Note

Simple formulas for lattice paths avoiding certain periodic staircase boundaries

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ABSTRACT

There is a strikingly simple classical formula for the number of lattice paths avoiding the line $x = ky$ when k is a positive integer. We show that the natural generalization of this simple formula continues to hold when the line $x = ky$ is replaced by certain periodic staircase boundaries—but only under special conditions. The simple formula fails in general, and it remains an open question to what extent our results can be further generalized.

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1. Background and main results

Throughout this paper, a *lattice path* will mean a lattice path in the plane whose only allowable steps are north $(0, 1)$ and east $(1, 0)$.

It is a classical theorem [1,2] that if k is a positive integer, then the number of lattice paths from $(0, 0)$ to $(a + 1, b)$ (where $a \geqslant kb$) that avoid touching or crossing the line $x = ky$ except at $(0, 0)$ is given by the formula

$$\binom{a + b}{b} - k\binom{a + b}{b - 1}. \tag{1}$$

In fact, more is true: There are

$$\binom{a}{c - 1}\binom{b}{c - 1} - k\binom{a - 1}{c - 2}\binom{b + 1}{c} \tag{2}$$

such paths with $c - 1$ northwest corners.[1] This stronger result appears explicitly in [5] and implicitly even earlier, but our favorite proofs of all these facts are the bijective proofs of Goulden and Serrano [4].

It is natural to ask if there are similar simple formulas for lattice paths from $(0, 0)$ to (a, b) that avoid the line $x = ky$, if k is allowed to be an arbitrary positive rational number. While one can write down a determinant formula (indeed, a determinantal formula exists for an arbitrarily shaped boundary), nothing as simple as (1) is known, and empirical investigation does not suggest any obvious conjecture.

Our first main result is that for certain periodic staircase boundaries (instead of straight-line boundaries), there *are* simple enumerative formulas that generalize (1) and (2), at least for certain special starting and ending points.

Definition 1. Given positive integers s and t, let $A_{s,t}$ be the infinite staircase path that starts at $(0, t)$, then takes s steps east, t steps north, s steps east, t steps north, and so on.

Definition 2. Given a set S of (finite) lattice paths, take each path $\pi \in S$, and augment it by prepending a north step to the beginning of π and appending a north step to the end of π. Let S^+ denote the resulting set of lattice paths.

Theorem 3. *Let s, t, n, and c be positive integers.*

(1) *Let S_1 be the set of lattice paths from $(0, 0)$ to $(sn + 1, tn)$ that avoid $A_{s,t}$. There are*

$$t \binom{sn}{c-1} \binom{tn}{c-1} - s \binom{sn-1}{c-2} \binom{tn+1}{c} \tag{3}$$

paths in S_1^+ with c northwest corners (equivalently, c southeast corners).

(2) *Let S_2 be the set of lattice paths from $(1, 0)$ to $(sn, tn - 1)$ that avoid $A_{s,t}$. There are*

$$t \binom{sn-1}{c-1} \binom{tn-1}{c-1} - s \binom{sn-2}{c-2} \binom{tn}{c} \tag{4}$$

paths in S_2^+ with c northwest corners (equivalently, c southeast corners).

The equivalence between counting northwest and southeast corners follows because in a lattice path that starts with a north step and ends with a north step, the first corner must be a northwest corner and the last corner must be a southeast corner, and northwest and southeast corners must alternate. Also, since $|S| = |S^+|$ for any S, summing over all c and applying Vandermonde convolution immediately yields the following corollary.

Corollary 4. *Let s, t, and n be positive integers. Then*

$$|S_1| = t \binom{sn+tn}{tn} - s \binom{sn+tn}{tn-1} \tag{5}$$

and

$$|S_2| = t \binom{sn+tn-2}{tn-1} - s \binom{sn+tn-2}{tn-2}. \tag{6}$$

Note that avoiding $A_{k,1}$ is the same as avoiding $x = ky$ except at $(0, 0)$, so our results generalize (1) and (2) in one direction, by allowing arbitrary s and t, but are simultaneously more special in

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[1] By a northwest corner we mean a corner formed by a north step followed by an east step. There is a similar-looking formula for paths with a given number of southeast corners.

another direction, since only certain special endpoints are allowed. More precisely, note that if we set $a = kn$ and $b = n$ in (1) and (2), then we get the same answers as if we set $s = k$ and $t = 1$ in (5) and (3). (When $t = 1$, the map $S \mapsto S^+$ simply adds a northwest corner to every path.)

Our proof of Theorem 3 is similar to Goulden and Serrano's in several ways but differs in one crucial way. Like Goulden and Serrano, we interpret (3) and (4) as counting *all* paths of a certain type, minus the *bad* paths. Another similarity is the idea of breaking the bad path into two halves ρ and σ at the first "bad point" so as to manipulate ρ and σ into something that is easier to count. The crucial difference is that Goulden and Serrano *rotate* ρ, whereas we *interchange ρ and σ.*[2]

We also give a second proof of Corollary 4, which is based on a well-known argument of Raney [8] regarding cyclic shifts of integer sequences.

It is frustrating that Theorem 3 applies only to special endpoints. Can anything be said about other endpoints? We do not have a satisfactory answer to this question, but our second main result is a tantalizing hint that more general theorems lie waiting to be found. It is best stated in the language of binary strings; we draw the connection to lattice paths afterwards.

Theorem 5. *For $n \geqslant 1$, $s \geqslant 0$, and $0 \leqslant r \leqslant 2n$, let $a(n, s, r)$ be the number of binary sequences of length $(s+2)n+1$ such that for all j, the jth occurrence of 10 (if it exists) appears in positions $(s+2)j+1$ and $(s+2)j+2$ or later, and such that the total number of occurrences of 10 and 01 is at most r. Then*

$$a(n, s, r) = 2\binom{(s+2)n - 1}{r} - (s-2)\sum_{i=0}^{r-1}\binom{(s+2)n - 1}{i}. \tag{7}$$

Our proof of Theorem 5 is again an application of Raney's argument, combined with a straightforward induction on n.

To convert Theorem 5 into lattice-path language, let $\beta = (b_1, b_2, \ldots, b_{(s+2)n+1})$ be a binary sequence, let $b_0 = 0$, and define $\Delta\beta$ by $(\Delta\beta)_i = |b_i - b_{i-1}|$, for $i \geqslant 1$. If we convert $\Delta\beta$ into a lattice path by turning 0's into east steps and 1's into north steps, then it is easily checked that the binary sequences in Theorem 5 turn into lattice paths avoiding B_s, as defined below.

Definition 6. *For $s \geqslant 0$, define B_s to be the staircase path that starts at $(0, 2)$, then takes $s + 1$ steps east, 2 steps north, s steps east, 2 steps north, s steps east, and so on, alternating between 2 steps north and s steps east except for the first segment of $s + 1$ steps east.*

For example, B_4 is the dashed line in the lower picture in either Fig. 2 or Fig. 3 below. Curiously, we have not been able to generalize Theorem 5 to more general staircase boundaries, or to refine the count according to northwest or southeast corners. But for the special case when $s = 2k$ is even, we have a second, purely bijective proof of the following corollary of Theorem 5.

Corollary 7. *For all $n \geqslant 1$ and $k \geqslant 0$, the number of lattice paths of length $2(k+1)n+1$ that start at $(0, 0)$ and that avoid touching or crossing B_{2k} equals the number of lattice paths of length $2(k+1)n+1$ that start at $(0, 0)$ and that avoid touching or crossing the line $x = ky$ except at $(0, 0)$. This number has the explicit formula*

$$\binom{2(k+1)n}{2n} - (k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n}{i}. \tag{8}$$

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[2] In an earlier draft of this paper, we stated, "therefore our bijection does not specialize to Goulden and Serrano's rotation principle nor to André's reflection principle." There are many problems with this remark. First, we learned from Marc Renault [9], Heinrich Niederhausen, and Katherine Humphreys that André did *not* use a reflection argument, but actually interchanged ρ and σ! We also overstated the difference between rotation and interchange; if we rotate ρ, then rotate σ, and then rotate the entire path, then we have simply interchanged ρ and σ. So when there is a proof by one technique, there is probably a proof by the other. Finally, independently and almost simultaneously with Goulden and Serrano, Loehr [6] used a rotation argument for a similar lattice-path enumeration problem.
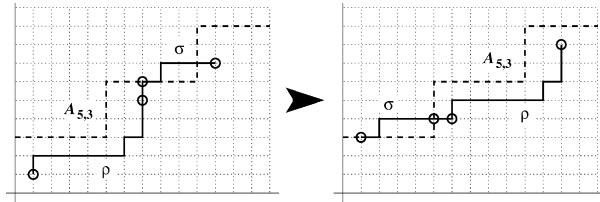
Fig. 1. Example of $\pi \mapsto \pi'$ with $s = 5$, $t = 3$, $n = 2$, and $\pi \in T_1 \cap U_2$.

Formula (8) is of course just obtained by summing over the appropriate instances of (1). These numbers also appear as A107027 in Sloane's Online Encyclopedia of Integer Sequences. This cries out for a combinatorial interpretation of each summand as counting lattice paths avoiding B_{2k} but with varying endpoints. Unfortunately, we do not know how to make this idea work.

Note that the case $s = 2$ of Theorem 5 is particularly simple:

Corollary 8. *For $n \geqslant 1$, there are $\binom{4n}{2n}$ binary sequences of length $4n + 1$ with the property that for all j, the jth occurrence of 10 appears in positions $4j + 1$ and $4j + 2$ or later (if it exists at all).*

We suspect that we have not yet found the "proof from the Book" of Corollary 8, and encourage the reader to find it.

The outstanding open question is whether our results generalize further. We should mention two papers [7] and [10] that consider staircase boundaries similar to $A_{s,t}$ and that prove results related to Corollary 4. See also Theorem 8.3 of [3]. Although our results do not seem to imply or be implied by these other results, perhaps it would be fruitful to investigate the precise relationships among them.

2. Proofs of Theorem 3 and Corollary 4

Proof of Theorem 3. We prove part (1) first. It will be convenient to first prove formula (5) bijectively, and then track corner counts through the bijection.

As we hinted above, we interpret (5) as counting the set T of *all* paths of a certain type, minus the set of *bad* paths. For $0 \leqslant i \leqslant t - 1$, let T_i be the set of all lattice paths from $(1, i)$ to $(sn + 1, tn + i)$, and let $T = \bigcup_i T_i$. Then

$$|T_i| = \binom{sn + tn}{tn} \quad \text{and} \quad |T| = t\binom{sn + tn}{tn}. \tag{9}$$

We regard S_1 as a subset of T as follows. Given any $\pi \in S_1$, find the smallest i such that $(1, i) \in \pi$; such an i must exist. Then there exists a unique $\pi' \in T_i$ that agrees exactly with the remainder of π, provided that we append i north steps to the end of π. Identifying π with π' embeds S_1 in T. It remains to show that the number of bad paths—i.e., the paths in $T \setminus S_1$—is $s\binom{sn+tn}{tn-1}$.

We partition the set $T \setminus S_1$ into s disjoint sets $U_1, \ldots, U_s$ as follows. By definition, every path in $T \setminus S_1$ must hit a bad point, i.e., a point on the boundary $A_{s,t}$. For $1 \leqslant j \leqslant s$, we let U_j be the set of all paths in $T \setminus S_1$ whose first bad point has an x-coordinate that is congruent to j modulo s. To prove formula (5), it suffices to show that $|U_j| = \binom{sn+tn}{tn-1}$, independent of j.

Fix any j. Given $\pi \in U_j$, observe that the step that terminates in the first bad point of π must be a north step. Let ρ be the portion of π prior to this fatal north step, and let σ be the portion of π after the bad point. Thus $\pi = (\rho, \text{north}, \sigma)$. Now comes the crucial part of the proof, where we interchange ρ and σ. More precisely, let π' be the lattice path that starts at $(j - 1, t)$ and takes steps $(\sigma, \text{east}, \rho)$. See Fig. 1 for an example.

We claim that $\pi \mapsto \pi'$ bijects U_j onto the set U_j' of all paths from $(j - 1, t)$ to $(sn + j, tn + t - 1)$. First note that since π and π' have the same total number of north steps and the same total number of east steps except that one north step of π has been changed into an east step of π', it follows that π' does in fact terminate at $(sn + j, tn + t - 1)$. Now, given any path $\pi' \in U_j'$, let σ be the initial

segment of π' up to the last point of π' that lies on the boundary $A_{s,t}$. The next step after that must be an east step; let ρ be the remainder of π' after that. It is straightforward to check that this allows us to construct a unique preimage π of π'. This proves formula (5).

Now we prove the stronger formula (3), for northwest corners. The initial north step of each lattice path in S_1^+ forces there to be a northwest corner with x-coordinate zero, whereas the final north step does not affect the northwest corner count. Therefore if we embed S_1 in T as above, we really want to count lattice paths with $c-1$ northwest corners (rather than c northwest corners). There are $t\binom{sn}{c-1}\binom{tn}{c-1}$ paths in T with $c-1$ northwest corners, because we can pick the x-coordinates and y-coordinates of the corners independently. It therefore suffices to show that for all j, there are $\binom{sn-1}{c-2}\binom{tn+1}{c}$ paths in U_j with $c-1$ northwest corners.

If α is a binary string, let $|\alpha|$ denote its length, and let $w(\alpha)$ denote its *weight*, i.e., the number of 1's in α. Let U_j'' be the set of ordered pairs (α, β) of binary strings such that $|\alpha| = sn-1$, $|\beta| = tn+1$, and $w(\beta) = w(\alpha) + 2$. It suffices to describe a bijection from U_j' to U_j'' such that the composite map $\pi \mapsto \pi' \mapsto (\alpha, \beta)$ sends paths with $c-1$ northwest corners to pairs (α, β) with $w(\beta) = c$.

Before describing this bijection, we make two observations. Let π, ρ, σ, and π' be as above. The first observation is that, because of the position of the endpoint of π relative to the boundary $A_{s,t}$, σ always has at least one east step. The second observation is that we lose a northwest corner when passing from π to π' iff σ starts with an east step, and we gain a northwest corner as we pass from π to π' iff σ ends with a north step. (Note that we can both gain a corner and lose a corner, leaving the total corner count unchanged.) So to track corners properly, we must watch the first and last steps of σ.

Now for the bijection. Given $\pi' \in U_j'$, construct α by first writing down a binary string of length sn whose ith digit ($1 \leqslant i \leqslant sn$) is 1 iff $j + i - 1$ is the x-coordinate of a northwest corner of π', and then deleting the digit corresponding to the point where π' intersects $A_{s,t}$ for the last time. This digit must exist, because σ has at least one east step. For example, in Fig. 1, we first write down 1000100001, and then delete the 4th digit to obtain $\alpha = 100100001$.

The first $tn-1$ digits of β are obtained by writing down the binary string of length $tn-1$ whose ith digit ($1 \leqslant i \leqslant tn-1$) is 1 iff $t+i$ is the y-coordinate of a northwest corner of π'. The next digit of β is 1 iff σ does not start with a north step, and the last digit of β is the complement of the deleted digit of α. For example, in Fig. 1, $\beta = 1110011$.

To see that $w(\beta) = w(\alpha) + 2$, first pair off the 1's in α and β arising from northwest corners that they both "see," and then note that β will have two extra 1's corresponding to the columns in which the first and last vertices of σ appear: Either β sees a northwest corner in that column (and α of course does not see it), or there is no such corner, in which case the appropriate trailing bit of β will be set. Either way, $w(\beta) = w(\alpha) + 2$.

Similarly, as we pass from π to π' to β, a corner that is lost from π to π' is "caught" by the penultimate bit of β, and β will gain an extra 1 either by catching a gained corner or, if no corner is gained, by setting its last bit. Thus $w(\beta)$ is one more than the number of northwest corners of π. Equivalently, $w(\alpha)$ is one *less* than the number of northwest corners of π.

It remains to show that $\pi' \mapsto (\alpha, \beta)$ is a bijection. Since $|U_j'| = |U_j''|$, it suffices to show that π' can be reconstructed from its image (α, β). To reconstruct π' it suffices to reconstruct the northwest corners. The penultimate digit of β is 0 iff π' has a northwest corner with x-coordinate $j-1$, so we need only reconstruct the deleted digit of α. The *value* of the deleted digit is the complement of the last digit of β, so we need only reconstruct its *position*. To do this, take (α, β) and begin constructing π' from the end backwards without regard to the deleted digit. At some point, the partially reconstructed path will touch or cross the boundary $A_{s,t}$. It is easy to check that the first such contact point with $A_{s,t}$ yields the position of the deleted digit of α.

This completes the proof of part (1). The proof of part (2) is very similar, so we focus only on the details that differ. For $0 \leqslant i \leqslant t-1$, let T_i be the set of all lattice paths from $(1, i)$ to $(sn, tn+i-1)$, and let $T = \bigcup_i T_i$. Then T is our set of *all* paths. Note that S_2 is already naturally a subset of T—in fact, $S_2 \subset T_0$—so we do not have to embed S_2 in T. The definition of the sets U_j is exactly analogous. However, π' now starts at $(j, t+1)$ rather than at $(j-1, t)$, and ends at $(sn+j, tn+t-1)$. The proof of (6) now goes through as before.

To do the corner count, we need to define the map $\pi' \mapsto (\alpha, \beta)$ in the case that σ is empty or vertical, i.e., has no east steps. In this case, we always delete the *first* digit of α. The definition of β is the same as before. The arguments that $w(\beta) = w(\alpha) + 2$ and that $\pi' \mapsto (\alpha, \beta)$ is a bijection still work.

However, $w(\alpha)$ is no longer always one less than the number of northwest corners of π. Let V denote the set of paths in $\bigcup_j U_j$ for which σ is vertical or empty and ρ starts with a horizontal step. Then it is straightforward to check that for $\pi \in V$, $w(\alpha)$ is *equal* to the number of northwest corners of π. So if we let X^c denote the members of X with c northwest corners, then pulling back U_j'' to U_j shows that formula (4) is the cardinality of the set

$$\left(T^{c-1} \setminus \bigcup_{j=1}^{s} U_j^{c-1} \right) \cup \left(V^{c-1} \setminus V^{c-2} \right) = S_2^{c-1} \cup \left(V^{c-1} \setminus V^{c-2} \right). \tag{10}$$

On the other hand, if we let N_2 denote the subset of S_2 consisting of paths that start with a north step, and observe that prepending a north step to $\pi \in S_2$ adds a northwest corner to π iff π starts with an east step, then we see that $(S_2^+)^c$ is equinumerous with

$$S_2^{c-1} \cup \left(N_2^c \setminus N_2^{c-1} \right). \tag{11}$$

Thus to show that (10) and (11) are equinumerous, it suffices to show that $N_2^c = V^{c-1}$ for any c. But this bijection is easily described: Given a path in N_2, simply move all the initial north steps to the end; this creates a path in V with one fewer northwest corner. This completes the proof.

Proof of Corollary 4. Of course this follows from Theorem 3, but we have another proof. The formula in Eq. (5) can be rewritten as $\frac{1}{n}\binom{sn+tn}{sn+1}$. Consider the set S_1' of all paths starting at the origin that end with a north step and that have a total of $sn + 1$ east steps and a total of tn north steps. Clearly $|S_1'| = \binom{sn+tn}{sn+1}$ and $S_1 \subset S_1'$. We need to show that S_1 comprises precisely $1/n$ of the paths in S_1'.

Decompose any path $\pi \in S_1'$ into n consecutive subpaths $\pi_1, \pi_2, \ldots, \pi_n$, where each π_j contains exactly t north steps and ends in a north step. Our desired result follows immediately from the following key claim: For *any* $\pi \in S_1'$, there is exactly one "cyclic shift" of π that lies in S_1, where by a cyclic shift we mean one of the n paths of the form

$$\pi_j, \pi_{j+1}, \ldots, \pi_n, \pi_1, \pi_2, \ldots, \pi_{j-1}$$

obtained from π by concatenating the subpaths in a cyclically permuted order.

To see the key claim, one first readily verifies that $\pi \in S_1$ iff for all $i > 0$, the total length of the first i subpaths $\pi_1, \ldots, \pi_i$ is at least $(s+t)i + 1$. Now we apply an argument patterned after a classic proof of Raney [8]. For all $j \geqslant 1$, let ℓ_j be the length of $\pi_{(j \bmod n)}$. Consider the graph in the xy plane with straight-line segments between vertices P_j and P_{j+1}, where

$$P_j = \left(j, \sum_{i=1}^{j} \ell_i \right).$$

The "average" slope of this graph is $(sn + tn + 1)/n = s + t + \frac{1}{n}$. The line of the form $y = (s+t+\frac{1}{n})x + C$ that is "tangent" to this graph from below intersects the graph exactly once every n points, because the graph has period n and the coefficient of x is an integer plus $1/n$. The points of intersection have the form P_j, P_{j+n}, P_{j+2n}, etc., and the value of j here yields the unique cyclic shift having the desired property. This proves the claim.

The proof of Eq. (6) is similar. Define S_2' to be the set of all paths from the origin that have a total of $sn - 1$ east steps and a total of $tn - 1$ north steps. Decompose any $\pi \in S_2'$ as follows:

$$\pi = \pi_1, \text{north}, \pi_2, \text{north}, \ldots, \text{north}, \pi_n,$$

where each π_j has $t - 1$ north steps. Then $\pi \in S_2$ iff for all $0 \leqslant i < n$ we have

$$|\pi_1| + |\pi_2| + \cdots + |\pi_i| \geqslant i(s + t - 1).$$

Exactly one "cyclic shift" of π has the equivalent property that for all i,

$$|\pi_1| + |\pi_2| + \cdots + |\pi_i| \geqslant (i/n)(sn + tn - n - 1).$$

Thus there are $\frac{1}{n}\binom{sn+tn-2}{tn-1}$ paths in S_2, which is equivalent to Eq. (6).

3. Proofs of Theorem 5 and Corollary 7

Proof of Theorem 5. It is easily verified that $a(1, s, 0) = 2$ and $a(1, s, 1) = a(1, s, 2) = s + 4$. If $n \geqslant 2$ and $r \leqslant 2n - 2$, then we claim that the following recursion holds:

$$a(n, s, r) = \sum_{d=0}^{s+2} \binom{s+2}{d} a(n-1, s, r-d). \tag{12}$$

The reason is that an admissible binary string of order $n-1$ can be extended by any sequence of $s+2$ bits without danger of causing inadmissibility, provided that the resulting string changes from 1 to 0 or vice versa at most $2n - 2$ times. The parameter d counts the number of changes introduced by the last $s + 2$ bits, and the binomial coefficient counts the number of ways to position the d changes.

By Vandermonde convolution, the recurrence (12) almost gives us a proof by induction on n, except that we need to handle the cases $r = 2n - 1$ and $r = 2n$. Note that no string of order n can have more than $2n - 1$ changes, and that Eq. (7) takes the same value for $r = 2n - 1$ and $r = 2n$. So to complete the proof of Theorem 5, it is enough to show that

$$a(n, s, 2n-1) - a(n, s, 2n-2) = 2\binom{(s+2)n - 1}{2n - 1} - s\binom{(s+2)n - 1}{2n - 2},$$

which can be rewritten as $\frac{1}{n}\binom{(s+2)n}{2n-1}$. The left-hand side counts the admissible strings with exactly $2n - 1$ changes, and we use the proof technique of Raney as before. Any such string σ must start with 0; we decompose it into substrings $\sigma_1, \sigma_2, \ldots, \sigma_n$, where each σ_j consists of a_j zeroes followed by b_j ones, and $a_j, b_j > 0$. The condition for admissibility can now be expressed as

$$|\sigma_1| + |\sigma_2| + \cdots + |\sigma_i| \geqslant i(s+2) + 1$$

for all $0 \leqslant i < n$. Exactly one cyclic shift of σ has the equivalent property that for all i,

$$|\sigma_1| + |\sigma_2| + \cdots + |\sigma_i| \geqslant (i/n)\big((s+2)n + 1\big).$$

Thus the number of admissible strings with exactly $2n - 1$ changes is equal to $1/n$ times the number of ways to partition $(s+2)n + 1$ into $2n$ positive integers, corresponding to the numbers a_j, b_j. This is well known to be $\binom{(s+2)n}{2n-1}$, and this completes the proof.

Proof of Corollary 7. We can deduce this easily from Theorem 5 just by showing that Eq. (7) reduces to Eq. (8) when $r = 2n$ and $s = 2k$. We have

$$a(n, 2k, 2n) = 2\binom{2(k+1)n - 1}{2n} - 2(k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n - 1}{i};$$

breaking up the sum, the right-hand side becomes

$$2\binom{2(k+1)n - 1}{2n} - (k-1)\binom{2(k+1)n - 1}{2n - 1}$$

$$- (k-1)\sum_{i=0}^{2n-1}\left[\binom{2(k+1)n - 1}{i} + \binom{2(k+1)n - 1}{i - 1}\right],$$

or

$$\left[2\frac{2kn}{2(k+1)n} - (k-1)\frac{2n}{2(k+1)n} \right]\binom{2(k+1)n}{2n} - (k-1)\sum_{i=0}^{2n-1}\binom{2(k+1)n}{i},$$

which then collapses to formula (8).

However, we also give a direct bijective proof. If $k = 0$ then formula (8) simplifies to 4^n, the boundary conditions are nearly vacuous, and the result is easy to prove. So fix $k \geqslant 1$ and $n \geqslant 1$.

Our bijection is actually between two sets of lattice paths that are slightly different from those mentioned in the corollary.

Let $\mathcal{F}_k(n)$ be the set of lattice paths of length $2(k+1)n$ (note the shorter length) that start at $(0,0)$ and avoid the line $x = ky$ except at $(0,0)$.

Let $\mathcal{G}_k(n)$ be the set of lattice paths of length $2(k+1)n + 1$ that start at $(0,0)$ and avoid B_{2k}, and that touch the line $x = ky + 1$ at least once for $y > 0$.

To see that a bijection between $\mathcal{F}_k(n)$ and $\mathcal{G}_k(n)$ implies the corollary, we make two observations. First, because $2(k+1)n$ is a multiple of $k+1$, every lattice path of length $2(k+1)n$ that avoids $x = ky$ can be extended by either an east step or a north step without hitting the line $x = ky$; therefore $|\mathcal{F}_k(n)|$ is exactly half the number of lattice paths of length $2(k+1)n + 1$ that avoid $x = ky$. Second, the paths excluded by the final condition on $\mathcal{G}_k(n)$ are precisely those that avoid the line $x = ky + 1$ after $(1,0)$, and therefore are in bijection with $\mathcal{F}_k(n)$−simply prepend an east step to each path in $\mathcal{F}_k(n)$.

The rest of the proof is devoted to describing a bijection $\varphi : \mathcal{F}_k(n) \to \mathcal{G}_k(n)$.

We define a procedure called *trisection* that we need in our construction of φ. Define the *potential* of a point (x, y) to be $x - ky$. Let P be a lattice path, not necessarily starting at $(0,0)$, but with the property that the potential difference of P−i.e., the potential of the last point of P minus the potential of the first point of P−is at least $k+1$ (and hence in particular comprises at least $k+1$ steps). To trisect P, first look at the last k steps of P. If all of these steps are east steps, then the trisection procedure fails. Otherwise, let b be the segment of P consisting of the last north step of P along with all the east steps after that. Let l be the length of b. Find the *last* lattice point $p \in P$ such that the initial segment a of P comprising everything up to p has potential difference exactly $k+1-l$. Such a point p must exist (since increases in potential can occur only one unit at a time) and must occur prior to b. Let P' be the segment of P between a and b. The decomposition $P = (a, P', b)$ is the trisection of P. Note that the potential difference of b is $l-1-k$ and so the combined potential difference of a and b is zero; thus the potential difference of P' is the same as that of P.

We are now ready to describe φ. Given $P \in \mathcal{F}_k(n)$, the construction of $\varphi(P)$ has two phases. In Phase 1, we decompose P into segments; in Phase 2, we build $\varphi(P)$ using the segments constructed in Phase 1.

The paths in $\mathcal{F}_k(n)$ with the smallest potential difference are those that terminate closest to the line $x = ky$; these are readily checked to have potential difference at least $k+1$. We begin Phase 1 by trying to trisect P into (a_1, P', b_1). If this fails, we proceed to Phase 2. Otherwise, if the *height* of a_1 (i.e., the y-coordinate of the last point of a_1 minus the y-coordinate of the first point of a_1) is even, then we proceed to Phase 2. Otherwise, we try to trisect P' into (a_2, P'', b_2), proceeding to Phase 2 if the trisection fails or if a_2 has even height. If we still do not reach Phase 2, then we try to trisect P'', and so on.

Each successful trisection preserves the potential difference of the middle section while shrinking its length, so we must eventually reach Phase 2, with a decomposition

$$P = (a_1, a_2, \ldots, a_{m-1}, a_m, Q, b_m, b_{m-1}, \ldots, b_2, b_1)$$

for some m, where Q denotes whatever remains in the middle. If we reach Phase 2 because the height of a_m is even, then we set

$$\varphi(P) = (\text{east}, a_m, b_m, a_1, b_1, a_2, b_2, a_3, b_3, \ldots, a_{m-1}, b_{m-1}, Q),$$

where the "east" means that we begin $\varphi(P)$ with an east step. For an example with $k = 2$ and $n = 7$, see Fig. 2.
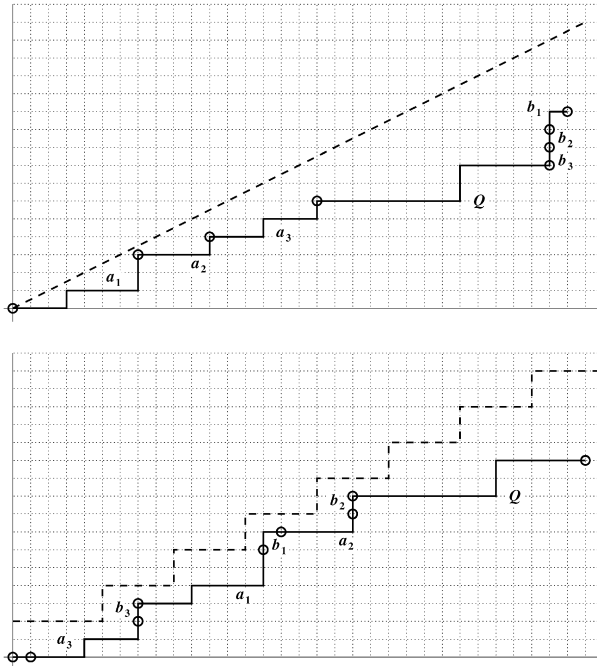
Fig. 2. Example of $P \mapsto \varphi(P)$ when a_m has even height.

The other way to reach Phase 2 is for the last k steps of Q to all be east steps. Decompose $Q = (Q', b_{m+1})$ where b_{m+1} comprises those final k east steps. Then set

$$\varphi(P) = (\text{north}, \text{east}, b_{m+1}, a_1, b_1, a_2, b_2, \ldots, a_m, b_m, Q').$$

For an example, again with $k = 2$ and $n = 7$, see Fig. 3.

One must check that $\varphi(P) \in \mathcal{G}_k(n)$. Define a *waypoint* to be a point on the line $x = ky + 1$ that is midway between two consecutive southeast corners of B_{2k} (i.e., it has the form $(2ik + k + 1, 2i + 1)$ for some i). The claim that $\varphi(P) \in \mathcal{G}_k(n)$ can be broken down into subclaims: (1) the part of $\varphi(P)$ preceding a_1 avoids B_{2k} and takes us to a waypoint; (2) each pair (a_1, b_1), (a_2, b_2), etc., resulting from successful trisections takes us from one waypoint to another and avoids B_{2k}; (3) the last part Q or Q' starts at a waypoint and avoids B_{2k}. Checking these subclaims is easier to do oneself than to write out in detail, so we will just indicate the key points. The claims about waypoints follow because an (a_i, b_i) pair has potential difference zero, and therefore if it starts on $x = ky$ then it ends on $x = ky$. If a_i has odd height then (a_i, b_i) ends on a waypoint if it starts on a waypoint. At most one a_i, namely a_m, has even height, and then (a_m, b_m) takes us from $(1, 0)$ to a waypoint. The other tricky claim is that (a_i, b_i) avoids B_{2k}, but this follows because by construction, the potential of a_i never drops below its initial potential so that it even stays below the line $x = ky + 1$; also, if we trace b_i backwards from its terminal waypoint, it takes at most $k - 1$ horizontal steps and therefore avoids hitting B_{2k}.

To invert φ, suppose we are given $P \in \mathcal{G}_k(n)$. Whether Fig. 2 or Fig. 3 applies depends on whether the first step of P is north or east. Mark all the waypoints of P; there must be at least one, since $P \in \mathcal{G}_k(n)$. By backing up from a waypoint until we find a north step, we can construct the b_i, and therefore also the a_i and Q. Hence φ is easily reversed. We leave the straightforward verification of the details to the reader.
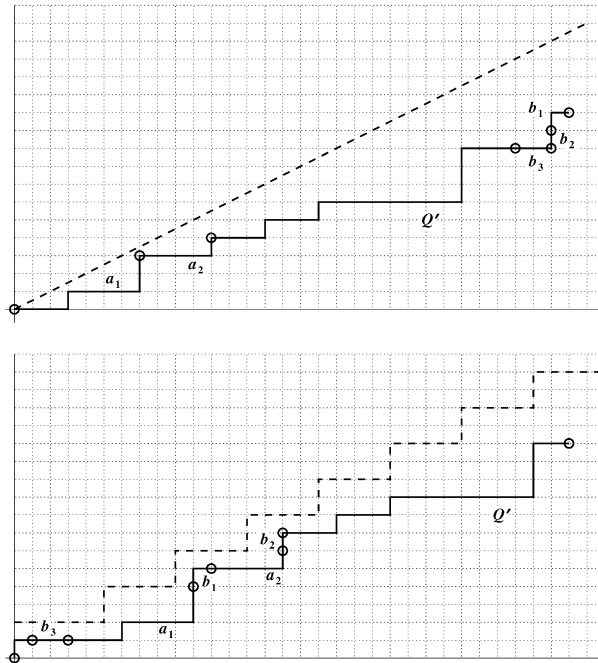
Fig. 3. Example of $P \mapsto \varphi(P)$ when b_{m+1} comprises k east steps.

Acknowledgments

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