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Submultiplicative and Subadditive Functions and Integral Inequalities of Bellman–Bihari Type

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We define two classes of functions which contain the classes of nondecreasing submultiplicative and subadditive functions. Then we discuss the properties of these classes and use them to give generalizations of some well-known integral inequalities like the Bellman–Bihari inequality. © 1986 Academic Press, Inc.

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The attractive Gronwall–Bellman inequality [16] plays a vital role in studying stability and asymptotic behavior of solutions of differential equations (see [7] and [8]). Many linear and nonlinear generalizations have appeared in the literature [30, 3]. An extensive survey of these generalizations is given by Beesack [3]. On the basis of a linear generalization given by Pachpatte [18] and its general version by Agarwal [2] various motivations have appeared in the literature (see, e.g., Agarwal and Thandapani [1]). In all these nonlinear generalizations, the nonlinear functions appearing in the right side are supposed to belong to the class of submultiplicative or (and) subadditive functions with some monotonicity properties. No attempt has been made to use another class of functions which relax the submultiplicativity and subadditivity conditions.

Dhongade and Deo [14] were the first who defined a class F of functions $w(u)$, which are continuous, positive and nondecreasing on $[0, \infty)$, and satisfy the condition

$$\frac{1}{\alpha} g(u) \leqq g\left(\frac{u}{\alpha}\right), \qquad u \geqq 0, v > 0. \tag{$*$}$$

In point of fact, the previous condition implies that $g(u) \equiv g(1) u$ for $u > 0$. To avoid this triviality, an essential modification has been given by Beesack [3], namely to require (*) to hold only for $u \geqq 0$, $v \geqq 1$.

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In Section 2 we define two classes of functions and discuss their properties. In Section 3 we extend some of the nonlinear generalizations of Gronwall–Bellman inequalities, where the nonlinear functions appearing in the right side belong to the classes of functions defined in Section 2. Also, we obtain several integral inequalities similar to the Bellman–Bihari inequality [9, Sect. 4] is devoted to the nonlinear versions of the main inequality of Pachpatte [18] and its extension by Agarwal [2].

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DEFINITION 1. A function $w: [0, \infty) \to [0, \infty)$ is said to belong to the class H if

(H_1) $w(u)$ is nondecreasing and continuous for $u \geqq 0$ and positive for $u > 0$.

(H_2) There exists a function ϕ, continuous on $[0, \infty)$ with $w(\alpha u) \leqq \phi(\alpha)\, w(u)$ for $\alpha > 0$, $u \geqq 0$.

Several examples and properties of the class H have been obtained by the author in [11]. In particular, H includes all functions $w \in F$, with corresponding function ϕ defined by $\phi(\alpha) = 1$ $(0 \leqq \alpha \leqq 1)$, $\phi(\alpha) = \alpha$ $(\alpha \geqq 1)$. Also H includes all submultiplicative functions which satisfy (H_1), with corresponding function $\phi = w$.

DEFINITION 2. A function $w: [0, \infty) \to [0, \infty)$ is said to belong to the class M if

(M_1) $w(u)$ is nondecreasing and continuous for $u \geqq 0$ and positive for $u > 0$.

(M_2) There exists a function ψ, continuous on $[0, \infty)$ with $w(\alpha + u) \leqq \psi(\alpha) + w(u)$ for $\alpha > 0$, $u \geqq 0$.

EXAMPLE 1. Every function w which is continuous and nondecreasing on $[0, \infty)$ with $w(u) > 0$ for $u > 0$ and which is subadditive is of class M with $\psi = w$.

EXAMPLE 2. Any nondecreasing continuous function w on $[0, \infty)$ with $w(u) > 0$ for $u > 0$ which satisfies a Lipschitz condition of order $n > 0$,

$$w(\alpha + u) - w(u) \leqq K\alpha^n,$$

is of class M with $\psi = K\alpha^n$, where K is a nonnegative constant.

EXAMPLE 3. The function $w(u)=\ln(\cosh u)$ belongs to M with $\psi(\alpha)=\ln(2\cosh\alpha)$.

EXAMPLE 4. The function $w(u)=u^3/(u^2+1)$ belongs to M with $\psi(\alpha)=(\alpha^3/(\alpha^2+1))+k\alpha$, for any $k\geqq\frac{3}{2}$.

Now we note some properties of the function $\psi(\alpha)$.

(a) $\psi(\alpha)\geqq 0$ for $\alpha\geqq 0$. This follows from

$$w(u)\leqq w(u+\alpha)\leqq w(u)+\psi(\alpha).$$

(b) If $w(0)=0$, then $w(\alpha)\leqq\psi(\alpha)$ for $\alpha\geqq 0$.

In what follows we give some properties of the class M.

LEMMA 1. *Let $w(u)\in M$ with corresponding function $\psi(\alpha)$. Then $T(u)=(1/u)\int_0^u w(s)\,ds$ for $u>0$ with $T(0)=w(0)$ is of class M.*

Proof. It follows from [10] that $T(u)$ satisfies M_1. Now we note that

$$\begin{aligned}T(\alpha+u)&=\frac{1}{\alpha+u}\left[\alpha T(\alpha)+\int_0^u w(\alpha+\theta)\,d\theta\right]\\&\leqq\frac{1}{\alpha+u}\left[\alpha T(\alpha)+\int_0^u (\psi(\alpha)+w(\theta))\,d\theta\right]\\&=\frac{\alpha}{\alpha+u}T(\alpha)+\frac{u\psi(\alpha)+uT(u)}{\alpha+u}\\&\leqq T(u)+[\psi(\alpha)+T(\alpha)];\end{aligned}$$

so T satisfies M_2 with corresponding function $(T+\psi)$.

LEMMA 2. *Let $F(u)$ be a convex continuous function on $[0,\infty)$ with $F(0)=0$ and $F(u)>0$ for $u>0$, which satisfies M_2 with corresponding function $\psi(\alpha)$. Assume also that $G(u)$ is a concave continuous function on $[0,\infty)$ with $G(0)=0$ for which there exists a function χ defined on $[0,\infty)$ such that $G(u+\alpha)\geqq\chi(\alpha)+G(u)$ for $u\geqq 0$, $\alpha>0$. If in addition, $\lim_{\alpha\to 0+}\psi(\alpha)/\chi(\alpha)=A$ exists (finite), then $F(u)/G(u)$ is of class M.*

Proof. Observe that $G(\alpha)\geqq\chi(\alpha)>0$ for $\alpha>0$, so $F(u)/G(u)$ is defined, positive, and continuous for $u>0$. By [10], F/G is also nondecreasing on $(0,\infty)$, and since $F(u)/G(u)>0$ for $u>0$, it follows that $B=\lim_{u\to 0+}F(u)/G(u)$ exists $(B\geqq 0)$. Now, we show that if the function

F/G is defined to have the value B for $u=0$, then it is of class M. For, as now proved, it satisfies M_1. For $\alpha>0$, $u\geqq 0$,

$$\frac{F(\alpha+u)}{G(\alpha+u)}\leqq\frac{\psi(\alpha)+F(u)}{\chi(\alpha)+G(u)}\leqq\frac{\psi(\alpha)}{\chi(\alpha)}+\frac{F(u)}{G(u)}.$$

A corresponding function for F/G is therefore the function $\phi(\alpha)$ defined by $\phi(0)=A$, $\phi(\alpha)=\psi(\alpha)/\chi(\alpha)$ for $\alpha>0$; ϕ is thus continuous on $[0,\infty)$ as required by M_2.

LEMMA 3. *Let $f(x)\in M$ with corresponding function $\psi(\alpha)$. Then*

$$f(\alpha x)\leqq([\alpha]+1)\,\psi(x)+f(0)\qquad \textit{for}\quad \alpha\geqq 0,\, x\geqq 0,$$

where $[\alpha]$ is the largest integer less than or equal to α.

Proof. Since $f\in M$, then

$$f(x+y)\leqq\psi(x)+f(y).$$

Putting $y=0$, one obtains

$$f(x)\leqq\psi(x)+f(0).$$

Therefore

$$f(x+y)\leqq\psi(x)+\psi(y)+f(0)$$

and

$$f(2x)\leqq 2\psi(x)+f(0).$$

It is easy to prove by induction that

$$f(\alpha x)\leqq\alpha\psi(x)+f(0),$$

for α belongs to natural numbers (N). If α does not belong to N, then $m<\alpha<m+1$, where $m\in N$. Hence

$$f(\alpha x)\leqq f((m+1)\,x)\leqq(m+1)\,\psi(x)+f(0).$$

The proof is complete.

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In this section further generalizations of the Bellman–Bihari inequality have been obtained. In what follows we prove a similar result to that of

Pachpatte [23], where the nonlinear function belongs to H rather than just to F.

THEOREM 1. *Let $x(t)$, $f(t)$, $g(t)$, $p(t)$, and $k(t)$ be real-valued positive functions defined on $I=[0, \infty)$, let $w(u) \in H$ with corresponding multiplier function ϕ and let $k(t)$ also be a monotonic, nondecreasing function, for which the inequality*

$$x(t) \leqq k(t) + p(t) \int_0^t f(s)\, x(s)\, ds + \int_0^t g(s)\, w(x(s))\, ds \tag{1}$$

holds for all $t \in I$. Then

$$x(t) \leqq k(t)\, r(t)\, W^{-1}\left[W(1) + \int_0^t \frac{1}{k(s)} g(s)\, \phi(k(s))\, \phi(r(s))\, ds\right], \tag{2}$$

for $t \in [0, b]$, where

$$r(t) = 1 + p(t)\left[\int_0^t f(s) \exp\left(\int_s^t p(\theta) f(\theta)\, d\theta\right) ds\right], \qquad t \in I, \tag{3}$$

$$W(r) = \int_{r_0}^{r} \frac{ds}{w(s)}, \qquad r>0 \quad (r_0>0) \tag{4}$$

and W^{-1} is the inverse function of W, and $t \in [0, b] \subset I$ so that

$$W(1) + \int_0^t \frac{1}{k(s)} g(s)\, \phi(k(s))\, \phi(r(s))\, ds \in \mathrm{Dom}(W^{-1}).$$

Proof. Since $k(t)$ is positive, monotonic, nondecreasing, we observe from (1) that

$$\frac{x(t)}{k(t)} \leqq 1 + p(t) \int_0^t f(s) \frac{x(s)}{k(s)}\, ds + \int_0^t \frac{g(s)}{k(s)}\, w(x(s))\, ds. \tag{5}$$

Let $z(t) = x(t)/k(t)$ and use the fact that $w \in H$. Then from (5) we have

$$z(t) \leqq 1 + p(t) \int_0^t f(s)\, z(s)\, ds + \int_0^t h(s)\, w(z(s))\, ds, \tag{6}$$

where $h(s) = g(s)\, \phi(k(s))/k(s)$.

Now, define

$$n(t) = 1 + \int_0^t h(s)\, w(z(s))\, ds, \qquad n(0) = 1 \tag{7}$$

and observe that $n(t)$ is positive monotonic nondecreasing. We obtain from Theorem 1 in [23] and (6) the following estimate for $z(t)$,

$$z(t) \leqq n(t)\, r(t). \tag{8}$$

Further,

$$w(z(t)) \leqq \phi(r(t))\, w(n(t))$$

since $w \in H$. Hence

$$\frac{h(t)\, w(z(t))}{w(n(t))} \leqq h(t)\, \phi(r(t)).$$

Because of (4) and (7), this reduces to

$$\frac{d}{dt} W(n(t)) \leqq h(t)\, \phi(r(t)).$$

Now, integrating from 0 to t, we obtain

$$W(n(t)) - W(1) \leqq \int_0^t h(s)\, \phi(r(s))\, ds. \tag{9}$$

The desired bound in (2) follows from (8) and (9).

Now we establish an extension of Theorem 5.6 [3], where the nonlinear function under the integral sign belongs to M and is not just subadditive as in [3].

We note that, somewhat earlier, Deo and Murdeshwar [13] had obtained the same estimate as that given in [3, Theorem 5.6], but the proof in [13] is unfortunately incorrect. See also Beesack [4], Theorem 1.

THEOREM 2. *Let $x(t)$, $a(t)$, $k(t)$, and $h(t)$ be real-valued nonnegative continuous functions defined on $J = [0, \beta)$. Let $g(u) \in M$ with corresponding function ϕ on an interval I such that $x(J) \subset I$ and $a(J) \subset I$. Suppose also that the function h is monotonic, nondecreasing on an interval K such that $0 \in K$, $h(K) \subset I$. Then*

$$x(t) \leqq a(t) + h\left[\int_0^t k(s)\, g(x(s))\, ds\right], \qquad t \in J, \tag{10}$$

implies

$$x(t) \leqq a(t) + h\left\{G^{-1}\left[\int_0^t k(s)\, ds + G\left(\int_0^t k(s)\, \phi(a(s))\, ds\right)\right]\right\} \tag{11}$$

for $0 \leqq t < \beta_1$, *where* $G(u) = \int_{u_0}^{u} dy/g(h(y))$ for $u, u_0 \in K$ *and* $\beta_1 = \min(u_1, u_2, u_3)$ *with*

$$u_1 = \sup \left\{ u \in J \colon a(t) + h\left(\int_0^t k(s)\, g(x(s))\, ds \right) \in I, \quad 0 \leqq t \leqq u \right\},$$

$$u_2 = \sup \left\{ u \in J \colon \int_0^u k(s) \left[\phi(a(s)) + goh\left(\int_0^s k(\theta)\, g(x(\theta))\, d\theta \right) \right] ds \in K \right\},$$

$$u_3 = \sup \left\{ u \in J \colon \int_0^t k(s)\, ds + G\left(\int_0^T k(s)\, \phi(a(s))\, ds \right) \in G(K), \quad 0 \leqq t \leqq T \leqq u \right\}.$$

The proof can be accomplished in a similar way to that of Theorem 5.6 [3] *or Theorem* 1 *of* [4].

Remark 1. It is not difficult to show that the same estimation for $x(t)$ can be obtained when $k(t)$ is nonpositive and $h(t)$ is nonincreasing.

Remark 2. When $h(u) = u$, Theorem 2 reduces to a generalization of Lemma 2 by Muldowney and Wong [17].

The case when the nonlinear function h in (10) is multiplied by $b(t)$ has been considered in detail by Beesack [5, Theorem 5.4; 4, Theorem 2; 3, p. 81–82]. Under several sets of conditions on (x, a, b, h, k, g), different incomparable estimates for $x(t)$ have been obtained.

COROLLARY. *Let* x, a, k, g *all be as in Theorem* 2 *and suppose* $b(t)$ *is nonnegative, continuous, and nondecreasing on* $I = [0, \beta]$. *If*

$$x(t) \leqq a(t) + b(t) \int_0^t k(s)\, g(x(s))\, ds, \qquad t \in I, \tag{12}$$

then

$$x(t) \leqq a(t) + G^{-1} \left[b(t) \int_0^t k(s)\, ds + G\left(b(t) \int_0^k k(s)\, \phi(a(s))\, ds \right) \right]$$

$$\vdots \tag{12'}$$

for $0 \leqq t \leqq t_0$, *where* G, G^{-1} *are defined in Theorem* 2, *but with* $h(u) \equiv u$ *there,*

$$b(t) \int_0^t k(s)\, ds + G\left(b(t) \int_0^t k(s)\, \phi(a(s))\, ds\right) \in \mathrm{Dom}(G^{-1}).$$

The proof of this corollary follows by an argument similar to that in the proof of Corollary 1 given in [11].

In what follows we give an estimate for $x(t)$ under different set of conditions on (x, a, b, h, k, g).

THEOREM 3. *Let* $x(t)$, $a(t)$, $k(t)$, $b(t)$ *be continuous and nonnegative on* $J = [0, \beta]$ *with* $b(t) > 0$ *and* $a(t)/b(t) \leqq \gamma$ *for some positive constant* γ. *Let* $g(u)$ *be of class* H *with corresponding function* ϕ. *Suppose that* $h(u)$ *is continuous, nonnegative and nondecreasing on* $[0, \infty)$. *If*

$$x(t) \leqq a(t) + b(t)\, h\left(\int_0^t k(s)\, g(x(s))\, ds\right), \qquad t \in J, \tag{13}$$

then

$$x(t) \leqq a(t) + b(t)\, h \circ L^{-1}\left(\int_0^t k(s)\, \phi(b(s))\, ds\right), \qquad 0 \leqq t < \beta_1 \tag{14}$$

where

$$L(u) = \int_0^u \frac{dz}{g(\gamma + h(z))}, \qquad u \geqq 0,$$

and

$$\beta_1 = \sup\left\{t \in J\colon \int_0^t k(s)\, g(b(s))\, ds \in L(R^+)\right\}.$$

Proof. Let

$$z(t) = \int_0^t k(s)\, g(x(s)\, ds.$$

Then from (13) and the hypotheses on g and a, b it follows that

$$\begin{aligned} \frac{dz}{dt} = k(t)\, g(x(t)) &\leqq k(t)\, g[a(t) + b(t)\, h(z(t))] \\ &\leqq k(t)\, \phi(b(t))\, g\left[\frac{a(t)}{b(t)} + h(z(t))\right] \end{aligned}$$

and

$$\frac{dz}{\gamma + h(z)} \leqq k(t)\, \phi(b(t))\, dt. \tag{15}$$

Integrating both sides of (15) from 0 to t, one obtains

$$L(z) \leqq \int_0^t k(s)\, \phi(b(s))\, ds$$

and

$$z \leqq L^{-1}\left[\int_0^t k(s)\, \phi(b(s))\, ds\right], \qquad 0 \leqq t < \beta_1. \tag{16}$$

The substitution of (16) in (13), implies (14).

Remark 3. In Theorem 3 it is clear that hypotheses $b > 0$ and $a/b \leqq \gamma$ can be replaced by $a > 0$ and $b/a \leqq \gamma$, therefore $\phi(b(s))$ in (14) will be replaced by $\phi(a(s))$ and

$$L(u) = \int_0^u dz/g(1 + \gamma h(z)).$$

Remark 4. *Let $g(u) = u^2/(1+u)$. Then $g(u)$ is not submultiplicative and does not satisfy the condition $g(u)/v \leqq g(u/v)$ for $u > 0$ and $v \geqq 1$. Therefore, all Theorems in* [4, 12, 5, 3] *are not applicable. Theorem* 3 *can be applied, since $u^2/(1+u)$ is of class H with corresponding function ϕ defined by* $\phi(\alpha) = \alpha$ $(0 \leqq \alpha \leqq 1)$, $\phi(\alpha) = \alpha^2$ $(\alpha \geqq 1)$.

Remark 5. In the case when g is strictly increasing and $h \equiv g^{-1}$ we obtain from Theorem 3 the following estimate for $x(t)$,

$$x(t) \leqq a(t) + b(t)\, g^{-1} \circ L^{-1}\left(\int_0^t k(s)\, \phi(b(s))\, ds\right),$$

where now $L(u) = \int_0^u dz/g(\gamma + g^{-1}(z))$.

This estimate is not comparable with a result obtained by Gollwitzer [15, Theorem 1].

We obtain another upper bound for $x(t)$ when g satisfies different, but general conditions. The following result essentially is the variation of Gollwitzer's Theorem 1 in which the conditions: g convex and submultiplicative, are replaced by: $g \in H \cap M$.

THEOREM 4. *Let $a(t)$, $b(t)$, and $k(t)$ be continuous, nonnegative functions* *on $J=[0,\beta]$ with $b(t)>0$ and $g\in H$ and M with corresponding functions ϕ and ψ, respectively. Assume also that g is strictly increasing. Then*

$$x(t)\leqq a(t)+b(t)\,g^{-1}\left(\int_0^t k(s)\,g(x(s))\,ds\right),\qquad t\in J, \tag{17}$$

implies

$$x(t)\leqq b(t)\,g^{-1}(B(t)),\qquad t\in[0,\beta_1], \tag{18}$$

where

$$B(t)=\psi\left(\frac{a(t)}{b(t)}\right)+\int_0^t k(s)\,\phi(b(s))\,\psi\left(\frac{a(s)}{b(s)}\right) \times\exp\left(\int_s^t k(\theta)\,\phi(b(\theta))\,d\theta\right)ds\cdots \tag{19}$$

and

$$\beta_1=\sup\left\{t\in J\colon \int_0^t k(s)\,\phi(b(s))\,B(s)\,ds\in g(R^+)\right\}.$$

Proof. From (17) it follows that

$$\frac{x(t)}{b(t)}\leqq\frac{a(t)}{b(t)}+g^{-1}\left(\int_0^t k(s)\,g\left(\frac{x(s)}{b(s)}\right)\phi(b(s))\,ds\right). \tag{20}$$

Let $x(t)/b(t)=z(t)$ and use the hypotheses on g, to obtain

$$g(z)\leqq\psi\left(\frac{a}{b}\right)+\int_0^t k\phi(b)\,g(z)\,ds. \tag{21}$$

Considering $g(z)$ as a function, using the most general linear Gronwall inequality (see, e.g., Beesack [3, Theorem 2.1]), it follows that $g(z(t))\leqq B(t)$, so $z(t)\leqq g^{-1}(B(t))$, but since $x(t)=b(t)\,z(t)$, (18) follows.

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Several integral inequalities similar to Bellman–Bihari type have been obtained by Pachpatte [18–29]. Most of these inequalities are based on a main inequality [18], in which an estimate for $x(t)$ has been obtained, when

$$x(t)\leqq x_0+\int_0^t f(s)\,x(s)\,ds+\int_0^t f(s)\left(\int_0^s g(\theta)\,x(\theta)\,d\theta\right)ds,$$

where $f(t)$, $g(t)$, and $x(t)$ are supposed to be nonnegative with x_0 is a positive constant and $t \in [0, \infty)$.

Later Agarwal [2] proved a general version of Pachpatte inequality, when $x(t)$ satisfies the inequality

$$x(t) \leqq p(t) + \int_0^t f_1(s)\, x(s)\, ds + \int_0^t f_2(s) \int_0^s f_3(\theta)\, x(\theta)\, d\theta, \quad t \geqq 0. \tag{22}$$

Several linear and nonlinear generalizations have been obtained by Agarwal and Thandapani in their interesting paper [1].

In the following two theorems we consider nonlinear versions of (22).

These two theorems are related to the special case $m = 2$ of Theorems 11 and 13 of [1], which dealt with $g, h \in F$ rather than $g, h \in H$ or $g, h \in M$. See also the case $k = 2$ of Theorem 1 of Beesack [6] for related results.

THEOREM 5. *Let $x(t)$, $a(t)$, $k(t)$, $l(t)$, and $m(t)$ be real-valued nonnegative, continuous functions on $I = [0, \infty)$ with $a(t)$ positive, nondecreasing. Assume that $g(u)$ and $h(u)$ belong to H with corresponding multiplier functions $\phi(u)$ and $\psi(u)$, respectively, with $\phi(u) \leqq cu$ for $u \geqq 1$, where c is a positive constant. Then*

$$x(t) \leqq a(t) + \int_0^t k(s)\, g(x(s))\, ds + \int_0^t l(s) \int_0^s m(\theta)\, h(x(\theta))\, d\theta\, ds, \qquad t \in I \tag{23}$$

implies that

$$x(t) \leqq a(t)\ )F(t)\, H^{-1} \left\{ H(1) + \int_0^t k_2(s)\, \psi(F(s))\, ds \right\}, \qquad 0 < t \leqq \beta, \tag{24}$$

where

$$F(t) = G^{-1} \left[G(1) + c \int_0^t k_1(s)\, ds \right], \tag{25}$$

$$k_1(t) = k(t)\, \phi(a(t))/a(t), \tag{26}$$

$$k_2(t) = l(t) \int_0^t \frac{m(\theta)\, \psi(a(\theta))}{a(\theta)}\, d\theta, \tag{27}$$

$$H(t) = \int_{u_0}^u ds/h(s), \qquad G(u) = \int_{u_0}^u ds/g(s), \quad u > 0\ (u_0 > 0). \tag{28}$$

H^{-1} and G^{-1} are the inverse functions of H and G, respectively, $\beta = \min(b_1, b_2)$,

$$b_1 = \sup \left\{ t \in I: G(1) + c \int_0^t k_1(s)\, ds \in \operatorname{Dom}(G^{-1}) \right\}$$

and

$$b_2 = \sup \left\{ t \in I: H(1) + \int_0^t k_2(s)\, \psi(F(s))\, ds \in \operatorname{Dom}(H^{-1}) \right\}.$$

Proof. Let $x(t)/a(t) \equiv y(t)$. Since g and h belong to H, from (23) it follows that

$$y(t) \leqq R(t), \tag{29}$$

where

$$R(t) = 1 + \int_0^t \frac{k(s)\, \phi(a(s))}{a(s)}\, g(y(s))\, ds + \int_0^t l(s) \int_0^s \frac{m(\theta)\, \psi(a(\theta))}{a(\theta)}\, h(y(\theta))\, d\theta\, ds, \qquad t \in I. \tag{30}$$

From (30) and the nondecreasing property of g and h one obtains

$$R' \leqq k_1(t)\, g(R) + k_2(t)\, h(R), \qquad R(0) = 1. \tag{31}$$

Integrating (31) from 0 to t we obtain

$$R(t) \leqq 1 + \int_0^t k_2(s)\, h(R(s))\, ds + \int_0^t k_1(s)\, g(R(s))\, ds, \qquad t \in I. \tag{32}$$

Putting

$$n(t) = 1 + \int_0^t k_2(s)\, h(R(s))\, ds \tag{33}$$

and using [11, Theorem 1], we obtain

$$R(t) \leqq n(t)\, G^{-1}\left[G(1) + \int_0^t k_1(s) \frac{\phi(n(s))}{n(s)}\, ds \right], \qquad 0 \leqq t \leqq b_1. \tag{34}$$

Since $\phi(n)/n \leqq c$, from (34) it follows that

$$R(t) \leqq n(t)\, F(t), \tag{35}$$

where $F(t)$ is defined by (25). Further,

$$h(R(t)) \leqq \psi(F(t))\, h(n(t))$$

since $h \in H$. Hence,

$$\frac{k_2(t)\, h(R(t))}{h(n(t))} \leqq k_2(t)\, \psi(F(t)).$$

Because of (29) and (33), this reduces to

$$\frac{d}{dt} H(n(t)) \leqq k_2(t)\, \psi(F(t)).$$

Now integrating from 0 to t, we obtain

$$H(n(t)) \leqq H(1) + \int_0^t k_2(s)\, \psi(F(s))\, ds. \tag{36}$$

The desired bound in (24) follows from (36), (35), and (29).

We point out that the conditions $g(\alpha u) \leqq \phi(\alpha)\, g(u) \leqq c\alpha g(u)$ for $u \geqq 0$, $\alpha \geqq 1$, imply that $g(u) \geqq g(1)\, u/c$ for $0 < u \leqq 1$, and $g(u) \leqq cg(u)u$ for $u \geqq 1$ (and that $c \geqq 1$).

Remark 6. We get a similar bound for $x(t)$, when the condition $\phi(u) \leqq cu$ is replaced by $\psi(u) \leqq cu$ for $u \geqq 1$.

THEOREM 6. *Let $x(t)$, $a(t)$, $k(t)$, $l(t)$, and $m(t)$ be real-valued nonnegative continuous functions on $I = [0, \infty)$, let g and h be of class M with corresponding functions $\phi(u)$ and $\psi(u)$ respectively and let* (i) *$g \in H$ or* (ii) *$h \in H$ with corresponding multiplier function χ such that $\chi(u) \leqq cu$, where $c > 0$ is a constant. Then*

$$x(t) \leqq a(t) + \int_0^t k(s)\, g(x(s))\, ds + \int_0^t l(s) \int_0^s m(\theta)\, h(x(\theta))\, d\theta\, ds$$

$$\vdots \tag{37}$$

for $t \in I$, implies for case (i) *that*

$$x(t) \leqq a(t) + r(t)\, N(t) + H^{-1} \left\{ r(t) \int_0^t p(s)\, ds + H\left[r(t) \int_0^t p(s)\, \psi(r(s)\, N(s))\, ds \right] \right\}, \tag{38}$$

while in case (ii), *one has*

$$x(t) \leqq a(t) + r_1(t)\, N(t) + G^{-1}\left\{ r_1(t) \int_0^t k(s)\, ds + G\left[r_1(t) \int_0^t k(s)\, \phi(r_1(s)\, N(s))\, ds \right] \right\} \tag{38'}$$

for $t \in [0, \beta]$, *where*

$$r_1(t) = H^{-1}\left[H(1) + c \int_0^t p(s)\, ds \right], \tag{39}$$

H and G are as defined in Theorem 5, H^{-1} *and* G^{-1} *are the inverse functions of H and G, respectively,*

$$N(t) = \int_0^t \left[k(s)\, \phi(a(s)) + l(s) \int_0^s m(\theta)\, \psi(a(\theta))\, d\theta \right] ds, \tag{40}$$

$$p(t) = l(t) \int_0^t m(s)\, ds, \tag{41}$$

$$r(t) = G^{-1}\left[G(1) + c \int_0^t k(s)\, ds \right], \tag{42}$$

$$\begin{aligned} &\beta = \min(\beta_1, \beta_2), \\ &\beta_1 = \sup\left\{ u \in I \colon G(1) + c \int_0^t k(s)\, ds \in \operatorname{Dom}(G^{-1}), \quad 0 \leqq t \leqq u \right\}, \\ &\beta_2 = \sup\left\{ u \in I \colon r(t) \int_0^t p(s)\, ds + H\left[r(t) \int_0^t p(s)\, \psi(r(s))\, N(s)\, ds \right] \in \operatorname{Dom}(H^{-1}), \quad 0 \leqq t \leqq u \right\}. \end{aligned} \tag{43}$$

Proof. It suffices to consider case (i), since case (ii) can be treated in a similar way. Let

$$R(t) = \int_0^t k(s)\, g(x(s))\, ds + \int_0^t l(s) \int_0^s m(\theta)\, h(x(\theta))\, d\theta\, ds, \qquad t \in I.$$

Since g and $h \in M$ one obtains

$$R'(t) \leqq k(t)\, \phi(a(t)) + k(t)\, g(R(t)) + \left[l(t) \int_0^t m(s)\, ds \right] h(R(t))$$

and

$$R(t) \leqq N(t) + \int_0^t k(s)\, g(R(s))\, ds + \int_0^t p(s)\, h(R(s))\, ds, \qquad t \in I. \tag{44}$$

If we put

$$M(t) = N(t) + \int_0^t p(s)\, h(R(s))\, ds, \tag{45}$$

then from [11, Theorem 1] it follows that

$$R(t) \leqq M(t)\, G^{-1}\left[G(1) + \int_0^t \frac{k(s)\, \chi(M(s))}{M(s)}\, ds\right] \tag{46}$$

for $t \in [0, \beta)$, where β is defined by (43). Since G and G^{-1} are strictly increasing and $\chi(M) \leqq cM$, then from (45) one obtains

$$R(t) \leqq r(t)\, N(t) + r(t) \int_0^t p(s)\, h(R(s))\, ds, \qquad t \in [0, \beta). \tag{47}$$

Now, the use of Corollary 1 completes the proof.

Remark 7. When $g(u) = h(u) = u$, Theorem 6 reduces to Lemma 1 of Agarwal [2].

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