mathbf{A} = \mathbf{T}^*\mathbf{T}$ the Cholesky decomposition of matrix $\mathbf{A}\in\mathfrak{P}_m^\beta$, with $\mathbf{T} = (t_{ij})\in\mathfrak{T}_U^\beta(m)$, then $\lambda_i = t_{ii}^2, t_{ii}\geqslant 0, i = 1,2,\dots,m$. See Hassairi and Lajmi (2001, p. 931, first paragraph), Hassairi et al. (2005, p. 390, lines -11 to -16) and Kołodziejek (2014, p.5, lines 1-6).

3. if $\kappa = (p,\dots,p)$, then

$$q_\kappa(\mathbf{A}) = |\mathbf{A}|^p, \tag{9}$$

in particular if $p = 0$, then $q_\kappa(\mathbf{A}) = 1$.

4. if $\tau = (t_1,t_2,\dots,t_m), t_1\geqslant t_2\geqslant\cdots\geqslant t_m\geqslant 0$, then

$$q_{\kappa+\tau}(\mathbf{A}) = q_\kappa(\mathbf{A})q_\tau(\mathbf{A}), \tag{10}$$

in particular if $\tau = (p,p,\dots,p)$, then

$$q_{\kappa+\tau}(\mathbf{A}) \equiv q_{\kappa+p}(\mathbf{A}) = |\mathbf{A}|^p q_\kappa(\mathbf{A}). \tag{11}$$

5. Finally, for $\mathbf{B}\in\mathfrak{T}_U^\beta(m)$ in such a manner that $\mathbf{C} = \mathbf{B}^*\mathbf{B}\in\mathfrak{S}_m^\beta$,

$$q_\kappa(\mathbf{B}^*\mathbf{A}\mathbf{B}) = q_\kappa(\mathbf{C})q_\kappa(\mathbf{A}) \tag{12}$$

and

$$q_\kappa(\mathbf{B}^{*-1}\mathbf{A}\mathbf{B}^{-1}) = (q_\kappa(\mathbf{C}))^{-1}q_\kappa(\mathbf{A}) = q_{-\kappa}(\mathbf{C})q_\kappa(\mathbf{A}), \tag{13}$$

see Hassairi et al. (2008, p. 776, Eq. (2.1)).

Remark 1. Let $\mathcal{P}(\mathfrak{S}_m^\beta)$ be the algebra of all polynomial functions on $\mathfrak{S}_m^\beta$, and $\mathcal{P}_k(\mathfrak{S}_m^\beta)$ the subspace of homogeneous polynomials of degree k and let $\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$ be an irreducible subspace of $\mathcal{P}(\mathfrak{S}_m^\beta)$ such that

$$\mathcal{P}_k(\mathfrak{S}_m^\beta) = \sum_\kappa \oplus\,\mathcal{P}^\kappa(\mathfrak{S}_m^\beta).$$

Note that q_κ is a homogeneous polynomial of degree k, moreover $q_\kappa\in\mathcal{P}^\kappa(\mathfrak{S}_m^\beta)$, see Gross and Richards (1987).

In (3), $[a]_\kappa^\beta$ denotes the generalised Pochhammer symbol of weight κ, defined as

$$[a]_\kappa^\beta = \prod_{i=1}^m (a-(i-1)\beta/2)_{k_i}$$

$$= \frac{\pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a+k_i-(i-1)\beta/2]}{\Gamma_m^\beta[a]}$$

$$= \frac{\Gamma_m^\beta[a,\kappa]}{\Gamma_m^\beta[a]},$$

where $\text{Re}(a) > (m-1)\beta/2 - k_m$ and

$$(a)_i = a(a+1)\cdots(a+i-1)$$

is the standard Pochhammer symbol.

An alternative definition of the generalised gamma function of weight κ is proposed by Khatri, 1966:

$$\Gamma_m^\beta[a,-\kappa] = \int_{\mathbf{A}\in\mathfrak{P}_m^\beta} \text{etr}\{-\mathbf{A}\}|\mathbf{A}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{A}^{-1})(d\mathbf{A}) \tag{14}$$

$$= \pi^{m(m-1)\beta/4}\prod_{i=1}^m \Gamma[a-k_i-(m-i)\beta/2]$$

$$= \frac{(-1)^k\Gamma_m^\beta[a]}{[-a+(m-1)\beta/2+1]_\kappa^\beta}, \tag{15}$$

where $\text{Re}(a) > (m-1)\beta/2 + k_1$.

Consider also the following generalised beta functions termed, *generalised c-beta function*, see Faraut and Korányi (1994, p. 130) and Díaz-García (2015b),

$$\mathcal{B}_m^\beta[a,\kappa;b,\tau] = \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S})|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m-\mathbf{S})(d\mathbf{S}) = \int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R})|\mathbf{I}_m+\mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}(\mathbf{I}_m+\mathbf{R})(d\mathbf{R}) = \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b,\kappa+\tau]},$$

where $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$, $\tau=(t_1,t_2,\dots,t_m)\in\Re^m$, $\mathrm{Re}(a)>(m-1)\beta/2-k_m$ and $\mathrm{Re}(b)>(m-1)\beta/2-t_m$. Similarly defined is the *generalised k-beta function* as, see Díaz-García (2015b),

$$\mathcal{B}_m^\beta[a,-\kappa;b,-\tau] = \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}^{-1})|\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau\left((\mathbf{I}_m-\mathbf{S})^{-1}\right)(d\mathbf{S}) = \int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}^{-1})|\mathbf{I}_m+\mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}\left((\mathbf{I}_m+\mathbf{R})^{-1}\right)(d\mathbf{R}) = \frac{\Gamma_m^\beta[a,-\kappa]\Gamma_m^\beta[b,-\tau]}{\Gamma_m^\beta[a+b,-\kappa-\tau]},$$

where $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$, $\tau=(t_1,t_2,\dots,t_m)\in\Re^m$, $\mathrm{Re}(a)>(m-1)\beta/2+k_1$ and $\mathrm{Re}(b)>(m-1)\beta/2+t_1$.

Finally, the following Jacobians involving the β parameter, reflects the generalised power of the algebraic technique; they can be seen as extensions of the full derived and unconnected results in the real, complex or quaternion cases, see Faraut and Korányi (1994) and Díaz-García and Gutiérrez-Jáimez (2011). These results are the base for several matrix and matric variate generalised analyses.

Proposition 1. *Let* $\mathbf{X}$ *and* $\mathbf{Y}\in\mathcal{L}_{n,m}^\beta$ *be matrices of functionally independent variables, and let* $\mathbf{Y}=\mathbf{AXB}+\mathbf{C}$, *where* $\mathbf{A}\in\mathcal{L}_{n,n}^\beta$, $\mathbf{B}\in\mathcal{L}_{m,m}^\beta$ *and* $\mathbf{C}\in\mathcal{L}_{n,m}^\beta$ *are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{m\beta/2}|\mathbf{B}^*\mathbf{B}|^{mn\beta/2}(d\mathbf{X}). \tag{16}$$

Proposition 2. *Let* $\mathbf{X}$ *and* $\mathbf{Y}\in\mathfrak{S}_m^\beta$ *be matrices of functionally independent variables, and let* $\mathbf{Y}=\mathbf{AXA}^*+\mathbf{C}$, *where* $\mathbf{A}\in\mathcal{L}_{m,m}^\beta$ *and* $\mathbf{C}\in\mathfrak{S}_m^\beta$ *are constant matrices. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{(m-1)\beta/2+1}(d\mathbf{X}). \tag{17}$$

Proposition 3. *Let* $\mathbf{X}\in\mathcal{L}_{n,m}^\beta$ *be matrix of functionally independent variables, and write* $\mathbf{X}=\mathbf{V}_1\mathbf{T}$, *where* $\mathbf{V}_1\in\mathcal{V}_{m,n}^\beta$ *and* $\mathbf{T}\in\mathfrak{T}_U^\beta(m)$ *with positive diagonal elements. Define* $\mathbf{S}=\mathbf{X}^*\mathbf{X}\in\mathfrak{P}_m^\beta$. *Then*

$$(d\mathbf{X}) = 2^{-m}|\mathbf{S}|^{\beta(n-m+1)/2-1}(d\mathbf{S})(\mathbf{V}_1^*d\mathbf{V}_1), \tag{18}$$

3. Matric variate Pearson type II-Riesz distribution

Two versions of the matric variate Pearson type II-Riesz distributions and the corresponding generalised beta type I distributions are obtained in this section.

A discussion of Riesz distribution may be found in Hassairi and Lajmi (2001) and Díaz-García (2015a); and a description of Kotz–Riesz distribution is given in Díaz-García (2015b). For convenience, we adhere to standard notation stated in Díaz-García (2015a,b). Now, consider the following two definitions.

Definition 1. Let $\mathbf{\Sigma}\in\mathbf{\Phi}_m^\beta$, $\mathbf{\Theta}\in\mathbf{\Phi}_n^\beta$, $\boldsymbol{\mu}\in\mathfrak{L}_{n,m}^\beta$ and $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$. And let $\mathbf{Y}\in\mathfrak{L}_{n,m}^\beta$ and $\mathcal{U}(\mathbf{B})\in\mathfrak{T}_U^\beta(n)$, such that $\mathbf{B}=\mathcal{U}(\mathbf{B})^*\mathcal{U}(\mathbf{B})$ is the Cholesky decomposition of $\mathbf{B}\in\mathfrak{S}_m^\beta$, then:

1. It is said that $\mathbf{Y}$ has a Kotz–Riesz distribution of type I and its density function is

$$\frac{\beta^{mn\beta/2+\sum_{i=1}^m k_i}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\kappa]|\mathbf{\Sigma}|^{n\beta/2}|\mathbf{\Theta}|^{m\beta/2}} \times \mathrm{etr}\left\{-\beta\mathrm{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\} \times q_\kappa\left[\mathcal{U}(\mathbf{\Sigma})^{*-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\mathcal{U}(\mathbf{\Sigma})^{-1}\right](d\mathbf{Y}) \tag{19}$$

with $\mathrm{Re}(n\beta/2)>(m-1)\beta/2-k_m$; denoting this distribution as

$$\mathbf{Y}\sim\mathcal{KR}_{n\times m}^{\beta,I}(\kappa,\boldsymbol{\mu},\mathbf{\Theta},\mathbf{\Sigma}).$$

2. And it is said that $\mathbf{Y}$ has a Kotz–Riesz distribution of type II and its density function is

$$\frac{\beta^{mn\beta/2-\sum_{i=1}^m k_i}\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,-\kappa]|\mathbf{\Sigma}|^{n\beta/2}|\mathbf{\Theta}|^{m\beta/2}} \times \mathrm{etr}\left\{-\beta\mathrm{tr}\left[\mathbf{\Sigma}^{-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\right]\right\} \times q_\kappa\left[\left(\mathcal{U}(\mathbf{\Sigma})^{*-1}(\mathbf{Y}-\boldsymbol{\mu})^*\mathbf{\Theta}^{-1}(\mathbf{Y}-\boldsymbol{\mu})\mathcal{U}(\mathbf{\Sigma})^{-1/2}\right)^{-1}\right](d\mathbf{Y}) \tag{20}$$

with $\mathrm{Re}(n\beta/2)>(m-1)\beta/2+k_1$; denoting this distribution as

$$\mathbf{Y}\sim\mathcal{KR}_{n\times m}^{\beta,II}(\kappa,\boldsymbol{\mu},\mathbf{\Theta},\mathbf{\Sigma}).$$

Definition 2. Let $\mathbf{\Xi}\in\mathbf{\Phi}_m^\beta$ and $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$, then:

1. It is said that $\mathbf{V}$ has a Riesz distribution of type I if its density function is

$$\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Xi}|^a q_\kappa(\mathbf{\Xi})}\mathrm{etr}\{-\beta\mathbf{\Xi}^{-1}\mathbf{V}\}|\mathbf{V}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{V})(d\mathbf{V}) \tag{21}$$

for $\mathbf{V}\in\mathfrak{P}_m^\beta$ and $\mathrm{Re}(a)\geqslant(m-1)\beta/2-k_m$; denoting this distribution as $\mathbf{V}\sim\mathcal{R}_m^{\beta,I}(a,\kappa,\mathbf{\Xi})$.

2. And, it is said that $\mathbf{V}$ has a Riesz distribution of type II if its density function is

$$\frac{\beta^{am-\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,-\kappa]|\mathbf{\Xi}|^a q_\kappa(\mathbf{\Xi}^{-1})}\mathrm{etr}\{-\beta\mathbf{\Xi}^{-1}\mathbf{V}\}|\mathbf{V}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{V}^{-1})(d\mathbf{V}) \tag{22}$$

for $\mathbf{V} \in \mathfrak{P}_m^\beta$ and $\text{Re}(a) > (m-1)\beta/2 + k_1$; denoting this distribution as $\mathbf{V} \sim \mathcal{R}_m^{\beta,II}(a,\kappa,\mathbf{\Xi})$.

Theorem 1. *Let* $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, *and* $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$. *Also define* $\mathbf{R} \in \mathcal{L}_{n,m}^\beta$ *as*

$$\mathbf{R} = \mathbf{X}\mathbf{L}^{-1},$$

where $\mathbf{L} \in \mathfrak{T}_U^\beta(m)$ *is such that* $\mathbf{U} = \mathbf{L}^*\mathbf{L} = \mathbf{U}_1 + \mathbf{X}^*\mathbf{X}$ *is the Cholesky decomposition of* $\mathbf{U}$,

1. *with* $\mathbf{U}_1 \sim \mathcal{R}_m^{\beta,I}(\nu\beta/2,\kappa,\mathbf{I}_m), \text{Re}(\nu\beta/2) > (m-1)\beta/2 - k_m$; *independent of* $\mathbf{X} \sim \mathcal{KR}_{n\times m}^{\beta,I}(\tau,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$, $\text{Re}(n\beta/2) > (m-1)\beta/2 - t_m$. *Then* $\mathbf{U} \sim \mathcal{R}_m^{\beta,I}((\nu+n)\beta/2,\kappa+\tau,\mathbf{I}_m)$ *independent of* $\mathbf{R}$ *with* $\text{Re}((\nu+n)\beta/2) > (m-1)\beta/2 - k_m - t_m$. *Furthermore, the density of* $\mathbf{R}$ *is*
$$\frac{\Gamma_m^\beta[n\beta/2]|\mathbf{I}_m - \mathbf{R}^*\mathbf{R}|^{(\nu-m+1)\beta/2-1}}{\pi^{mn\beta/2}\mathcal{B}_m^\beta[\nu\beta/2,\kappa;n\beta/2,\tau]} q_\kappa(\mathbf{I}_m - \mathbf{R}^*\mathbf{R})q_\tau(\mathbf{R}^*\mathbf{R})(d\mathbf{R}), \tag{23}$$
which shall be termed the matric variate Pearson type II-Riesz distribution type I, where $\mathbf{I}_m - \mathbf{R}^*\mathbf{R} \in \mathfrak{P}_m^\beta$.
2. *with* $\mathbf{U}_1 \sim \mathcal{R}_m^{\beta,II}(\nu\beta/2,\kappa,\mathbf{I}_m), \text{Re}(\nu\beta/2) > (m-1)\beta/2 + k_1$; *independent of* $\mathbf{X} \sim \mathcal{KR}_{n\times m}^{\beta,II}(\tau,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$, $\text{Re}(n\beta/2) > (m-1)\beta/2 + t_1$. *Then* $\mathbf{U} \sim \mathcal{R}_m^{\beta,II}((\nu+n)\beta/2,\kappa+\tau,\mathbf{I}_m)$ *independent of* $\mathbf{R}$ *with* $\text{Re}((\nu+n)\beta/2) > (m-1)\beta/2 + k_1 + t_1$. *Furthermore, the density of* $\mathbf{R}$ *is*
$$\frac{\Gamma_m^\beta[n\beta/2]|\mathbf{I}_m - \mathbf{R}^*\mathbf{R}|^{(\nu-m+1)\beta/2-1}}{\pi^{mn\beta/2}\mathcal{B}_m^\beta[\nu\beta/2,-\kappa;n\beta/2,-\tau]} q_\kappa\left[(\mathbf{I}_m - \mathbf{R}^*\mathbf{R})^{-1}\right] \times q_\tau\left[(\mathbf{R}^*\mathbf{R})^{-1}\right](d\mathbf{R}), \tag{24}$$
which shall be termed the matric variate Pearson type II-Riesz distribution type II, where $\mathbf{I}_m - \mathbf{R}^*\mathbf{R} \in \mathfrak{P}_m^\beta$.

Proof.

1. From Definitions 1 and 2, the joint density of $\mathbf{U}_1$ and $\mathbf{X}$ is
$$\propto |\mathbf{U}_1|^{(\nu-m+1)\beta/2-1}\text{etr}\{-\beta(\mathbf{U}_1 + \mathbf{X}^*\mathbf{X})\}q_\kappa(\mathbf{U}_1)q_\tau(\mathbf{X}^*\mathbf{X})(d\mathbf{U}_1)(d\mathbf{X}),$$
where the constant of proportionality given by
$$c = \frac{\beta^{\nu m\beta/2+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[\nu\beta/2,\kappa]} \cdot \frac{\beta^{mn\beta/2+\sum_{i=1}^m t_i}\,\Gamma_m^\beta[n\beta/2]}{\pi^{mn\beta/2}\Gamma_m^\beta[n\beta/2,\tau]}.$$
Making the change of variable $\mathbf{U}_1 = (\mathbf{U} - \mathbf{X}^*\mathbf{X})$ and $\mathbf{X} = \mathbf{R}\mathbf{L}$, where $\mathbf{U} = \mathbf{L}^*\mathbf{L}$, then by (16)
$$(d\mathbf{U}_1)(d\mathbf{X}) = |\mathbf{L}^*\mathbf{L}|^{n\beta/2}(d\mathbf{U})(d\mathbf{R}) = |\mathbf{U}|^{n\beta/2}(d\mathbf{U})(d\mathbf{R}),$$
and observing that $|\mathbf{U}_1| = |\mathbf{U} - \mathbf{X}^*\mathbf{X}| = |\mathbf{U} - \mathbf{L}^*\mathbf{R}^*\mathbf{R}\mathbf{L}| = |\mathbf{U}||\mathbf{I}_m - \mathbf{R}^*\mathbf{R}|$, the joint density of $\mathbf{U}$ and $\mathbf{R}$ is
$$\propto |\mathbf{U}|^{(\nu+n-m+1)\beta/2-1}\text{etr}\{-\beta\mathbf{U}\}q_{\kappa+\tau}(\mathbf{U})|\mathbf{I}_m - \mathbf{R}^*\mathbf{R}|^{(\nu-m+1)\beta/2-1} \times q_\kappa(\mathbf{I}_m - \mathbf{R}^*\mathbf{R})q_\tau(\mathbf{R}^*\mathbf{R})(d\mathbf{U})(d\mathbf{R}).$$
Finally, note that the joint density of $\mathbf{U}$ and $\mathbf{R}$ is
$$= \frac{\beta^{(\nu+n)m\beta/2+\sum_{i=1}^m (k_i+t_i)}}{\Gamma_m^\beta[(\nu+n)\beta/2,\kappa+\tau]}|\mathbf{U}|^{(\nu+n-m+1)\beta/2-1} \text{etr}\{-\beta\mathbf{U}\}q_{\kappa+\tau}(\mathbf{U})(d\mathbf{U}) \times \frac{\Gamma_m^\beta[\nu\beta/2,\kappa]|\mathbf{I}_m - \mathbf{R}^*\mathbf{R}|^{(\nu-m+1)\beta/2-1}}{\pi^{mn\beta/2}\mathcal{B}_m^\beta[\nu\beta/2,\kappa;n\beta/2,\tau]} q_\kappa(\mathbf{I}_m - \mathbf{R}^*\mathbf{R})q_\tau(\mathbf{R}^*\mathbf{R})(d\mathbf{R})$$
which shows that $\mathbf{U} \sim \mathcal{R}_m^{\beta,I}((\nu+n)\beta/2,\kappa+\tau,\mathbf{I}_m)$ and is independent of $\mathbf{R}$.
2. The proof follows the same method used for proving item 1. □

An alternative way to define the matric variate Pearson type II-Riesz distributions is collected in the following result.

Corollary 1. *Let* $\kappa_1 = (k_{11}, k_{12}, \dots, k_{1n}) \in \Re^n$, *and* $\tau_1 = (t_{11}, t_{12}, \dots, t_{1n}) \in \Re^n$. *Also define* $\mathbf{R}_1 \in \mathcal{L}_{n,m}^\beta$ *as*

$$\mathbf{R}_1 = \mathbf{L}_1^{-1}\mathbf{Y},$$

with $\mathbf{L}_1^* \in \mathfrak{T}_U^\beta(n)$ *is such that* $\mathbf{V} = \mathbf{L}_1\mathbf{L}_1^* = \mathbf{V}_1 + \mathbf{Y}\mathbf{Y}^*$ *is the Cholesky decomposition of* $\mathbf{V}$,

1. *where* $\mathbf{V}_1 \sim \mathcal{R}_n^{\beta,I}(a\beta/2,\kappa_1,\mathbf{I}_n), \text{Re}(a\beta/2) > (n-1)\beta/2 - k_{1n}$; *independent of* $\mathbf{Y} = \mathbf{X}^* \sim \mathcal{KR}_{n\times m}^{\beta,I}(\tau_1,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$, $\text{Re}(m\beta/2) > (n-1)\beta/2 - t_{1n}$. *Then* $\mathbf{U} \sim \mathcal{R}_n^{\beta,I}((a+m)\beta/2,\kappa_1+\tau_1,\mathbf{I}_n)$ *independent of* $\mathbf{R}_1$ *with* $\text{Re}((a+m)\beta/2) > (n-1)\beta/2 - k_{1n} - t_{1n}$. *Furthermore, the density of* $\mathbf{R}_1$ *is*
$$\frac{\Gamma_n^\beta[m\beta/2]|\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^*|^{(a-n+1)\beta/2-1}}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[a\beta/2,\kappa_1;n\beta/2,\tau_1]} q_{\kappa_1}(\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^*) \times q_{\tau_1}(\mathbf{R}_1\mathbf{R}_1^*)(d\mathbf{R}_1), \tag{25}$$
which shall be termed the matric variate Pearson type II-Riesz distribution type I, where $\mathbf{I}_m - \mathbf{R}_1\mathbf{R}_1^* \in \mathfrak{P}_m^\beta$.
2. *where* $\mathbf{V}_1 \sim \mathcal{R}_n^{\beta,II}(a\beta/2,\kappa_1,\mathbf{I}_n), \text{Re}(\nu\beta/2) > (n-1)\beta/2 + k_{11}$; *independent of* $\mathbf{Y} = \mathbf{X}^* \sim \mathcal{KR}_{n\times m}^{\beta,II}(\tau_1,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$, $\text{Re}(m\beta/2) > (n-1)\beta/2 + t_{11}$. *Then* $\mathbf{V} \sim \mathcal{R}_n^{\beta,II}((a+m)\beta/2,\kappa_1+\tau_1,\mathbf{I}_n)$ *independent of* $\mathbf{R}_1$ *with* $\text{Re}((a+m)\beta/2) > (n-1)\beta/2 + k_{11} + t_{11}$. *Furthermore, the density of* $\mathbf{R}_1$ *is*
$$\frac{\Gamma_n^\beta[m\beta/2]|\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^*|^{(a-n+1)\beta/2-1}}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[a\beta/2,-\kappa_1;n\beta/2,-\tau_1]} q_{\kappa_1}\left[(\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^*)^{-1}\right] \times q_{\tau_1}\left[(\mathbf{R}_1\mathbf{R}_1^*)^{-1}\right](d\mathbf{R}_1), \tag{26}$$
which shall be termed the matric variate Pearson type II-Riesz distribution type II, where $\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^* \in \mathfrak{P}_n^\beta$.

Proof. The proof is a verbatim copy of the proof of Theorem 1. Alternatively, observe that densities (25) and (26) can be obtained from densities (23) and (24), respectively, making the following substitutions,

$$\mathbf{R} \rightarrow \mathbf{R}_1^* \quad m \rightarrow n, \quad n \rightarrow m, \quad \nu \rightarrow a, \tag{27}$$

and thus, $\kappa \rightarrow \kappa_1, \quad \tau \rightarrow \tau_1$, and $k_i \rightarrow k_{1i} \quad t_i \rightarrow t_{1i}$. $\square$

Corollary 2. *Let* $\mathbf{Q} = \mathcal{U}(\mathbf{\Omega})^{-1}\mathbf{R}\mathcal{U}(\mathbf{\Xi}) + \boldsymbol{\mu}$, $\mathbf{R}$ *as in Theorem 1, and* $\mathcal{U}(\mathbf{\Omega}) \in \mathfrak{T}_U^\beta(n)$ *and* $\mathcal{U}(\mathbf{\Xi}) \in \mathfrak{T}_U^\beta(m)$ *are constant matrices such that* $\mathbf{\Omega} = \mathcal{U}(\mathbf{\Omega})^*\mathcal{U}(\mathbf{\Omega}) \in \mathfrak{P}_m^\beta$ *and* $\mathbf{\Xi} = \mathcal{U}(\mathbf{\Xi})^*\mathcal{U}(\mathbf{\Xi}) \in \mathfrak{P}_n^\beta$, *respectively, and* $\boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$ *is constant.*

1. *Then, from (23) the density of* $\mathbf{Q}$ *is*

$$\propto |\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu})|^{(\nu-m+1)\beta/2-1} \times q_\kappa[\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu})]q_\tau[(\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu})](d\mathbf{Q}),$$

with constant of proportionality

$$\frac{\Gamma_m^\beta[n\beta/2]|\mathbf{\Omega}|^{m\beta/2}}{\pi^{mn\beta/2}\mathcal{B}_m^\beta[\nu\beta/2,\kappa;n\beta/2,\tau]|\mathbf{\Xi}|^{(\nu+n-m+1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Xi})}$$

where $\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu}) \in \mathfrak{P}_m^\beta$. *This distribution is denoted as*

$$\mathbf{Q} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{n\times m}^{\beta,I}(\nu,\kappa,\tau,\boldsymbol{\mu},\mathbf{\Omega},\mathbf{\Xi}).$$

2. *And from (24) the density of* $\mathbf{Q}$ *is*

$$\propto |\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu})|^{(\nu-m+1)\beta/2-1} \times q_\kappa\left[(\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu}))^{-1}\right]q_\tau\left[((\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu}))^{-1}\right](d\mathbf{Q}),$$

with constant of proportionality

$$\frac{\Gamma_m^\beta[n\beta/2]|\mathbf{\Omega}|^{m\beta/2}}{\pi^{mn\beta/2}\mathcal{B}_m^\beta[\nu\beta/2,-\kappa;n\beta/2,-\tau]|\mathbf{\Xi}|^{(\nu+n-m+1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Xi}^{-1})}$$

where $\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu}) \in \mathfrak{P}_m^\beta$. *This distribution is denoted as*

$$\mathbf{Q} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{m\times n}^{\beta,II}(\nu,\kappa,\tau,\boldsymbol{\mu},\mathbf{\Omega},\mathbf{\Xi}).$$

Proof.

1. The proof follows from (23) and (24), respectively, observing that, by (16)

$$(d\mathbf{R}) = |\mathbf{\Omega}|^{m\beta/2}|\mathbf{\Xi}|^{-n\beta/2}(d\mathbf{Q}),$$

and

$$(\mathbf{I}_m - \mathbf{R}^*\mathbf{R}) = (\mathbf{I}_m - \mathcal{U}(\mathbf{\Xi})^{*-1}(\mathbf{Q}-\boldsymbol{\mu})^*\mathcal{U}(\mathbf{\Omega})^*\mathcal{U}(\mathbf{\Omega})(\mathbf{Q}-\boldsymbol{\mu})\mathcal{U}(\mathbf{\Xi})^{-1}) = \mathcal{U}(\mathbf{\Xi})^{*-1}(\mathbf{\Xi} - (\mathbf{Q}-\boldsymbol{\mu})^*\mathbf{\Omega}(\mathbf{Q}-\boldsymbol{\mu}))\mathcal{U}(\mathbf{\Xi})^{-1}.$$

2. It can be obtained by applying a similar procedure for proving item 1. $\square$

Next some basic properties of the matric variate Pearson type II-Riesz distributions are studied.

Corollary 3. *Let* $\mathbf{Q}_1 = \mathcal{U}(\mathbf{\Omega})\mathbf{R}\mathcal{U}(\mathbf{\Xi})^{-1} + \boldsymbol{\mu}$, $\mathbf{R}$ *as in Corollary 1, and* $\mathcal{U}(\mathbf{\Omega})^* \in \mathfrak{T}_U^\beta(n)$ *and* $\mathcal{U}(\mathbf{\Xi})^* \in \mathfrak{T}_U^\beta(m)$ *are constant matrices such that* $\mathbf{\Omega} = \mathcal{U}(\mathbf{\Omega})\mathcal{U}(\mathbf{\Omega})^* \in \mathfrak{P}_m^\beta$ *and* $\mathbf{\Xi} = \mathcal{U}(\mathbf{\Xi})\mathcal{U}(\mathbf{\Xi})^* \in \mathfrak{P}_n^\beta$, *respectively, and* $\boldsymbol{\mu} \in \mathcal{L}_{m,n}^\beta$ *is constant.*

1. *From (25) the density of* $\mathbf{Q}_1$ *is*

$$\propto |\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*|^{(a-n+1)\beta/2-1}q_{\kappa_1}[\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*] \times q_{\tau_1}[(\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*](d\mathbf{Q}_1),$$

with constant of proportionality

$$\frac{\Gamma_n^\beta[m\beta/2]|\mathbf{\Xi}|^{n\beta/2}}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[a\beta/2,\kappa_1;m\beta/2,\tau_1]|\mathbf{\Omega}|^{(a+m-n+1)\beta/2-1}q_{\kappa_1+\tau_1}(\mathbf{\Omega})}$$

where $\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*) \in \mathfrak{P}_n^\beta$. *This distribution is denoted as*

$$\mathbf{Q}_1 \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{n\times m}^{\beta,I}(a,\kappa_1,\tau_1,\boldsymbol{\mu},\mathbf{\Omega},\mathbf{\Xi}).$$

2. *Similarly, from (26) the density of* $\mathbf{Q}_1$ *is*

$$\propto |\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*|^{(a-n+1)\beta/2-1}q_{\kappa_1}\left[(\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*)^{-1}\right] \times q_{\tau_1}\left[((\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*)^{-1}\right](d\mathbf{Q}_1),$$

with constant of proportionality

$$\frac{\Gamma_n^\beta[m\beta/2]|\mathbf{\Xi}|^{n\beta/2}}{\pi^{mn\beta/2}\mathcal{B}_n^\beta[a\beta/2,-\kappa_1;m\beta/2,-\tau_1]|\mathbf{\Omega}|^{(a+m-n+1)\beta/2-1}q_{\kappa_1+\tau_1}(\mathbf{\Omega}^{-1})}$$

where $\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*) \in \mathfrak{P}_n^\beta$. *This distribution is denoted as*

$$\mathbf{Q}_1 \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{m\times n}^{\beta,II}(a,\kappa,\tau,\boldsymbol{\mu},\mathbf{\Omega},\mathbf{\Xi}).$$

Proof.

1. The proof follows from (25) and (26), respectively, observing that, by (16)

$$(d\mathbf{R}_1) = |\mathbf{\Omega}|^{-m\beta/2}|\mathbf{\Xi}|^{n\beta/2}(d\mathbf{Q}_1),$$

and

$$(\mathbf{I}_n - \mathbf{R}_1\mathbf{R}_1^*) = (\mathbf{I}_m - \mathcal{U}(\mathbf{\Omega})^{-1}(\mathbf{Q}_1 - \boldsymbol{\mu})\mathcal{U}(\mathbf{\Xi})\mathcal{U}(\mathbf{\Xi})^*(\mathbf{Q}_1-\boldsymbol{\mu})^*\mathcal{U}(\mathbf{\Omega})^{*-1}) = \mathcal{U}(\mathbf{\Omega})^{-1}(\mathbf{\Omega} - (\mathbf{Q}_1-\boldsymbol{\mu})\mathbf{\Xi}(\mathbf{Q}_1-\boldsymbol{\mu})^*)\mathcal{U}(\mathbf{\Omega})^{*-1}.$$

2. It can be obtained by applying a similar procedure for proving item 1. $\square$

Now c-beta-Riesz type I and k-beta-Riesz type I distributions can be obtained, see Díaz-García (2015b). Let $n \geqslant m$ and let $\mathbf{B} \in \mathfrak{P}_m^\beta$ defined as $\mathbf{B} = \mathbf{R}^*\mathbf{R}$ then, under the conditions of Theorem 1, we have

$$\mathbf{B} = \mathbf{R}^*\mathbf{R} = \mathbf{L}^{*-1}\mathbf{X}^*\mathbf{X}\mathbf{L}^{-1} = \mathbf{L}^{*-1}\mathbf{W}\mathbf{L}^{-1}$$

where $\mathbf{W} = \mathbf{X}^*\mathbf{X}$, $\mathbf{L} \in \mathfrak{T}_U^\beta(m)$ and $\mathbf{U} = \mathbf{L}^*\mathbf{L} = \mathbf{U}_1 + \mathbf{X}^*\mathbf{X}$ is the Cholesky decomposition of $\mathbf{U}$. Therefore:

Theorem 2.

1. *Assuming that* $\mathbf{R} \sim \mathcal{P}_{\mathcal{II}}\mathcal{R}_{n\times m}^{\beta,I}(\nu,\kappa,\tau,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$. *Then, the density of* $\mathbf{B}$, *such that* $\mathbf{I}_m - \mathbf{B} \in \mathfrak{P}_m^\beta$ *is*

$$\frac{|\mathbf{B}|^{(n-m+1)\beta/2-1}}{\mathcal{B}_m^\beta[\nu\beta/2,\kappa;n\beta/2,\tau]}|\mathbf{I}_m-\mathbf{B}|^{(\nu-m+1)\beta/2-1}q_\kappa(\mathbf{I}_m-\mathbf{B})q_\tau(\mathbf{B})(d\mathbf{B}). \tag{28}$$

$\mathbf{B}$ is said to have a matric variate c-beta-Riesz type I distribution.

2. *Suppose that $\mathbf{R}\sim\mathcal{P}_{\mathcal{II}}\mathcal{R}^{\beta,II}_{n\times m}(\nu,\kappa,\tau,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$. Then the density of $\mathbf{B}$, such that $\mathbf{I}_m-\mathbf{B}\in\mathfrak{P}_m^\beta$ is*

$$\frac{|\mathbf{B}|^{(n-m+1)\beta/2-1}}{\mathcal{B}_m^\beta[\nu\beta/2,-\kappa;n\beta/2,-\tau]}|\mathbf{I}_m-\mathbf{B}|^{(\nu-m+1)\beta/2-1}q_\kappa[(\mathbf{I}_m-\mathbf{B})^{-1}] \tag{29}$$

$\times q_\tau[(\mathbf{B})^{-1}](d\mathbf{B})$. *$\mathbf{B}$ is said to have a matric variate k-beta-Riesz type I distribution.*

Proof.

1. From (23) the density function of $\mathbf{R}$ is

$$\propto|\mathbf{I}_m-\mathbf{R}^*\mathbf{R}|^{(\nu-m+1)\beta/2-1}q_\kappa(\mathbf{I}_m-\mathbf{R}^*\mathbf{R})q_\tau(\mathbf{R}^*\mathbf{R})(d\mathbf{R}).$$

Now make the change of variable $\mathbf{B}=\mathbf{R}^*\mathbf{R}$, so that

$$(d\mathbf{R})=2^{-m}|\mathbf{B}|^{(n-m+1)\beta/2-1}(d\mathbf{B})(\mathbf{V}_1^*d\mathbf{V}_1),$$

with $\mathbf{V}_1\in\mathcal{V}^\beta_{m,n}$. The joint density of $\mathbf{B}$ and $\mathbf{V}_1$ is then

$$\propto|\mathbf{I}_m-\mathbf{B}|^{(\nu-m+1)\beta/2-1}q_\kappa(\mathbf{I}_m-\mathbf{B})q_\tau(\mathbf{B})|\mathbf{B}|^{(n-m+1)\beta/2-1}(d\mathbf{R})\times(\mathbf{V}_1^*d\mathbf{V}_1).$$

Integrating with respect to $\mathbf{V}_1$ using (1), gives the stated marginal density of $\mathbf{B}$.

2. This is obtained in a similar way to the obtained in item 1. □

In addition, assume that $n<m$ and let $\mathbf{B}_1\in\mathfrak{P}_n^\beta$ defined as $\mathbf{B}_1=\mathbf{R}_1\mathbf{R}_1^*$ then, under the conditions of Corollary 1 we have

$$\widetilde{\mathbf{B}}=\mathbf{L}_1^{-1}\mathbf{Y}\mathbf{Y}^*\mathbf{L}_1^{*-1}=\mathbf{L}_1^{-1}\mathbf{W}_1\mathbf{L}_1^{*-1},$$

where $\mathbf{W}_1=\mathbf{Y}\mathbf{Y}^*$. Hence:

Theorem 3.

1. *Assuming that $\mathbf{R}\sim\mathcal{P}_{\mathcal{II}}\mathcal{R}^{\beta,I}_{n\times m}(a,\kappa_1,\tau_1,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$. Then, the density of $\mathbf{B}_1$ is*

$$\frac{|\mathbf{B}_1|^{(m-n+1)\beta/2-1}}{\mathcal{B}_n^\beta[a\beta/2,\kappa_1;m\beta/2,\tau_1]}|\mathbf{I}_n-\mathbf{B}_1|^{(a-n+1)\beta/2-1}q_{\kappa_1}(\mathbf{I}_n-\mathbf{B}_1)q_{\tau_1}(\mathbf{B}_1)(d\mathbf{B}_1), \tag{30}$$

where $\mathbf{I}_n-\mathbf{B}_1\in\mathfrak{P}_n^\beta$, also, we say that $\mathbf{B}_1$ has a matric variate c-beta-Riesz type I distribution.

2. *Similarly, assuming that $\mathbf{R}\sim\mathcal{P}_{\mathcal{II}}\mathcal{R}^{\beta,II}_{n\times m}(a,\kappa_1,\tau_1,\mathbf{0},\mathbf{I}_n,\mathbf{I}_m)$. Then the density of $\mathbf{B}_1$ is*

$$\frac{|\mathbf{B}_1|^{(m-n+1)\beta/2-1}}{\mathcal{B}_n^\beta[a\beta/2,-\kappa_1;m\beta/2,-\tau_1]}|\mathbf{I}_n-\mathbf{B}_1|^{(a-n+1)\beta/2-1}q_{\kappa_1}[(\mathbf{I}_n-\mathbf{B}_1)^{-1}] \tag{31}$$

$\times q_{\tau_1}[(\mathbf{B}_1)^{-1}](d\mathbf{B}_1)$, *where $\mathbf{I}_n-\mathbf{B}_1\in\mathfrak{P}_n^\beta$. We say that $\mathbf{B}_1$ has a matric variate k-beta-Riesz type I distribution.*

Proof. The proof follows a similar procedure given for Theorem 2. □

Alternatively, observe that densities (30) and (31) can be obtained from densities (28) and (29), respectively, by making the following substitutions

$$\mathbf{B}\to\mathbf{B}_1,\quad m\to n,\quad n\to m,\quad \nu\to a, \tag{32}$$

and consequently $\kappa\to\kappa_1$, $\tau\to\tau_1$, and $k_i\to k_{1i}$ $t_i\to t_{1i}$.

We end this section, deriving the non-standardised densities of the c-, and k-beta distributions.

Corollary 4. *Define $\mathbf{C}=\mathcal{U}(\mathbf{\Theta})^*\mathbf{B}\mathcal{U}(\mathbf{\Theta})$, where $\mathcal{U}(\mathbf{\Theta})\in\mathfrak{T}_U^\beta(m)$ is such that $\mathbf{\Theta}=\mathcal{U}(\mathbf{\Theta})^*\mathcal{U}(\mathbf{\Theta})$ is the Cholesky decomposition of $\mathbf{\Theta}$.*

1. *Assume that $\mathbf{B}$ has the density (28), then the density of the random matrix $\mathbf{C}$ is*

$$\propto|\mathbf{C}|^{(n-m+1)\beta/2-1}|\mathbf{\Theta}-\mathbf{C}|^{(\nu-m+1)\beta/2-1}q_\kappa(\mathbf{\Theta}-\mathbf{C})q_\tau(\mathbf{C})(d\mathbf{C}), \tag{33}$$

with constant of proportionally

$$\frac{1}{\mathcal{B}_m^\beta[\nu\beta/2,\kappa;n\beta/2,\tau]|\mathbf{\Theta}|^{(\nu+n-m+1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Theta})},$$

for $\mathbf{\Theta}-\mathbf{C}\in\mathfrak{P}_m^\beta$.

2. *Suppose that $\mathbf{B}$ has the density (29), then the density of the random matrix $\mathbf{C}$ is*

$$|\mathbf{C}|^{(n-m+1)\beta/2-1}|\mathbf{\Theta}-\mathbf{C}|^{(\nu-m+1)\beta/2-1}q_\kappa[(\mathbf{\Theta}-\mathbf{C})^{-1}]q_\tau(\mathbf{C}^{-1})(d\mathbf{C}), \tag{34}$$

with constant of proportionally

$$\frac{1}{\mathcal{B}_m^\beta[\nu\beta/2,-\kappa;n\beta/2,-\tau]|\mathbf{\Theta}|^{(\nu+n-m+1)\beta/2-1}q_{\kappa+\tau}(\mathbf{\Theta}^{-1})},$$

for $\mathbf{\Theta}-\mathbf{C}\in\mathfrak{P}_m^\beta$.

Proof. This immediate from (17). □

4. Conclusions

Modern, integrated and unified statistics requires a number of concepts and results of abstract algebra; the generalised theory has a robust, concise and elegant exposition; but it is out of the common language of statisticians. In opposite context, a notorious tendency about unconnected translations of matrix distribution results in real-Gaussian to real-non Gaussian, complex-Gaussian, complex-non Gaussian, ruled the statistical theory for decades. We expect that publications in the line proposed in this work will increase their impact on statistical theory. Some of these statistical results can be cited, for example Micheas et al. (2006) addressed the problem of point estimation of parameters in complex shape theory. Also, Khatri (1965) considered the estimation of parameters of a complex matrix multivariate normal distribution and established a test of hypothesis about the mean. In the quaternionic context, Bhavsar (2000) set test statistics and their corresponding asymptotic distributions for two interesting particular hypothesis. As suggested by the reviewer of this work, classical and influential statistical results provided by Muirhead (1982) and Fang and Zhang (1990) can be studied in the context of real normed division algebras; but first we need to research upon several aspects, in fact, some of them were obtained here. In particular, Pearson type II distribution in the context of real normed division algebras and Riesz theory, performs a crucial role in the addressed generalised theory; taking into account the published parallel results involving the Kotz type distribution.

Acknowledgements

The authors wish to thank the Editor and the anonymous reviewers for their constructive comments on the preliminary version of this paper. This article was written under the existing research agreement between the first author and the Universidad Autónoma Agraria Antonio Narro, Saltillo, México. The second author was supported by University of Medellin, in the context of a joint research project with University of Toulouse and University of Bordeaux, France.

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