Discrete Mathematics 97 (1991) 243–250
North-Holland

The existence of C_k-factorizations of $K_{2n} - F$

D.G. Hoffman*

Department of Algebra, Combinatorics and Analysis, Auburn University, Auburn, AL 36849-3501, USA

P.J. Schellenberg**

Department of Combinatorics and Optimization, University of Waterloo, Waterloo, Ont., Canada N2L 3G1

In memory of Egmont Köhler.

Abstract

Hoffman, D.G. and P.J. Schellenberg, The existence of C_k-factorizations of $K_{2n} - F$, Discrete Mathematics 97 (1991) 243–250.

A necessary condition for the existence of a C_k-factorization of $K_{2n} - F$ is that k divides $2n$. It is known that neither $K_6 - F$ nor $K_{12} - F$ admit a C_3-factorization. In this paper we show that except for these two cases, the necessary condition is also sufficient.

1. Introduction

Let G be a graph and let $k > 2$ be an integer. A *k-cycle decomposition* of G is an edge-decomposition of G into cycles of length k. If the cycles of a k-cycle decomposition can be partitioned into 2-factors of G, we call this 2-factorization a C_k-factorization of G. More generally, for any graphs G and H, an *H-factor* of G is a spanning subgraph D of G such that every component of D is isomorphic to H. If G can be expressed as an edge-disjoint sum of H-factors, then this sum is called an *H-factorization* of G.

Let C_k denote the cycle of length k, let K_n denote the complete graph with n vertices, and let $K_{2n} - F$ denote the graph obtained by deleting the edges of a 1-factor F from K_{2n}. A necessary condition for the existence of a C_k-factorization of K_{2n+1} is that k divides $2n + 1$. Alspach, Schellenberg, Stinson and Wagner [2] show that this necessary condition is also sufficient.

* Research supported by NSA grant MDA-904-88-H-2005.
** Research supported by NSERC grant A 8509.

Of course K_{2n} does not have a C_k-factorization because each vertex has odd degree. The best one can hope to do is find a C_k-factorization of $K_{2n} - F$. A necessary condition for the existence of such a factorization is that k divides $2n$. In this paper we show that, for $k > 3$ an odd integer, $K_{4k} - F$ has a C_k-factorization. As an immediate consequence of this, we establish that the necessary condition for the existence of a C_k-factorization of $K_{2n} - F$ is sufficient except when $k = 3$ and $2n \in \{6, 12\}$.

2. The construction

We use Bose's [4] method of symmetrically repeated differences (see, for example, [3]) to give a direct construction of a C_k-factorization of $K_{4k} - F$ for all odd integers $k > 3$. It is easy to see that any such factorization consists of $2k - 1$ C_k-factors. Our approach is to find an initial C_k-factor which generates the desired C_k-factorization under the action of the group $\mathbb{Z}_{2k-1}$, the integers modulo $2k - 1$. Therefore, let the vertices of $K_{4k} - F$ consist of two copies $\mathbb{Z}_{2k-1}$, namely $\{0, 1, \ldots, 2k-2\}$ and $\{\bar{0}, \bar{1}, \ldots, \overline{2k-2}\}$, together with the two fixed points $\{\infty_1, \infty_2\}$.

Let us first treat two special cases to which the general construction does not apply.

Lemma 2.1. *There exists a C_5-factorization of $K_{20} - F$ and a C_7-factorization of $K_{28} - F$.*

Proof. Let the vertices of K_{20} be

$$\{0, 1, \ldots, 8\} \cup \{\bar{0}, \bar{1}, \ldots, \bar{8}\} \cup \{\infty_1, \infty_2\}.$$

Consider the C_5-factor

$$\{0, 1, 3, 6, \bar{0}, 0\}\{2, 7, \bar{5}, 8, \bar{1}, 2\}\{\infty_1, 4, \bar{8}, \bar{2}, \bar{4}, \infty_1\}\{\infty_2, 5, \bar{6}, \bar{7}, \bar{3}, \infty_2\}$$

where, for example, $\{0, 1, 3, 6, \bar{0}, 0\}$ represents the cycle made up of the edges $\{0, 1\}$, $\{1, 3\}$, $\{3, 6\}$, $\{6, \bar{0}\}$ and $\{\bar{0}, 0\}$. The differences corresponding to the edges of this C_5-factor include every pure and mixed difference except for the mixed difference 5 (i.e. $\bar{g} - h$ where $g - h = 5$ in $\mathbb{Z}_9$), and every infinite difference except for the differences $\pm(\infty_1 - \infty_2)$. Letting $\mathbb{Z}_9$ act on this factor generates a C_5-factorization of $K_{20} - F$. The 1-factor F consists of the edges

$$\{\bar{5}, 0\}, \{\bar{6}, 1\}, \ldots, \{\bar{4}, 8\}, \{\infty_1, \infty_2\}.$$

Let the vertices of K_{28} be

$$\{0, 1, \ldots, 12\} \cup \{\bar{0}, \bar{1}, \ldots, \overline{12}\} \cup \{\infty_1, \infty_2\}.$$

The edges of the C_7-factor

$$\{0, 1, 3, 6, 10, 5, \bar{0}, 0\}\{\bar{1}, \bar{2}, \bar{4}, \overline{10}, 9, \overline{11}, 7, \bar{1}\}\{\infty_1, 11, 4, \bar{3}, \bar{7}, \overline{12}, \bar{9}, \infty_1\}$$
$$\{\infty_2, 12, \bar{8}, 2, \bar{5}, 8, \bar{6}, 1_2\}$$

represent every pure and mixed difference, with respect to $\mathbb{Z}_{13}$, except for the mixed difference 5, and every infinite difference except for the differences $\pm(\infty_1 - \infty_2)$. Letting $\mathbb{Z}_{13}$ act on this factor generates a C_7-factorization of $K_{28} - F$. □

We now proceed to construct such initial C_k-factors for every odd integer k. To this end, let $k = 2t + 1$, and consider the graph K_{8t+4} having vertex set

$$\{0, 1, \ldots, 4t\} \cup \{\bar{0}, \bar{1}, \ldots, \overline{4t}\} \cup \{\infty_1, \infty_2\}.$$

The first cycle is

$$\{0, \bar{1}, -1, \bar{2}, -2, \bar{3}, -3, \ldots, \bar{t}, -t, 0\}.$$

The edges of this $(2t + 1)$-cycle have mixed differences $1, 2, \ldots, 2t$ as well as the pure difference $\pm t$. (As before, we say $\bar{g} - h$ is the mixed difference d if $g - h = d$ in $\mathbb{Z}_{4t+1}$.) The second cycle is

$$\{-t-2, \overline{t+1}, -t-3, \overline{t+2}, \ldots, -2t+1, \overline{2t-2}, -2t,$$
$$\overline{2t-1}, \bar{0}, \overline{2t}, -2t-1, -t-2\}.$$

Its edges have mixed differences $2t + 3$, $2t + 4$, $2t + 5, \ldots, 4t - 1$ and 0, and pure differences $\pm(\overline{2t-1})$, $\pm\overline{2t}$ and $\pm(t-1)$. The third cycle is

$$\{\infty_1, \overline{4t-1}, \overline{2t+1}, \overline{4t-2}, \overline{2t+2}, \ldots, \overline{3t-1}, \overline{3t}, t-2, \infty_1\}$$

having mixed difference $2t + 2$ and pure differences $\pm\bar{1}$, $\pm\bar{2}, \ldots, \pm(\overline{2t-2})$. It also has the two differences involving ∞_1.

These three cycles are represented schematically in Fig. 1. This diagram may help the reader recognize the pattern being employed in the above construction. Observe that the vertices in the right-hand column are in ascending order $\bar{0}$, $\bar{1}$, $\bar{2}, \ldots, \overline{4t}$, whereas those in the left-hand column are in descending order $0, -1, -2, \ldots, -4t$. Note that Fig. 1 also indicates how we get $t + 3$ of the edges of the fourth cycle.

Table 1 summarizes the differences that are contained in these three cycles.

Let K' be the subgraph of K_{8t+4} induced by the set of vertices which are *not* in any of these three cycles, namely the vertices

$$(\{1, 2, \ldots, 2t-1\} \setminus \{t-2\}) \cup \{-t-1, \overline{4t}, \infty_2\}.$$

Since no edge in K' has mixed difference $4t$, the edges

$$\{\bar{0}, 1\}, \{\bar{1}, 2\}, \ldots, \{\overline{4t-1}, 4t\}, \{\overline{4t}, 0\} \quad \text{and} \quad \{\infty_1, \infty_2\}$$

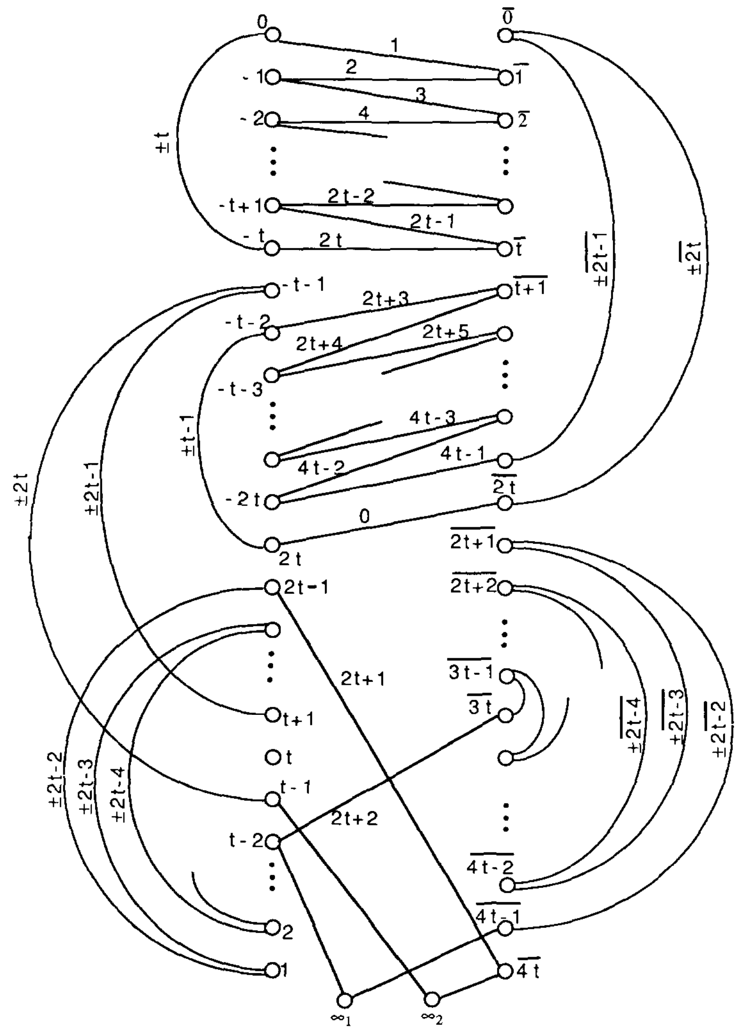

Fig. 1.

constitute the 1-factor F of $K_{8t+4} - F$. Let the fourth cycle contain the path

$$\left\{t+1,\ -t-1,\ t-1,\ \infty_2,\ \overline{4t},\ 2t-1,\ 1,\ 2t-2,\ 2,\ \ldots,\ \frac{t-4}{2},\ \frac{3t+2}{2},\ \frac{t-2}{2},\ \frac{3t}{2}\right\}$$

when t is even, and the path

$$\left\{t+1,\ -t-1,\ t-1,\ \infty_2,\ \overline{4t},\ 2t-1,\ 1,\ 2t-2,\ 2,\ \ldots,\ \frac{3t+3}{2},\ \frac{t-3}{2},\ \frac{3t+1}{2},\ \frac{t-1}{2}\right\}$$

Table 1

	pure differences	mixed differences
1st cycle	$\pm t$	$1, 2, \ldots, 2t$
2nd cycle	$\pm(t-1),\ \pm\overline{(2t-1)},\ \pm\overline{2t}$	$0,\ 2t+3,\ 2t+4, \ldots, 4t-1$
3rd cycle	$\pm\bar{1},\ \pm\bar{2}, \ldots, \pm\overline{(2t-2)}$	$2t+2$

when t is odd. The edges of this path represent the two differences involving ∞_2, the mixed differences $2t+1$, and the pure differences

$$\pm 2t,\ \pm(2t-1),\ \pm(2t-2), \ldots, \pm(t+1).$$

When t is even, we adjoin the path

$$\left\{\frac{3t}{2}, \frac{3t-2}{2}, \frac{3t-6}{2}\right\}$$

which has differences ± 1 and ± 2. Then adjoin paths of the form

$$P_{a,b} = \{a, b, a+1, b+2, a-1, b+1, a-3\}.$$

Path $P_{a,b}$ starts at vertex a, ends at vertex $a-3$, and passes through vertices $a+1$, $a-1$, b, $b+1$, $b+2$; furthermore, its edges represent the six consecutive pure differences

$$\pm(a-b+1),\ \pm(a-b),\ \pm(a-b-1),\ \pm(a-b-2),$$
$$\pm(a-b-3),\ \pm(a-b-4).$$

We string together paths $P_{a,b}$ for

$$(a, b) = \left(\frac{3t-6}{2}, \frac{t}{2}\right), \left(\frac{3t-12}{2}, \frac{t+6}{2}\right), \left(\frac{3t-18}{2}, \frac{t+12}{2}\right), \ldots$$

as long as $a > t+4$ and $b < t-7$ or $(a, b) = (t+3, t-6)$. We now consider three cases.

Case 1: $t \equiv 0 \pmod 6$.

Consider the union of paths $P_{a,b}$ for all

$$(a, b) \in \left\{\left(\frac{3t-6i}{2}, \frac{t-6+6i}{2}\right) : i = 1, 2, \ldots, \frac{t-6}{6}\right\}.$$

(Notice that, when $t = 6$, this set of pairs is empty and, hence, no paths $P_{a,b}$ are used.) This union is a path of length $(t-6)$ which starts at vertex $(3t-6)/2$, ends at the vertex t, and passes through all the vertices

$$\frac{3t-4}{2}, \frac{3t-8}{2}, \frac{3t-10}{2}, \frac{3t-12}{2}, \ldots, t+2,$$

and

$$\frac{t}{2}, \frac{t+2}{2}, \frac{t+4}{2}, \ldots, t-4.$$

Furthermore, the edges of this path represent all the pure differences

$$\pm(t-2), \ \pm(t-3), \ \pm(t-4), \ldots, \pm 5.$$

When $(a, b) = (t, t-3)$, complete the cycle by adding the path $\{t, t-3, t+1\}$. It can be seen that this is, in fact, a cycle of length $2t+1$ which spans the graph K' and includes all the required differences. Letting $\mathbb{Z}_{4t+1}$ act on this cycle and the three defined earlier produces a C_{2t+1}-factorization of $K_{8t+4} - F$ when $t \equiv 0 \pmod 6$.

Case 2: $t \equiv 2 \pmod 6$.

The argument proceeds as in Case 1, taking the union of paths $P_{a,b}$ for all

$$(a, b) \in \left\{\left(\frac{3t-6i}{2}, \frac{t-6+6i}{2}\right): i = 1, 2, \ldots, \frac{t-14}{6}\right\}.$$

When $(a, b) = (t+4, t-7)$, complete the cycle by adding the path

$$\{t+4, t-7, t+5, t-5, t+2, t-6, t+3, t-3, t, t-4, t+1\}.$$

As before, it can be shown that this cycle spans K' and includes all the required differences.

When $t = 8$, $(3t-6)/2 = 9 < 12 = t+4$. Hence, this technique cannot be applied. In this case, one possible fourth cycle is

$$\{\infty_2, \overline{32}, 15, 1, 14, 2, 13, 3, 12, 10, 11, 5, 8, 4, 9, 24, 7, \infty_2\}.$$

When $t = 2$, $8t + 4 = 20$ and existence is resolved in Lemma 2.1.

Case 3: $t \equiv 4 \pmod 6$.

When $(a, b) = (t+2, t-5)$, complete the fourth cycle by adding the path

$$\{t+2, t-5, t+3, t-3, t, t-4, t+1\}.$$

When $t = 4$, $(3t-6)/2 = 3 < 6 = t+2$. As before, this case must be handled separately. A fourth cycle is

$$\{\infty_2, \overline{16}, 7, 1, 6, 4, 3, 12, 5, \infty_2\}.$$

When t is odd, we adjoin the path

$$\left\{\frac{t-1}{2}, \frac{3t-5}{2}, \frac{3t-1}{2}, \frac{3t-3}{2}, \frac{t+3}{2}\right\},$$

which has pure differences $\pm(t-2)$, ± 2, ± 1, $\pm(t-3)$, to the path

$$\left\{t+1, -t-1, t-1, \infty_2, \overline{4t}, 2t-1, 1, 2t-2, \right.$$
$$\left. 2, \ldots, \frac{3t+3}{2}, \frac{t-3}{2}, \frac{3t+1}{2}, \frac{t-1}{2}\right\}.$$

Then adjoin paths of the form

$$Q_{a,b} = \{b, a, b-1, a-2, b+1, a-1, b+3\}.$$

Path $Q_{a,b}$ starts at vertex b, ends at vertex $b+3$, and passes through vertices $b-1$, $b+1$, a, $a-1$, $a-2$; furthermore, its six edges represent the consecutive pure differences

$$\pm(a-b+1),\ \pm(a-b),\ \pm(a-b-1),\ \pm(a-b-2),$$
$$\pm(a-b-3),\ \pm(a-b-4).$$

We string together such paths $Q_{a,b}$ for

$$(a, b) = \left(\frac{3t-7}{2}, \frac{t+3}{2}\right), \left(\frac{3t-13}{2}, \frac{t+9}{2}\right), \left(\frac{3t-19}{2}, \frac{t+15}{2}\right), \ldots$$

as long as $a > t+3$ and $b < t-5$. We now consider three more cases.

Case 4: $t \equiv 1 \pmod 6$.

When $(a, b) = (t+3, t-5)$, complete the cycle by adding the path

$$\{t-5, t+2, t-6, t+3, t-3, t, t-4, t+1\}.$$

When $t = 7$, $(3t-7)/2 = 7 < 10 = t+3$. Hence, this approach fails. In this case, a possible fourth cycle is

$$\{\infty_2, \overline{28}, 13, 1, 12, 2, 11, 3, 8, 21, 6, 10, 9, 7, 4, \infty_2\}.$$

When $t = 1$, Kotzig and Rosa [5] have shown that a C_3-factorization of $K_{12} - F$ does not exist.

Case 5: $t \equiv 3 \pmod 6$.

When $(a, b) = (t+1, t-3)$, complete the fourth cycle by adding the path

$$\{t-3, t, t-4, t+1\}.$$

This technique fails when $t = 3$, since $(3t-7)/2 = 1 < 4 = t+1$. However, as $8t + 4 = 28$, Lemma 2.1 resolves the question of existence.

Case 6: $t \equiv 5 \pmod 6$.

When $(a, b) = (t+2, t-4)$, complete the cycle by adding the path

$$\{t-4, t+2, t-5, t, t-3, t+1\}.$$

This does not work when $t = 5$ since $(3t-7)/2 = 4 < 7 = t+2$. In this case,

$$\{\infty_2, \overline{20}, 9, 1, 8, 2, 5, 7, 6, 15, 4, \infty_2\}$$

is a suitable fourth cycle.

This establishes the following result.

Theorem 2.2. *If $k > 3$ is an odd integer, then $K_{4k} - F$ has a C_k-factorization.*

This also completely resolves the existence question for C_k-factorizations of $K_{2n} - F$.

Theorem 2.3. *For any integer $k > 2$, if k divides $2n$, then $K_{2n} - F$ has a C_k-factorization, except when $k = 3$ and $2n \in \{6, 12\}$.*

Proof. Alspach and Häggkvist [1] establish this result when k is an even integer. Kotzig and Rosa [5] show that $K_6 - F$ and $K_{12} - F$ do not have C_3-factorizations. Alspach, Schellenberg, Stinson and Wagner [2] resolve the existence question for k an odd integer and $n \neq 2k$. The techniques of [2] do not apply to the case when $n = 2k$ because $K_4 - F$ cannot be decomposed into 2-factors consisting of cycles of odd length. Theorem 2.2 resolves the case when k is odd and $n = 2k$. □

References

[1] B. Alspach and R. Häggkvist, Some observations on the Oberwolfach problem, J. Graph Theory 9 (1985) 177–187.

[2] B. Alspach, P.J. Schellenberg, D.R. Stinson and D. Wagner, The Oberwolfach problem and factors of uniform odd length cycles, J. Combin. Theory Ser. A 52 (1989) 20–43.

[3] T. Beth, D. Jungnickel and H. Lenz, Design Theory (Bibliographisches Institut, Mannheim, 1985) and (Cambridge Univ. Press, Cambridge, 1986).

[4] R.C. Bose, On the construction of balanced incomplete block designs, Ann. Eugenics 9 (1939) 353–399.

[5] A. Kotzig and A. Rosa, Nearly Kirkman systems, in: Proc. 5th SE Conf. on Combinatorics, Graph Theory and Computing, Boca Raton, FL (1974) 607–614.