On a periodic predator–prey system with time delays

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Abstract

This paper deals with discuss the global existence of positive periodic solutions for a predator–prey system with time delays based on the theory of coincidence degree.

Keywords: Coincidence degree; Logistic equation; Global existence; Fredholm operator

1. Introduction

We consider the following predator–prey system

\[
\begin{aligned}
\dot{N}_1(t) &= N_1(t) \left[ a_1(t) - b_1(t) \int_{-\infty}^{t} k(t - u)N_1(u) \, du - \frac{c(t)N_2(t - \tau(t))}{1 + mN_1(t)} \right], \\
\dot{N}_2(t) &= N_2(t) \left[ -a_2(t) + \frac{b_2(t)N_1(t - \tau(t))}{1 + mN_1(t - \tau(t))} \right],
\end{aligned}
\]

where \( N_i \ (i = 1, 2) \) stand for the prey’s and the predator’s density at time \( t \), respectively; \( m \) is a positive constant that denotes the half capturing saturation constant; \( a_1 \in C(R, R), \ a_2, \ b_i \ (i = 1, 2), \ c, \tau \in C(R, R^+) \) are \( T \)-periodic functions; \( k(s) : R^+ \to R^+ \) is a measurable,

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$T$-periodic, normalized function such that $\int_{0}^{+\infty} k(s) \, ds = 1$, corresponding to a delay kernel or a weighting factor, which says how much emphases should be given to the size of the prey population at earlier times to determine the present effect on resource availability, and also formally yields $\int_{-\infty}^{t} \delta_{0}(t - u) x(u) \, du = x(t)$ if $k(s) = \delta_{0}(s)$ and $\int_{-\infty}^{t} \delta_{0}(t - \tau - u) x(u) \, du = x(t - \tau)$ if $k(s) = \delta_{0}(s - \tau)$, where $\delta_{0}(s)$ is the Dirac delta function at $s = 0$.

Considerable attention has been given to several special cases and similar cases of the predator–prey system (1.1) by several authors including Hsu and Huang [1], Arditi and Ginzburg [2], He [3], Beretta and Kuang [4], Ginzburg and Akcakaya [5], Zhao and Wang [7,8]. They also observed that the Rosenzweig–Macarthur model:

$$
\begin{align*}
\frac{dx}{dt} &= x(a - bx) - \frac{cxy}{1 + mx}, \\
\frac{dy}{dt} &= y \left( -d + k \frac{fx}{1 + mx} \right),
\end{align*}
$$

where $a, b, c, d, k, m$ and $f$ are positive constants, represents a fairly reasonable predator–prey dynamical system.

Although the literature is replete with many papers on the predator–prey systems with or without time delay, almost all systems are concerned with constant environment so, the main objective of this paper is to study a more general predator–prey model described by Eq. (1.1).

2. Global existence of positive periodic solution

We will first summarize below a few important concepts and a result from Gaines and Mawhin [6].

Let $X, Z$ be normed vector spaces, $L: \text{Dom} \, L \subset X \rightarrow Z$ be a linear operator and $N: X \rightarrow Z$ be a continuous operator. The operator $L$ will be called a Fredholm operator of index zero if $\dim \text{Ker} \, L = \text{codim} \, \text{Im} \, L < +\infty$ and $\text{Im} \, L$ is closed in $Z$. If $L$ is a Fredholm operator of index zero, there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that $\text{Im} \, P = \text{Ker} \, L$, $\text{Ker} \, Q = \text{Im} \, L = \text{Im}(I - Q)$. It follows that $L | \text{Dom} \, L \cap \text{Ker} \, P: (I - P)X \rightarrow \text{Im} \, L$ is invertible. We denote the inverse of that operator by $K$. If $\Omega$ is an open bounded subset of $X$, the operator $N$ will be called $L$-compact on $\bar{\Omega}$, if $QN(\bar{\Omega})$ is bounded and $K(I - Q)N: \bar{\Omega} \rightarrow X$ is compact. Since $\text{Im} \, Q$ is isomorphic to $\text{Ker} \, L$, there exist isomorphisms $J: \text{Im} \, Q \rightarrow \text{Ker} \, L$.

Lemma 1. Let $L$ be a Fredholm operator of index zero and let $N$ be $L$-compact on $\bar{\Omega}$. Suppose:

1. For each $\lambda \in (0, 1]$, every solution $x$ of $Lx = \lambda Nx$ is such that $x \notin \partial \Omega$.
2. $QNx \neq 0$ for each $x \in \partial \Omega \cap \text{Ker} \, L$.
3. $\text{Deg}\{JQN, \Omega \cap \text{Ker} \, L, 0\} \neq 0$.

Then the equation $Lx = Nx$ has at least one solution lying in $\text{Dom} \, L \cap \bar{\Omega}$.

Our main result is given in following theorem.

For convenience, we introduce the following notation.

If $f$ is a $T$-periodic continuous function then we make the notation

$$
F := \int_{0}^{T} f(t) \, dt \quad \text{and} \quad \bar{F} := \int_{0}^{T} |f(t)| \, dt.
$$
Theorem 2. If

1. \( A_1 > 0 \); 
2. \( A_2 B_1 \cdot \exp\{A_1 + B_1\} < A_1 (B_2 - m A_2) \) and \( B_2 > m A_2 \),

then Eq. (1.1) has at least one positive \( T \)-periodic solution.

Proof. We have

\[
N_1(t) = N_1(0) \exp \left\{ \int_0^t \left[ a_1(s) - b_1(s) \int_{-\infty}^s k(s-u)N_1(u) \, du \right. \\
- \frac{c(s)}{1 + m N_1(s)} N_2(s - \tau(s)) \bigg] \, ds \right\},
\]

where \( N_1(0) > 0 \),

\[
N_2(t) = N_2(0) \exp \left\{ \int_0^t \left[ -a_2(s) + \frac{b_2(s) N_1(s - \tau(s))}{1 + m N_1(s - \tau(s))} \right] \, ds \right\}, \quad N_2(0) > 0.
\]

By making the change

\[
N_i(t) = \exp\{x_i(t)\}, \quad i = 1, 2,
\]

Eqs. (1.1)–(1.2) are reformulated as

\[
\dot{x}_1(t) = a_1(t) - b_1(t) \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du - \frac{c(t) \exp\{x_2(t - \tau(t))\}}{1 + m \exp\{x_1(t)\}},
\]

\[
\dot{x}_2(t) = -a_2(t) + \frac{b_2(t) \exp\{x_1(t - \tau(t))\}}{1 + m \exp\{x_1(t - \tau(t))\}}.
\]

Let \( X = Z = \{x(t) = (x_1(t), x_2(t))^T \in (\mathbb{R}, \mathbb{R}^2), \ x(t + T) = x(t)\} \) and \( \|x\| = \|(x_1(t), x_2(t))^T\| = \max_{t \in [0,T]} |x_1(t)| + \max_{t \in [0,T]} |x_2(t)| \). Then \( X \) and \( Z \) are both Banach spaces with the norm \( \|\cdot\| \).

Let \( Lx = \dot{x}, \)

\[
Nx = \left( a_1(t) - b_1(t) \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du - \frac{c(t) \exp\{x_2(t - \tau(t))\}}{1 + m \exp\{x_1(t)\}}, \right.
\]

\[
- a_2(t) + \frac{b_2(t) \exp\{x_1(t - \tau(t))\}}{1 + m \exp\{x_1(t - \tau(t))\} \right)^T,
\]

\[
Px = \frac{1}{T} \int_0^T x(t) \, dt, \quad x \in X; \quad Qz = \frac{1}{T} \int_0^T z(t) \, dt, \quad z \in Z.
\]
Hence
\[
\text{Ker } L = \{ x \in X : x = h \in \mathbb{R}^2 \}, \quad \text{Im } L = \left\{ z \in Z : \int_0^T z(t) \, dt = 0 \right\}.
\]

It is clear that \( \text{Im } L \) is closed in \( Z \).

On the other hand, \( \text{dim Ker } L = \text{codim Im } L = 2 \). Hence \( L \) is a Fredholm operator of index zero, and it is easy to find that \( P, Q \) are continuous projectors such that
\[
\text{Im } P = \text{Ker } L, \quad \text{Im } L = \text{Ker } Q = \text{Im}(I - Q).
\]

Obviously, \( L \mid \text{Dom } L \cap \text{Ker } P : (I - P)X \to \text{Im } L \) is one-to-one. Consequently, it is invertible. We denote the inverse of that operator by \( K \).

Let \( z \in \text{Im } L \) and let
\[
Lx = z.
\]

Integrating both sides of above identity over \( s \) from 0 to \( t \), we have
\[
x(t) = x(0) + \int_0^t z(s) \, ds,
\]
and since
\[
\frac{1}{T} \int_0^T x(t) \, dt = 0,
\]
we obtain
\[
K(z) = \int_0^t z(s) \, ds - \frac{1}{T} \int_0^T \int_0^t z(s) \, ds \, dt,
\]
and
\[
QN x = \left( \frac{1}{T} \int_0^T \left[ a_1(s) - b_1(s) \int_{-\infty}^s k(s-u) \exp\{x_1(u)\} \, du - \frac{c(s) \exp\{x_2(s - \tau(s))\}}{1 + m \exp\{x_1(s)\}} \right] \, ds, \right. \]
\[
\left. \frac{1}{T} \int_0^T \left[ -a_2(s) + \frac{b_2(s) \exp\{x_1(s - \tau(s))\}}{1 + m \exp\{x_1(s - \tau(s))\}} \right] \, ds \right)^T.
\]

Now \( N \) denotes the Niemytski operator for Eqs. (2.4)–(2.5). Let \( \tilde{\Omega} \) be an open bounded subset of \( X \). Clearly, \( K(I - Q)N : \tilde{\Omega} \to X \) is compact that follows from the complete continuity of \( K \), the continuity (i.e. the boundedness) of \( I - Q \), and the boundedness of the set \( N(\tilde{\Omega}) \). Hence, \( N \) is \( L \)-compact on \( \tilde{\Omega} \) which follows from [6].

On the other hand, we consider operator equation \( Lx = \lambda Nx, \lambda \in (0, 1] \) and
\[
\dot{x}_1(t) = \lambda \left( a_1(t) - b_1(t) \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du - \frac{c(t) \exp\{x_2(t - \tau(t))\}}{1 + m \exp\{x_1(t)\}} \right), \]
\[
\dot{x}_2(t) = \lambda \left( -a_2(t) + \frac{b_2(t) \exp\{x_1(t - \tau(t))\}}{1 + m \exp\{x_1(t - \tau(t))\}} \right), \quad \lambda \in (0, 1].
\]
Suppose that \( x(t) \in X \) is a solution of system (2.11)–(2.12) for a certain \( \lambda \in (0, 1] \). Integrating both sides of (2.12)–(2.13) over \( t \) from 0 to \( T \) gives

\[
\int_0^T \left( b_1(t) \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du + \frac{c(t) \exp\{x_2(t-\tau(t))\}}{1 + m \exp\{x_1(t)\}} \right) \, dt = \int_0^T a_1(t) \, dt
\]

\( = A_1 > 0, \) \hspace{1cm} (2.14)

\[
\int_0^T \left( \frac{b_2(t) \exp\{x_1(t-\tau(t))\}}{1 + m \exp\{x_1(t-\tau(t))\}} \right) \, dt = \int_0^T a_2(t) \, dt = A_2, \quad \lambda \in (0, 1]. \)

(2.15)

It follows from (2.12)–(2.15) that

\[
\int_0^T |\dot{x}_1(t)| \, dt
\]

\[= \lambda \int_0^T \left| a_1(t) - b_1(t) \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du - \frac{c(t) \exp\{x_2(t-\tau(t))\}}{1 + m \exp\{x_1(t)\}} \right| \, dt
\]

\[\leq \int_0^T |a_1(t)| \, dt + |b_1(t)| \int_{-\infty}^t k(t-u) \exp\{x_1(u)\} \, du + \frac{c(t) \exp\{x_2(t-\tau(t))\}}{1 + m \exp\{x_1(t)\}} \int_0^T \, dt
\]

\[= \bar{A}_1 + A_1 \] \hspace{1cm} (2.16)

and

\[
\int_0^T |\dot{x}_2(t)| = \lambda \int_0^T \left| -a_2(t) + \frac{b_2(t) \exp\{x_1(t-\tau(t))\}}{1 + m \exp\{x_1(t-\tau(t))\}} \right| \, dt
\]

\[\leq \int_0^T |a_2(t)| \, dt + \int_0^T \frac{b_2(t) \exp\{x_1(t-\tau(t))\}}{1 + m \exp\{x_1(t-\tau(t))\}} \, dt
\]

\[= \bar{A}_2 + A_2. \] \hspace{1cm} (2.17)

Since \( x(t) = (x_1(t), x_2(t))^T \in X \), then there exist \( \xi_i, \tau_i \in [0, T] \) such that

\[
x_i(\xi_i) = \min_{t \in [0, T]} x_i(t); \quad x_i(\tau_i) = \max_{t \in [0, T]} x_i(t), \quad i = 1, 2. \]

(2.18)

Hence

\[
A_2 \geq \frac{\exp\{x_1(\xi_1)\}}{1 + m \exp\{x_1(\xi_1)\}} \int_0^T b_2(t) \, dt,
\]

that is,

\[
x_1(\xi_1) \leq \ln \frac{A_2}{B_2 - mA_2}.
\]
Consequently,
\[ x_1(t) \leq x_1(\xi_1) + \int_0^T |\dot{x}_1(t)| \leq \ln \left( \frac{A_2}{B_2 - mA_2} \right) + \bar{A}_1 + A_1. \]  \hspace{1cm} (2.19)

Similarly,
\[ A_2 \leq \exp \left\{ \frac{x_1(\tau_1)}{1 + m \exp \{x_1(\tau_1)\}} \right\} \int_0^T b_2(t) \, dt, \]
which yields
\[ x_1(\tau_1) \geq \ln \frac{A_2}{B_2 - mA_2}. \]

Thus,
\[ x_1(t) \geq x_1(\tau_1) - \int_0^T |\dot{x}_1(t)| \geq \ln \left( \frac{A_2}{B_2 - mA_2} \right) - (\bar{A}_1 + A_1). \]  \hspace{1cm} (2.20)

It follows from (2.19) and (2.20) that
\[
\max_{t \in [0, T]} |x_1(t)| \\
\leq \max \left\{ \left| \ln \left( \frac{A_2}{B_2 - mA_2} \right) + \bar{A}_1 + A_1 \right|, \left| \ln \left( \frac{A_2}{B_2 - mA_2} \right) - \bar{A}_1 - A_1 \right| \right\} =: M_1.
\]

On the other hand,
\[ A_1 \geq \int_0^T c(t) \exp \{x_2(\xi_2)\} \, dt = C \frac{\exp \{x_2(\xi_2)\}}{1 + m \exp \{x_1(\tau_1)\}}, \]
that is,
\[ \exp \{x_2(\xi_2)\} \leq \frac{A_1}{C} \left( 1 + m \exp \{x_1(\tau_1)\} \right) \leq \frac{A_1}{C} \left( 1 + m \exp \{M_1\} \right) \]
which yields
\[ x_2(\xi_2) \leq \ln \left( \frac{A_1}{C} \left( 1 + m \exp \{M_1\} \right) \right). \]

Thus,
\[ x_2(t) \leq x_2(\xi_2) + \int_0^T |\dot{x}_2(t)| \, dt \]
\[ \leq \ln \left( \frac{A_1}{C} \left( 1 + m \exp \{M_1\} \right) \right) + \bar{A}_2 + A_2 =: M_2. \]  \hspace{1cm} (2.21)

Furthermore, it follows from (2.14)–(2.19) that
Furthermore, in view of the assumption in Theorem 2.1, it is easy to see that
\[ \ker L \]
Therefore
\[ Lx \]
has a unique solution
\[ (v) \]
quirements in Lemma 2.1. Hence system (2.4)–(2.5) has at least one solution in \( \text{Dom} L \)

Then it follows that
\[ \exp\{x_2(\tau_2)\} \geq \frac{A_1}{C} - \frac{B_1A_2}{C(B_2 - mA_2)} \exp\{\bar{A}_1 + A_1\} \]
that is,
\[ x_2(\tau_2) \geq \ln\left\{ \frac{A_1}{C} - \frac{B_1A_2}{C(B_2 - mA_2)} \exp\{\bar{A}_1 + A_1\} \right\} . \]
Hence,
\[ x_2(t) \geq x_2(\tau_2) - \int_0^T |\dot{x}_2(t)| dt \]
\[ \geq \ln\left\{ \frac{A_1}{C} - \frac{B_1A_2}{C(B_2 - mA_2)} \exp\{\bar{A}_1 + A_1\} \right\} - (\bar{A}_2 + A_2) =: M_3. \]

It follows from (2.11) and (2.23) that
\[ \max_{t \in [0, T]} |x_2(t)| \leq \max\{M_2, M_3\} =: M_4. \]

Clearly, \( M_i \) (\( i = 1, 2, 3, 4 \)) are independent of the choice of \( \lambda \). Under the assumptions in Theorem 2.1 it is easy to show that the system of algebraic equations
\[ A_1 - B_1v_1 - \frac{Cv_2}{1 + mv_1} = 0, \quad A_2 - \frac{B_2v_1}{1 + mv_1} = 0 \]
has a unique solution \((v^*_1, v^*_2)^T \in \mathbb{R}^2\) with \( v^*_i > 0 \). Take \( M = M_1 + M_4 + M_5 \) where \( M_5 > 0 \) is taken sufficiently large such that \( \|\ln\{v^*_1\}, \ln\{v^*_2\}\| = |\ln\{v^*_1\}| + |\ln\{v^*_2\}| < M_5 \) and define
\[ \Omega = \{ x(t) = (x_1(t), x_2(t))^T \in X : \|x\| < M \} . \]

Therefore \( Lx \neq \lambda Nx \) for any \( \lambda \in (0, 1) \), \( x \in \text{Dom} L \cap \partial \Omega \). On the other hand, when \( x \in \partial \Omega \cap \text{Ker} L = \partial \Omega \cap \mathbb{R}^2 \), \( x \) is a constant vector in \( \mathbb{R}^2 \) with \( \|x\| = M \), then
\[ QNx = \left( \frac{1}{T} \left( A_1 - B_1 \exp\{x_1\} - C \frac{\exp\{x_2\}}{1 + m \exp\{x_1\}} \right), \frac{1}{T} \left( -A_2 + B_2 \frac{\exp\{x_1\}}{1 + m \exp\{x_1\}} \right) \right)^T \]
\[ \neq 0. \]
Furthermore, in view of the assumption in Theorem 2.1, it is easy to see that
\[ \deg\{JQN, \Omega \cap \text{Ker} L, 0\} \neq 0, \]
where \( J \) is the identity mapping since \( \text{Im} P = \text{Ker} L \). Thus we proved that \( \Omega \) satisfies all the requirements in Lemma 2.1. Hence system (2.4)–(2.5) has at least one solution in \( \text{Dom} L \cap \Omega \). Consequently, system (1.1) has at least one positive \( T \)-periodic solution. The proof is complete. \( \Box \)
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