Available online at www.sciencedirect.com
ScienceDirect
Theoretical Computer Science 396 (2008) 294–300
Theoretical Computer Science
www.elsevier.com/locate/tcs

Note

On the obfuscation complexity of planar graphs

Oleg Verbitsky

Institute for Applied Problems of Mechanics and Mathematics, Naukova St.3b, Lviv 79060, Ukraine

Received 31 May 2007; received in revised form 14 December 2007; accepted 14 February 2008

Communicated by G. Ausiello

Abstract

Being motivated by John Tantalo's Planarity Game, we consider straight line plane drawings of a planar graph G with edge crossings and wonder how obfuscated such drawings can be. We define $obf(G)$, the *obfuscation complexity* of G, to be the maximum number of edge crossings in a drawing of G. Relating $obf(G)$ to the distribution of vertex degrees in G, we show an efficient way of constructing a drawing of G with at least $obf(G)/3$ edge crossings. We prove bounds $(\delta(G)^2/24 - o(1))\, n^2 \le obf(G) < 3\, n^2$ for an n-vertex planar graph G with minimum vertex degree $\delta(G) \ge 2$.

The *shift complexity* of G, denoted by *shift*(G), is the minimum number of vertex shifts sufficient to eliminate all edge crossings in an arbitrarily obfuscated drawing of G (after shifting a vertex, all incident edges are supposed to be redrawn correspondingly). If $\delta(G) \ge 3$, then *shift*(G) is linear in the number of vertices due to the known fact that the matching number of G is linear. However, in the case $\delta(G) \ge 2$ we notice that *shift*(G) can be linear even if the matching number is bounded. As for computational complexity, we show that, given a drawing D of a planar graph, it is NP-hard to find an optimum sequence of shifts making D crossing-free.

Keywords: Combinatorial games; Planar graphs; Straight line drawings; Computational complexity

1. Introduction

This note is inspired by John Tantalo's Planarity Game [10] (another implementation is available at [13]). An instance of the game is a straight line drawing of a planar graph with many edge crossings. In a move the player is able to shift one vertex of the graph to a new position; the incident edges will be redrawn correspondingly. The objective is to achieve a crossing-free drawing in a possibly smaller number of moves.

Let us fix some relevant terminology. By a *drawing* we will always mean a straight line plane drawing of a graph where no vertex is an inner point of any edge. An *edge crossing* in a drawing D is a pair of edges having a common inner point. The number of edge crossings in D will be denoted by $obf(D)$. We define the *obfuscation complexity* of a graph G to be the maximum $obf(D)$ over all drawings D of G. This graph parameter will be denoted by $obf(G)$.

Given a drawing D of a planar graph G, let *shift*(D) denote the minimum number of vertex shifts making D crossing-free. The *shift complexity* of G, denoted by *shift*(G), is the maximum *shift*(D) over all drawings of G.

Our aim is a combinatorial and a complexity-theoretic analysis of the Planarity Game from the standpoint of a game designer. The latter should definitely have a library of planar graphs G with large *shift*(G). Generation of

E-mail address: verbitsk@informatik.hu-berlin.de.

0304-3975/$ - see front matter
doi:10.1016/j.tcs.2008.02.032

planar graphs with large $obf(G)$ is also of interest. Though large obfuscation complexity does not imply large shift complexity (see the discussion in Section 4), the designer can at least expect that a large $obf(D)$ will be a psychological obstacle for a player to play optimally on D.

A result of direct relevance to the topic is obtained by Pach and Tardos [8]. Somewhat surprisingly, they prove that even cycles have large shift complexity, namely, $n - O((n \log n)^{2/3}) \leq shift(C_n) \leq n - \lfloor \sqrt{n} \rfloor$.

We first address the obfuscation complexity. In Section 2 we relate this parameter of a graph to the distribution of its vertex degrees. This gives us an efficient way of constructing a drawing D of a given graph G so that $obf(D) \geq obf(G)/3$. As another consequence, we prove that $obf(G) \geq (\delta(G)^2/24 - o(1))n^2$ for an n-vertex planar graph with minimum vertex degree $\delta(G) \geq 2$. On the other hand, we prove an upper bound $obf(G) < 3n^2$. In Section 3 we discuss the relationship between the shift complexity of a planar graph and its matching number. We also show that the shift complexity of a drawing is NP-hard to compute. Section 4 contains concluding remarks and questions.

Related work. Investigation of the parameter $shift(G)$ is well motivated from a graph drawing perspective. Several results were obtained in this area independently of our work and appeared in [3,9,2] soon after the present note was submitted to the journal. The Planarity Game is also mentioned in [3,9] as a source of motivation.

Goaoc et al. [3] independently prove that computing $shift(D)$ for a given drawing D is an NP-hard problem, the same result as stated in our Theorem 8. They use a different reduction, allowing them to show that $shift(D)$ is even hard to approximate. Our reduction has another advantage: It shows that it is NP-hard to untangle even drawings of as simple graphs as matchings.

Spillner and Wolff [9] and Bose et al. [2] obtain general upper bounds for $shift(G)$, which quantitatively improve the classical Wagner–Fáry–Stein theorem (cf. Theorem 4 in Section 3). The stronger of their bounds [2] claims that $shift(G) \leq n - \sqrt[4]{n/9}$ for any planar G. Even better bounds are established for trees [3] and outerplanar graphs [9]. The series of papers [3,9,2] gives also lower bounds on the variant of $shift(G)$ for a broader notion of a "bad drawing".

Notation. We reserve n and m for, respectively, the number of vertices and the number of edges in a graph under consideration. We use the standard notation K_n, $K_{s,t}$, and C_n for, respectively, complete graphs, complete bipartite graphs, and cycles. The vertex set of a graph G will be denoted by $V(G)$. By kG we mean the disjoint union of k copies of G. The number of edges emanating from a vertex v is called the *degree* of v and denoted by $\deg v$. The *minimum degree* of a graph G is defined by $\delta(G) = \min_{v \in V(G)} \deg v$. A set of pairwise non-adjacent vertices (resp., edges) are called an *independent set* (resp., a *matching*). The maximum cardinality of an independent set (resp., a matching) in a graph G is denoted by $\alpha(G)$ (resp., $\nu(G)$) and called the *independence number* (resp., the *matching number*) of G. A graph is *k-connected* if it stays connected after removal of any $k - 1$ vertices.

2. Estimation of the obfuscation complexity

Note that $obf(G)$ is well defined for an arbitrary, not necessary planar graph G. As a warm-up, consider a few examples.

$obf(K_n) = \binom{n}{4}$. Indeed, let D be a drawing of K_n. $obf(D)$ is computable as follows. We start with the initial value 0 and, tracing through all pairs $\{e, e'\}$ of non-adjacent edges, increase it by 1 once e and e' cross. Consider the set S of 4 endpoints of e and e'. In fact, S corresponds to exactly 3 pairs of edges. If the convex hull of S is a triangle, then none of these three pairs is crossing. If it is a quadrangle, then 1 of the three pairs is crossing and 2 are not. It follows that $obf(D)$ does not exceed the number of all possible S. This upper bound is attained if every S has a quadrangular hull, for instance, if the vertices of D lie on a circle.

$obf(K_{s,t}) = \binom{s}{2}\binom{t}{2}$. The upper bound is provable by the same argument as above, where a 4-point set S has 2 points in the s-point part of $V(D)$ and 2 points in the t-point part. Such an S corresponds to 2 pairs of non-adjacent edges, at most 1 of which is crossing. This upper bound is attained if we put the two vertex parts of $K_{s,t}$ on two parallel lines.

$obf(C_n) = n(n-3)/2$ if n is odd. The value of $n(n-3)/2$ is attained by the n-pointed star drawing of C_n. This is the maximum by a simple observation: $n(n-3)/2$ is the total number of pairs of non-adjacent edges in C_n.

Let us state the upper bound argument we just used for the odd cycles in a general form. Given a graph G with m edges, let

$$\epsilon(G) = \binom{m}{2} - \sum_{v \in V(G)} \binom{\deg v}{2}.$$

Note that $\epsilon(G) = \frac{1}{2}(m(m+1) - \sum_v \deg^2 v)$, where the latter term is closely related to the variance of the vertex degrees. Since $\epsilon(G)$ is equal to the number of pairs of non-adjacent edges in G, we have $obf(G) \le \epsilon(G)$. Notice also a lower bound in terms of $\epsilon(G)$.

Theorem 1. *$\epsilon(G)/3 \le obf(G) \le \epsilon(G)$. Moreover, a drawing D of G with $obf(D) \ge \epsilon(G)/3$ is efficiently constructible.*

Proof. Fix an arbitrary n-point set V on a circle. We use the probabilistic method to prove that there is a drawing D with $V(D) = V$ having at least $\epsilon(G)/3$ edge crossings. Let $\mathbf{D}$ be a random straight line embedding of G with $V(\mathbf{D}) = V$, which is determined by a random map of $V(G)$ onto V. For each pair e, e' of non-adjacent vertices of G, we define a random variable $X_{e,e'}$ by $X_{e,e'} = 1$ if e and e' cross in $\mathbf{D}$ and $X_{e,e'} = 0$ otherwise. Let S be a 4-point subset of V. Under the condition that the set of endpoints of e and e' in $\mathbf{D}$ is S, these edges cross one another in $\mathbf{D}$ with probability $1/3$. It follows that $X_{e,e'} = 1$ with probability $1/3$. Note that $obf(\mathbf{D}) = \sum_{\{e,e'\}} X_{e,e'}$. By linearity of the expectation, we have $\mathbb{E}[obf(\mathbf{D})] = \sum_{\{e,e'\}} \mathbb{E}\left[X_{e,e'}\right] = \frac{1}{3}\epsilon(G)$ and hence $obf(D) \ge \frac{1}{3}\epsilon(G)$ for at least one instance D of $\mathbf{D}$. Such a D is efficiently constructible by standard derandomization techniques, namely, by the method of conditional expectations, see, e.g., [1, Chapter 15]. ■

As a consequence of Theorem 1, we have $obf(G) = \Theta(n^2)$ for a planar G whenever $\delta(G) \ge 2$ (the latter condition excludes the cases like $obf(K_{1,s}) = 0$). Indeed, $\epsilon(G) < \frac{9}{2}n^2$ because $m < 3n$ for any planar graph. This bound is sharp in the sense that $\epsilon(G) \ge \frac{9}{2}n^2 - O(n)$ for maximal planar graphs of bounded vertex degree. A sharp lower bound for $\epsilon(G)$ is stated below.

Theorem 2. *$\epsilon(G) \ge \left(\frac{\delta(G)^2}{8} - o(1)\right)n^2$ for a planar graph G with $\delta(G) \ge 2$. The constant $\delta(G)^2/8$ cannot be better here.*

Proof. Let $A_k(G) = \{v \in V(G) : \deg v < k\}$ and denote

$$a_k(G) = |A_k(G)| \quad \text{and} \quad s_k(G) = \sum_{v \in V(G) \setminus A_k(G)} \deg v.$$

West and Will [12] prove that, if $k \ge 12$, then for every planar G on $n \ge \frac{3}{2}k - 1$ vertices we have

$$a_k(G) \ge \frac{(k-8)n + 16}{k-6}$$

and

$$s_k(G) < 2n - 16 + \frac{12(n-8)}{k-6}.$$

We begin with the bound

$$\epsilon(G) > \frac{1}{2}\left(m^2 - \sum_{v \in V(G)} \deg^2 v\right).$$

Set $\delta = \delta(G)$. Let $\sigma = s_k(G)/n$ (to simplify the notation, we do not indicate the dependence of σ on k). Suppose that k is large enough, namely, $k \ge 14$. Note that $0 \le \sigma < 2 + 12/(k-6)$. We now estimate m from below and $\sum_v \deg^2 v$ from above.

$$\begin{aligned} m &= \frac{1}{2}\sum_v \deg v = \frac{1}{2}\left(\sum_{v \in A_k(G)} \deg v + \sum_{v \notin A_k(G)} \deg v\right) \\ &\ge \frac{1}{2}\left(\delta(G)a_k(G) + s_k(G)\right) > \frac{1}{2}\left(\frac{\delta(k-8)}{k-6} + \sigma\right)n. \end{aligned}$$

Furthermore,

$$\sum_v \deg^2 v = \sum_{v \in A_k(G)} \deg^2 v + \sum_{v \notin A_k(G)} \deg^2 v < (k-1)^2 n + f(\sigma) n^2,$$

where

$$f(\sigma) = \begin{cases} 2 + (\sigma - 2)^2 & \text{if } 2 \le \sigma < 2 + 12/(k-6), \\ 1 + (\sigma - 1)^2 & \text{if } 1 \le \sigma < 2, \\ \sigma^2 & \text{if } 0 \le \sigma < 1. \end{cases}$$

Thus,

$$\epsilon(G) > g(\sigma)\, n^2 - \frac{(k-1)^2}{2} n, \quad \text{where } g(\sigma) = \frac{1}{2}\left(\frac{1}{4}\left(\frac{\delta(k-8)}{k-6} + \sigma\right)^2 - f(\sigma)\right).$$

A routine calculation shows that

$$\min\left\{ g(\sigma) : 0 \le \sigma < 2 + \frac{12}{k-6} \right\} = g(0) = \frac{\delta^2}{8}\left(\frac{k-8}{k-6}\right)^2.$$

We conclude that

$$\epsilon(G) > \frac{\delta^2}{8}\left(\frac{k-8}{k-6}\right)^2 n^2 - \frac{(k-1)^2}{2} n > \left(\frac{\delta^2}{8} - \frac{\delta^2}{2(k-6)} - \frac{(k-1)^2}{2n}\right) n^2$$

whenever $k \ge 14$ and $n \ge \frac{3}{2}k - 1$. Recall that $\delta(G) \le 5$ for any planar G. If we make k a function of n that grows to the infinity slower than $\sqrt{n}$, then the factor in front of n^2 becomes $\delta^2/8 - o(1)$ and we arrive at the claimed bound.

The optimality of the constant $\delta^2/8$ is ensured by regular planar graphs (i.e., cycles and cubic, quartic, and quintic planar graphs). ■

As was already mentioned, for planar graphs we have $obf(G) \le \epsilon(G) < \frac{9}{2} n^2$, where the bound for $\epsilon(G)$ cannot be improved. However, for $obf(G)$ we can do somewhat better.

Theorem 3. *$obf(G) < 3\, n^2$ for a planar graph G on n vertices.*

Proof. Note that, if K is a subgraph of H, then $obf(K) \le obf(H)$. It therefore suffices to prove the theorem for the case that G is a maximal planar graph, that is, a triangulation. Let E be a (crossing-free, not necessary straight line) plane embedding of G. Denote the number of triangular faces in E by t and note that $3t = 2m$. Based only on facial triangles, let us estimate from below the number of non-crossing edge pairs in an arbitrary straight line drawing D of G. Let P denote the set of all pairs of adjacent edges occurring in facial triangles. Here we have $|P| = 3t$ edge pairs which are non-crossing in D. Furthermore, for each pair of edge-disjoint facial triangles $\{T, T'\}$ we take into account pairs of non-crossing edges $\{e, e'\}$ with e from T and e' from T'. Since at most $3t/2$ pairs of facial triangles can share an edge, there are at least $\binom{t}{2} - \frac{3t}{2}$ such $\{T, T'\}$. We split this amount into two parts. Let A consist of vertex-disjoint $\{T, T'\}$ and B consist of $\{T, T'\}$ sharing one vertex. As easily seen, every $\{T, T'\}$ in A gives us at least 3 edge pairs $\{e, e'\}$ which are non-crossing in D. Every $\{T, T'\}$ in B contributes at least 2 pairs of non-adjacent edges and exactly 4 pairs of adjacent edges. However, 2 of the latter 4 edge pairs can participate in P. We conclude that in D there are at least $|P| + (3|A| + 4|B|)/4$ non-crossing edge pairs. The factor of $1/4$ in the latter term is needed because an edge pair $\{e, e'\}$ can be contributed by 4 triangle pairs $\{T, T'\}$. Thus,

$$obf(D) \le \binom{m}{2} - 3t - \frac{3}{4}\left(\binom{t}{2} - \frac{3t}{2}\right) < \frac{1}{2} m^2 - \frac{3}{8} t^2 = \frac{1}{3} m^2.$$

Since $m < 3n$ as a simple consequence of Euler's formula, we have $obf(D) < 3n^2$. As D is arbitrary, the bound for $obf(G)$ follows. ■

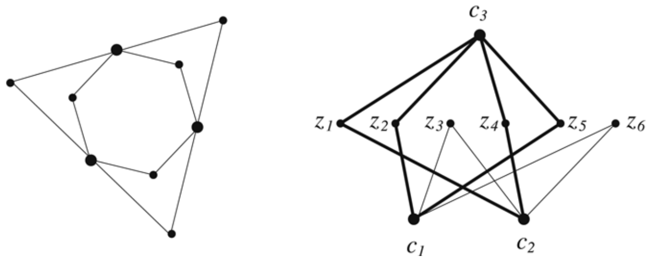

Fig. 1. G_2 and F in D_2.

3. Estimation of the shift complexity

A basic fact about *shift*(G) is that this number is well defined.

Theorem 4 (*Wagner, Fáry, Stein (see, e.g., [6])*). *Every planar graph G has a straight line plane drawing. In other words, shift*$(G) \leq n-3$ *if* $n \geq 3$.

If we seek for lower bounds, the following example is instructive despite its simplicity: *shift*$(mK_2) = m-1$. It immediately follows that

$$\textit{shift}(G) \geq \nu(G) - 1.$$

Theorem 5. *Let G be a connected planar graph on n vertices.*

1. *If* $\delta(G) \geq 3$ *(in particular, if G is 3-connected) and* $n \geq 10$, *then shift*$(G) \geq (n-1)/3$.
2. *If G is* 4*-connected, then shift*$(G) \geq (n-3)/2$.
3. *There is an infinite family of connected planar graphs G with* $\delta(G) = 2$ *and shift*$(G) \leq 2$.

Proof. Item 1 follows from the fact that, under the stated conditions on G, we have $\nu(G) \geq (n+2)/3$ (Nishizeki–Baybars [5]). Item 2 is true because every 4-connected planar G is Hamiltonian (Tutte [11]) and hence $\nu(G) \geq (n-1)/2$ in this case. Item 3 is due to the bound *shift*$(K_{2,s}) \leq 2$. The latter follows from the elementary fact of plane geometry stated in Lemma 6 below. ■

Lemma 6. *For any finite set of points Z there are two points x and y such that the segments with one endpoint in* $\{x, y\}$ *and the other in Z do not cross each other and have no inner points in Z.*

Proof. Let L denote the set of all lines going through at least two points in Z. Fix the direction "upward" not in parallel to any line in L. Pick up x above every line in L and y below every line in L. ■

The next question we address is this: How close is the relationship between *shift*(G) and $\nu(G)$? By Theorem 5, if $\delta(G) \geq 3$ then both graph parameters are linear. However, if $\delta(G) \leq 2$, the existence of a large matching is not the only cause of large shift complexity.

Theorem 7. *There is a planar graph* G_s *on* $3s+3$ *vertices with* $\delta(G_s) = 2$ *such that* $\nu(G_s) = 3$ *and shift*$(G_s) \geq 2s-6$.

Proof. A suitable G_s can be obtained as follows: take the multigraph which is a triangle with multiplicity of every edge s and make it a graph by inserting a new vertex in each of the $3s$ edges (see Fig. 1). Using Lemma 6, it is not hard to show that *shift*$(G_s) \leq 2s+3$. We now construct a drawing D_s of G_s with *shift*$(D_s) \geq 2s-6$. Put vertices $z_1, \ldots, z_{3s}$ in this order in a line and the remaining vertices c_0, c_1, c_2 somewhere else in the plane. Connect z_i with c_j iff $j \neq i$ mod 3. Therewith D_s is specified. Denote the fragment of D_s induced on $\{z_1, z_2, z_4, z_5, c_0, c_1, c_2\}$ by F. It is not hard to see that F cannot be disentangled by moving only c_0, c_1, and c_2. In fact, if in place of z_1, z_2, z_4, z_5 we take any quadruple z_i, z_j, z_k, z_l with $i < j < k < l$, $i \equiv k \pmod 3$, and $j \equiv l \pmod 3$, this will give us a fragment completely similar to F. To destroy all such fragments, we need to move at least two vertices in every triple $z_{3h+1}, z_{3h+2}, z_{3h+3}$ $(0 \leq h < s)$ with possible exception for at most 3 of them. Therefore, making $2(s-3)$ shifts is unavoidable. ■

Finally, we prove a complexity result.

Theorem 8. *Computing the shift complexity of a given drawing is an NP-hard problem.*

Proof. In fact, this hardness result is true even for drawings of graphs mK_2. Given such a drawing D, consider its intersection graph S_D whose vertices are the edges of D with e and e' adjacent in S_D iff they cross one another in D. Since computing the independence number of intersection graphs of segments in the plane is known to be NP-hard (Kratochvíl–Nešetřil [4]), it suffices for us to express $\alpha(S_D)$ as a simple function of $shift(D)$. Fix an optimal way of untangling D and denote the set of edges whose position was not changed by E. Clearly, E is an independent set in S_D and hence $shift(D) \geq m - |E| \geq m - \alpha(S_D)$. On the other hand, $shift(D) \leq m - \alpha(S_D)$. Indeed, fix an independent set I in S_D of the maximum size $\alpha(S_D)$. Then D can be untangled this way: we leave the edges in I unchanged and shrink each edge not in I by shifting one endpoint sufficiently close to the other endpoint. Thus, $\alpha(S_D) = m - shift(D)$, as desired. ■

4. Concluding remarks and problems

1. By Theorem 1 we have $\frac{1}{3}\epsilon(G) \leq obf(G) \leq \epsilon(G)$. The upper bound cannot be improved in general as $obf(C_n) = \epsilon(C_n)$ for odd n. Can one improve the factor of $\frac{1}{3}$ in the lower bound?
2. By Theorems 1–3 we have $(\delta(G)^2/24 - o(1))n^2 \leq obf(G) \leq 3n^2$ where $\delta(G) \geq 2$ is necessary for the lower bound. Optimize the factors in the left- and the right-hand sides.
3. As follows from the proof of Theorem 1, there is an n-point set V (in fact, this can be an arbitrary set on the border of a convex body) with the following property: Every graph G of order n has a drawing D with $V(D) = V$ such that $obf(D) \geq \frac{1}{3}obf(G)$. Can this uniformity result be strengthened? Is there an n-point set V on which one can attain $obf(D) = obf(G)$ for all n-vertex G?
4. The following remarks show that the obfuscation and the shift complexity of a drawing have, in general, rather independent behavior.

   Maximum $obf(D)$ does not imply maximum $shift(D)$. Consider $3K_{1,s}$, the union of 3 disjoint copies of the s-star. It is not hard to imagine how a drawing attaining $obf(3K_{1,s}) = 3s^2$ should look (where every two non-adjacent edges cross) and it becomes clear that such a drawing can be untangled just by 2 shifts. However, $shift(3K_{1,s}) \geq s$ is provable similarly to Theorem 7 (an upper bound $shift(3K_{1,s}) \leq s + 2$ follows from Lemma 6).

   Maximum $shift(D)$ does not imply maximum $obf(D)$. The simplest example is given by a drawing of the disjoint union of K_2 and $K_{1,2}$ with only one edge crossing.

   Large $obf(D)$ does not imply large $shift(D)$. This can be shown by drawings of $obf(K_{2,s})$. Indeed, we know that $obf(K_{2,s}) = \binom{s}{2}$ from Section 2 and $shift(K_{2,s}) \leq 2$ from Section 3 (the latter bound is exact if $s \geq 4$).

   Large $shift(D)$ does not imply large $obf(D)$. Pach and Tardos [8, Fig. 2] show a drawing D of the cycle C_n with linear $shift(D)$ and $obf(D) = 1$.
5. In spite of the observation we just made that large $obf(D)$ does not imply large $shift(D)$, in some interesting cases it does. Pach and Solymosi [7] prove that every system S of m segments in the plane with $\Omega(m^2)$ crossings has two disjoint subsystems S_1 and S_2 with both $|S_1| = \Omega(m)$ and $|S_2| = \Omega(m)$ such that every segment in S_1 crosses all segments in S_2. As $shift(S) \geq \min\{|S_1|, |S_2|\}$, this result has an interesting consequence: If D is a drawing of mK_2 with $obf(D) = \Omega(m^2)$, then $shift(D) = \Omega(m)$.
6. Theorem 8 shows that computing $shift(D)$ for a drawing D of a graph G can be hard even in the cases when computing $shift(G)$ is easy. Is $shift(G)$ hard to compute in general? Theorem 1 shows that $obf(G)$ is polynomial-time approximable within a factor of 3. Is the exact computation of $obf(G)$ NP-hard (Amin Coja-Oghlan)?

Acknowledgments

The author is thankful to the members of the 'Algorithms and Complexity' group at the Humboldt University of Berlin and to Taras Banakh for helpful discussions. He also thanks Sasha Ravsky for a simplification of the proof of Theorem 3 and an anonymous referee for useful comments, in particular, for drawing his attention to the recent work done in [3,9,2]. The author was supported by an Alexander von Humboldt return fellowship.

References

[1] N. Alon, J. Spencer, The Probabilistic Method, John Wiley & Sons, 1992.
[2] P. Bose, V. Dujmovic, F. Hurtado, S. Langerman, P. Morin, D.R. Wood, A polynomial bound for untangling geometric planar graphs. E-print: http://arxiv.org/abs/0710.1641, October 2007.
[3] X. Goaoc, J. Kratochvíl, Y. Okamoto, C.S. Shin, A. Wolff, Moving vertices to make a drawing plane, in: Proc. of the 15-th International Symposium Graph Drawing, in: Lecture Notes in Computer Science, vol. 4875, Springer-Verlag, 2007, pp. 101–112.
[4] J. Kratochvíl, J. Nešetřil, Independent set and clique problems in intersection-defined classes of graphs, Comment. Math. Univ. Carolin. 31 (1) (1990) 85–93.
[5] T. Nishizeki, I. Baybars, Lower bounds on the cardinality of the maximum matchings of planar graphs, Discrete Math. 28 (3) (1979) 255–267.
[6] T. Nishizeki, Md.S. Rahman, Planar Graph Drawing, World Scientific, 2004.
[7] J. Pach, J. Solymosi, Crossing patterns of segments, J. Combin. Theory, Ser. A 96 (2) (2001) 316–325.
[8] J. Pach, G. Tardos, Untangling a polygon, Discrete Comput. Geom. 28 (2002) 585–592.
[9] A. Spillner, A. Wolff, Untangling a planar graph, in: Proc. of the 34-th International Conference on Current Trends Theory and Practice of Computer Science, in: Lecture Notes in Computer Science, vol. 4910, Springer-Verlag, 2008, pp. 473–484.
[10] J. Tantalo, Planarity game. http://www.planarity.net/.
[11] W.T. Tutte, A theorem on planar graphs, Trans. Amer. Math. Soc. 82 (1956) 99–116.
[12] D.B. West, T.G. Will, Vertex degrees in planar graphs, in: DIMACS Series in Discrete Mathematics and Theoretical Computer Science, vol. 9, 1993, pp. 139–149.
[13] Lange Nacht der Wissenschaften 2006, Humboldt Universität zu Berlin. http://www2.informatik.hu-berlin.de/alcox/lndw/planar.html.