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Matrix representations of Sturm–Liouville problems with transmission conditions

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ABSTRACT

We identify a class of Sturm–Liouville equations with transmission conditions such that any Sturm–Liouville problem consisting of such an equation with transmission condition and an arbitrary separated or real coupled self-adjoint boundary condition has a representation as an equivalent finite dimensional matrix eigenvalue problem. Conversely, given any matrix eigenvalue problem of certain type and an arbitrary separated or real coupled self-adjoint boundary condition and transmission condition, we construct a class of Sturm–Liouville problems with this specified boundary condition and transmission condition, each of which is equivalent to the given matrix eigenvalue problem.

1. Introduction

In this paper we explore relationship between regular self-adjoint Sturm–Liouville problems with transmission conditions of Atkinson type and matrix eigenvalue problems in the form

$$VX = \lambda WX, \tag{1.1}$$

where V and W are $l \times l\,(l \in \mathbb{Z}, l > 3)$ matrices over the reals $\mathbb{R}$ and W is diagonal.

In [1], the authors Kong et al. considered the relationship between a class of Sturm–Liouville problems of Atkinson type and matrix eigenvalue problems and show that, the eigenvalue problem of this class of Sturm–Liouville problems with arbitrary self-adjoint boundary conditions (either separated or coupled) are equivalent to the matrix eigenvalue problems. Such a connection can be used to “transfer” results from one problem to the other. For the corresponding applications for S–L case please see [2,3]. Recently, we prove that the Sturm–Liouville problems(SLPs) with transmission conditions also have a finite spectrum, see [4].

Following [5,1,4], in this paper, we consider the Sturm–Liouville equation

$$-(py')' + qy = \lambda wy, \quad \text{on } J = (a, c) \cup (c, b),\ c \in (a, b), \quad \text{with } -\infty < a < b < +\infty \tag{1.2}$$

together with boundary conditions of the form

$$AY(a) + BY(b) = 0, \qquad Y = \begin{bmatrix} y \\ py' \end{bmatrix}, \quad A, B \in M_2(\mathbb{C}), \tag{1.3}$$

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and the transmission conditions

$$CY(c-) + DY(c+) = 0, \tag{1.4}$$

where c is inner discontinuity point, λ is the spectral parameter, and $C = (c_{ij})_{2\times 2}, D = (d_{ij})_{2\times 2}$ are real valued 2×2 matrices satisfying $det(C) = \rho > 0, det(D) = \theta > 0$. $M_2(\mathbb{C})$ denotes the set of square matrices of order 2 over $\mathbb{C}$. The coefficients satisfy the conditions

$$r = 1/p, q, w \in L(J, \mathbb{R}), \tag{1.5}$$

where $L(J, \mathbb{R})$ denotes the real valued functions which are Lebesgue integrable on J [6–8].

The BC (1.3) is said to be self-adjoint if the following two conditions are satisfied:

$$rank(A, B) = 2, \qquad AEA^* = BEB^* \quad \text{with } E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.6}$$

The SLP with transmission condition (1.2)–(1.4) is said to be self-adjoint if its associated operator T is self-adjoint. The operator T is self-adjoint if and only if

$$rank(A, B) = 2, \qquad \frac{1}{\rho}AE^{-1}A^* = \frac{1}{\theta}BE^{-1}B^* \quad \text{with } E = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{1.7}$$

For details on the self-adjointness of SLPs with transmission conditions please see [9,10].

It is well known [11] that under the condition (1.5), the BCs (1.3) fall into two disjoint classes: separated and coupled. The separated boundary conditions have the canonical representation:

$$\begin{aligned} &\cos\alpha y(a) - \sin\alpha(py')(a) = 0, \quad 0 \le \alpha < \pi \\ &\cos\beta y(b) - \sin\beta(py')(b) = 0, \quad 0 < \beta \le \pi. \end{aligned} \tag{1.8}$$

The real coupled boundary conditions have the canonical representation:

$$Y(b) = KY(a) \quad \text{with } K = (k_{ij}),\ k_{ij} \in \mathbb{R},\ 1 \le i, j \le 2, \qquad det(K) = 1. \tag{1.9}$$

Let $u = y$ and $v = (py')$. Then we have the system representation of equation (1.2) [7]:

$$u' = rv, \qquad v' = (q - \lambda w)u, \quad \text{on } J. \tag{1.10}$$

In the classical case when r and w in (1.2) are positive a.e. on J and the BCs (1.3) and transmission conditions (1.4) are self adjoint, SLPs with transmission conditions (1.2)–(1.4) have a discrete spectrum consisting of an infinite number of real eigenvalues [5,9,10,12]. In our recent works, we proved SLPs with transmission conditions of Atkinson type have a finite number of eigenvalues. Hence, in this paper, we identify this class of Sturm–Liouville equations with transmission conditions of Atkinson type, and show that, given any member of this class, and an arbitrary self-adjoint BC (1.3), either separated or coupled, and proper transmission condition (1.4), there is a matrix eigenvalue problem in the form of (1.1) with exactly the same eigenvalues as the corresponding SLP with transmission condition (1.2)–(1.4). Conversely, given a matrix eigenvalue problem (1.1) satisfying appropriate conditions, there exist SLPs with arbitrary self-adjoint separated or real coupled BC and transmission condition having exactly the same eigenvalues as the matrix problem. This reveals a connection between the SLPs with transmission conditions of Atkinson type and certain matrix eigenvalue problems. Such a connection can be used to 'transfer' results from one problem to the other as illustrated below.

2. Matrix representations of SLPs with transmission conditions of Atkinson type

Following [1,4,13] we still associate a special class of SLPs with transmission conditions with the name of Atkinson.

Definition 2.1. A Sturm–Liouville equation (1.2) with transmission condition is said to be of Atkinson type if, for some positive integers $m, n \ge 1$, there exists a partition of the interval J

$$a = a_0 < a_1 < a_2 < \cdots < a_{2m+1} = c, \qquad c = b_0 < b_1 < b_2 < \cdots < b_{2n+1} = b \tag{2.1}$$

such that

$$\begin{aligned} &r = \frac{1}{p} = 0 \quad \text{on } [a_{2k}, a_{2k+1}], \quad k = 0, 1, \ldots, m-1, \quad \text{and} \quad [a_{2m}, a_{2m+1}), \qquad \int_{a_{2k}}^{a_{2k+1}} w \ne 0, \quad k = 0, 1, \ldots, m, \\ &r = \frac{1}{p} = 0 \quad \text{on } (b_0, b_1] \quad \text{and} \quad [b_{2i}, b_{2i+1}], i = 1, 2, \ldots, n, \qquad \int_{b_{2i}}^{b_{2i+1}} w \ne 0, \quad i = 0, 1, \ldots, n; \end{aligned} \tag{2.2}$$

and

$$
\begin{aligned}
&q = 0 = w \quad \text{on } [a_{2k+1}, a_{2k+2}], \qquad \int_{a_{2k+1}}^{a_{2k+2}} r \neq 0, \quad k = 0, 1, \ldots, m-1, \\
&q = 0 = w \quad \text{on } [b_{2i+1}, b_{2i+2}], \qquad \int_{b_{2i+1}}^{b_{2i+2}} r \neq 0, \quad i = 0, 1, \ldots, n-1.
\end{aligned} \tag{2.3}
$$

In this section, we construct matrix eigenvalue problems which have exactly the same eigenvalues as the corresponding SLPs with transmission conditions of Atkinson type.

Definition 2.2. A SLP with a transmission condition of Atkinson type is said to be equivalent to a matrix eigenvalue problem if the former has exactly the same eigenvalues as the latter.

Before we can state our theorems we need to introduce some additional notation. Given (2.1)–(2.3), let

$$
\begin{aligned}
&p_k = \left(\int_{a_{2k-1}}^{a_{2k}} r\right)^{-1}, \quad k = 1, 2, \ldots, m; \qquad q_k = \int_{a_{2k}}^{a_{2k+1}} q, \qquad w_k = \int_{a_{2k}}^{a_{2k+1}} w, \quad k = 0, 1, \ldots, m; \\
&\tilde{p}_i = \left(\int_{b_{2i-1}}^{b_{2i}} r\right)^{-1}, \quad i = 1, 2, \ldots, n; \qquad \tilde{q}_i = \int_{b_{2i}}^{b_{2i+1}} q, \qquad \tilde{w}_i = \int_{b_{2i}}^{b_{2i+1}} w, \quad i = 0, 1, \ldots, n.
\end{aligned} \tag{2.4}
$$

We note from (2.2) and (2.3) that $p_k, w_k, \tilde{p}_i, \tilde{w}_i \in \mathbb{R} \setminus \{0\}$, and no sign restrictions are imposed on them.

From (2.2) and (2.3) we see that, for any solution u, v of (1.10), u is constant on the intervals where r is identically zero and v is constant on the intervals where q and w are both identically zero. Let

$$
\begin{aligned}
&u_k = u(x), \quad x \in [a_{2k}, a_{2k+1}],\ k = 0, \ldots, m-1, \qquad u_m = u(x), \quad x \in [a_{2m}, a_{2m+1}), \\
&\tilde{u}_0 = u(x), \quad x \in (b_0, b_1], \qquad \tilde{u}_i = u(x), \quad x \in [b_{2i}, b_{2i+1}],\ i = 1, \ldots, n; \\
&v_k = v(x), \quad x \in [a_{2k-1}, a_{2k}],\ k = 1, \ldots, m, \qquad \tilde{v}_i = v(x), \quad x \in [b_{2i-1}, b_{2i}],\ i = 1, \ldots, n,
\end{aligned} \tag{2.5}
$$

and set

$$
\begin{aligned}
&v_0 = v(a_0) = v(a), \qquad \tilde{v}_{n+1} = v(b_{2n+1}) = v(b), \\
&v_{m+1} = v(a_{2m+1}-) = v(c-), \qquad \tilde{v}_0 = v(b_0+) = v(c+).
\end{aligned} \tag{2.6}
$$

Lemma 2.1. *Assume Eq.* (1.2) *is of Atkinson type. Then for any solution* u, v *of Eq.* (1.10) *we have*

$$p_k(u_k - u_{k-1}) = v_k, \quad k = 1, 2, \ldots, m, \tag{2.7}$$

$$v_{k+1} - v_k = u_k(q_k - \lambda w_k), \quad k = 0, 1, \ldots, m, \tag{2.8}$$

$$\tilde{p}_i(\tilde{u}_i - \tilde{u}_{i-1}) = \tilde{v}_i, \quad i = 1, 2, \ldots, n, \tag{2.9}$$

$$\tilde{v}_{i+1} - \tilde{v}_i = \tilde{u}_i(\tilde{q}_i - \lambda \tilde{w}_i), \quad i = 0, 1, \ldots, n. \tag{2.10}$$

Conversely, for any solution u_k, $k = 0, 1, \ldots, m$, v_k, $k = 0, 1, \ldots, m+1$ *and* $\tilde{u}_i$, $i = 0, 1, \ldots, n$, $\tilde{v}_i$, $i = 0, 1, \ldots, n+1$, *of system* (2.7)–(2.10), *there is a unique solution* $u(x)$ *and* $v(x)$ *of Eq.* (1.10) *satisfying* (2.5) *and* (2.6).

Proof. From the first equation of (1.10), for $k = 1, 2, \ldots, m$, we have

$$u_k - u_{k-1} = u(a_{2k}) - u(a_{2k-2}) = \int_{a_{2k-2}}^{a_{2k}} u' = \int_{a_{2k-2}}^{a_{2k}} rv = \int_{a_{2k-1}}^{a_{2k}} rv = v_k \int_{a_{2k-1}}^{a_{2k}} r = v_k/p_k.$$

This establishes (2.7).

Similarly, from second equation of (1.10), for $k = 1, 2, \ldots, m$, we have

$$
\begin{aligned}
v_{k+1} - v_k &= v(a_{2k+1}) - v(a_{2k-1}) = \int_{a_{2k-1}}^{a_{2k+1}} v' = \int_{a_{2k-1}}^{a_{2k+1}} (q - \lambda w)u \\
&= \int_{a_{2k}}^{a_{2k+1}} (q - \lambda w)u = u_k \int_{a_{2k}}^{a_{2k+1}} (q - \lambda w) = u_k(q_k - \lambda w_k),
\end{aligned}
$$

and for $k = 0$, we still have the same result. This establishes (2.8).

(2.9) and (2.10) can be proved in the same way.

On the other hand, if u_k, v_k satisfy (2.7) and (2.8), $\tilde{u}_i$, $\tilde{v}_i$ satisfy (2.9) and (2.10), then we define $u(x)$ and $v(x)$ according to (2.5) and (2.6), and then extend them continuously to the whole interval J as a solution of (1.10) by integrating the equations in (1.10) over subintervals. □

To discuss the matrix representation of the Sturm–Liouville equation with transmission condition (1.4) and BC (1.3), let $G = (g_{ij}) = -D^{-1}C$ and for each of the separated BC (1.8) and real coupled BC (1.9), we will state our theorems in two cases when $g_{12} \neq 0$ or $g_{12} = 0$.

First, we consider SLP with transmission condition (1.2)–(1.4) with separated BC (1.8).

Theorem 2.1. *Assume $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$ and $g_{12} \neq 0$. Define an $(m+n+2) \times (m+n+2)$ tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 \sin\alpha + \cos\alpha & -p_1 \sin\alpha & & & & & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & & & & & \\ \cdots & \cdots & \cdots & & & & & & & \\ & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & & \\ & & -p_m & p_m + \frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} & & & & & \\ & & & -\frac{\rho}{\theta}\frac{1}{g_{12}} & \tilde{p}_1 + \frac{g_{22}}{g_{12}} & -\tilde{p}_1 & & & & \\ & & & & -\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & & \\ & & & & \cdots & \cdots & \cdots & & & \\ & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n & -\tilde{p}_n & & \\ & & & & & & -\tilde{p}_n \sin\beta & \tilde{p}_n \sin\beta - \cos\beta & & \end{bmatrix}, \tag{2.11}$$

and diagonal matrices

$$\begin{aligned} Q_{\alpha\beta} &= diag(q_0 \sin\alpha, q_1, \ldots, q_m, \tilde{q}_0, \ldots, \tilde{q}_{n-1}, \tilde{q}_n \sin\beta), \\ W_{\alpha\beta} &= diag(w_0 \sin\alpha, w_1, \ldots, w_m, \tilde{w}_0, \ldots, \tilde{w}_{n-1}, \tilde{w}_n \sin\beta). \end{aligned} \tag{2.12}$$

Then SLP with transmission condition (1.2), (1.4), (1.8) is equivalent to matrix eigenvalue problem

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U, \tag{2.13}$$

where $U = [u_0, u_1, \ldots, u_m, \tilde{u}_0, \ldots, \tilde{u}_n]^T$. Moreover, all eigenvalues are geometrically simple, and unique up to constant multiples, the eigenfunction $u(x)$ of SLP with transmission condition (1.2), (1.4), (1.8) and the corresponding eigenvector U of the matrix eigenvalue problem (2.13) associated with the same eigenvalue are related by $u(x) = u_k, x \in [a_{2k}, a_{2k+1}], k = 0, \ldots, m-1, u(x) = u_m, x \in [a_{2m}, a_{2m+1}), u(x) = \tilde{u}_0, x \in (b_0, b_1], u(x) = \tilde{u}_i, x \in [b_{2i}, b_{2i+1}], k = 1, 2, \ldots, n.$

Proof. There is a one-to-one correspondence between the solutions of system (2.7)–(2.10) and the solutions of the following system:

$$p_1(u_1 - u_0) - v_0 = u_0(q_0 - \lambda w_0), \tag{2.14}$$
$$p_{k+1}(u_{k+1} - u_k) - p_k(u_k - u_{k-1}) = u_k(q_k - \lambda w_k), \quad k = 1, 2, \ldots, m-1, \tag{2.15}$$
$$v_{m+1} - p_m(u_m - u_{m-1}) = u_m(q_m - \lambda w_m), \tag{2.16}$$
$$\tilde{p}_1(\tilde{u}_1 - \tilde{u}_0) - \tilde{v}_0 = \tilde{u}_0(\tilde{q}_0 - \lambda \tilde{w}_0), \tag{2.17}$$
$$\tilde{p}_{i+1}(\tilde{u}_{i+1} - \tilde{u}_i) - \tilde{p}_i(\tilde{u}_i - \tilde{u}_{i-1}) = \tilde{u}_i(\tilde{q}_i - \lambda \tilde{w}_i), \quad i = 1, 2, \ldots, n-1, \tag{2.18}$$
$$\tilde{v}_{n+1} - \tilde{p}_n(\tilde{u}_n - \tilde{u}_{n-1}) = \tilde{u}_n(\tilde{q}_n - \lambda \tilde{w}_n). \tag{2.19}$$

In fact, assume that $u_k, k = 0, 1, 2, \ldots, m$, and $v_k, k = 0, 1, 2, \ldots, m+1$, is a solution of system (2.7), (2.8). Then (2.14)–(2.16) follow from (2.7), (2.8). Similarly, (2.17)–(2.19) follow from (2.9), (2.10) by assuming that $\tilde{u}_i, i = 0, 1, 2, \ldots, n$, and $\tilde{v}_i, i = 0, 1, 2, \ldots, n+1$, is a solution of system (2.9), (2.10). On the other hand, assume $u_k, k = 0, 1, 2, \ldots, m$, is a solution of system (2.14)–(2.16). Then v_0 and v_{m+1} are determined by (2.14) and (2.16), respectively. Let $v_k, k = 1, 2, \ldots, m$, be defined by (2.7). Then using (2.14) and by induction on (2.15) we obtain (2.8). Similarly for (2.9) and (2.10).

Therefore by Lemma 2.1, any solution of Eq. (1.10), and hence of (1.2), is uniquely determined by a solution of system (2.14)–(2.19). Note that from BC (1.8) we have

$$u_0 \cos\alpha = v_0 \sin\alpha, \quad \tilde{u}_n \cos\beta = \tilde{v}_{n+1} \sin\beta, \tag{2.20}$$

and from the transmission condition (1.4) we have

$$\tilde{u}_0 = g_{11} u_m + g_{12} v_{m+1}, \qquad \tilde{v}_0 = g_{21} u_m + g_{22} v_{m+1}, \tag{2.21}$$

since $g_{12} \neq 0$ and $det(G) = -\frac{\rho}{\theta}$, thus from (2.21) we have

$$v_{m+1} = \frac{1}{g_{12}} \tilde{u}_0 - \frac{g_{11}}{g_{12}} u_m, \qquad \tilde{v}_0 = \frac{g_{22}}{g_{12}} \tilde{u}_0 - \frac{\rho}{\theta} \frac{1}{g_{12}} u_m, \tag{2.22}$$

The equivalence follows from (2.14)–(2.22). $\square$

Theorem 2.2. *Assume $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$ and $g_{12} = 0$. Define an $(m + n + 1) \times (m + n + 1)$ tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 \sin\alpha + \cos\alpha & -p_1 \sin\alpha & & & & & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & \\ & & & -g_{22}p_m & \begin{smallmatrix} g_{22}p_m \\ +g_{11}\tilde{p}_1+g_{21} \end{smallmatrix} & -\tilde{p}_1 & & & & \\ & & & & -g_{11}\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & & \\ & & & & & -\tilde{p}_2 & \tilde{p}_2 + \tilde{p}_3 & -\tilde{p}_3 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n & -\tilde{p}_n \\ & & & & & & & & -\tilde{p}_n \sin\beta & \tilde{p}_n \sin\beta - \cos\beta \end{bmatrix}, \tag{2.23}$$

and diagonal matrices

$$\begin{aligned} Q_{\alpha\beta} &= diag(q_0 \sin\alpha, q_1, \dots, q_{m-1}, g_{22}q_m + g_{11}\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_{n-1}, \tilde{q}_n \sin\beta), \\ W_{\alpha\beta} &= diag(w_0 \sin\alpha, w_1, \dots, w_{m-1}, g_{22}w_m + g_{11}\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{n-1}, \tilde{w}_n \sin\beta). \end{aligned} \tag{2.24}$$

Then SLP with transmission condition (1.2), (1.4), (1.8) is equivalent to matrix eigenvalue problem

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U, \tag{2.25}$$

where $U = [u_0, u_1, \dots, u_m, \tilde{u}_1, \dots, \tilde{u}_n]^T$. Moreover, all eigenvalues are geometrically simple, and unique up to constant multiples, the eigenfunction $u(x)$ of SLP with transmission condition (1.2), (1.4), (1.8) and the corresponding eigenvector U of the matrix eigenvalue problem (2.25) associated with the same eigenvalue are related by $u(x) = u_k, x \in [a_{2k}, a_{2k+1}], k = 0, \dots, m-1, u(x) = u_m, x \in [a_{2m}, a_{2m+1}), u(x) = g_{11}u_m, x \in (b_0, b_1], u(x) = \tilde{u}_i, x \in [b_{2i}, b_{2i+1}], k = 1, 2, \dots, n.$

Proof. Since $g_{12} = 0$, transmission condition (1.4) is the same as

$$\tilde{u}_0 = g_{11}u_m, \qquad \tilde{v}_0 = g_{21}u_m + g_{22}v_{m+1}. \tag{2.26}$$

From (2.16), (2.17), (2.18) and (2.26) we have

$$\tilde{p}_1(\tilde{u}_1 - g_{11}u_m) - g_{21}u_m - g_{22}[p_m(u_m - u_{m-1}) + u_m(q_m - \lambda w_m)] = g_{11}u_m(\tilde{q}_0 - \lambda\tilde{w}_0), \tag{2.27}$$

$$p_2(\tilde{u}_2 - \tilde{u}_2) - \tilde{p}_1(\tilde{u}_1 - g_{11}u_m) = \tilde{u}_1(\tilde{q}_1 - \lambda\tilde{w}_1). \tag{2.28}$$

Then the equivalence follows from (2.14)–(2.16), (2.18)–(2.20) and (2.27)–(2.28). □

Corollary 2.1. (i) *Assume $\alpha, \beta \in (0, \pi)$ and $g_{12} \neq 0$. Define an $(m + n + 2) \times (m + n + 2)$ tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & & & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & \\ & & & -p_m & p_m + \frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} & & & & \\ & & & & -\frac{\rho}{\theta}\frac{1}{g_{12}} & \tilde{p}_1 + \frac{g_{22}}{g_{12}} & -\tilde{p}_1 & & & \\ & & & & & -\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n & -\tilde{p}_n \\ & & & & & & & & -\tilde{p}_n & \tilde{p}_n - \cot\beta \end{bmatrix}, \tag{2.29}$$

and diagonal matrices

$$Q_{\alpha\beta} = diag(q_0, q_1, \dots, q_m, \tilde{q}_0, \dots, \tilde{q}_n), \qquad W_{\alpha\beta} = diag(w_0, w_1, \dots, w_m, \tilde{w}_0, \dots, \tilde{w}_n). \tag{2.30}$$

Then SLP with transmission condition (1.2), (1.4), (1.8) is equivalent to matrix eigenvalue problem

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta} U, \tag{2.31}$$

where $U = [u_0, u_1, \dots, u_m, \tilde{u}_0, \dots, \tilde{u}_n]^T$.

(ii) *If $\alpha = 0$ and/or $\beta = \pi$ then a similar statement holds with matrices P, Q, W obtained from matrices (2.29), (2.30) by deleting their first row and column if $\alpha = 0$ and/or the last row and column if $\beta = \pi$.*

Proof. (i) In this case, we divide the first and the last rows of system (2.13) by $\sin\alpha$ and $\sin\beta$, respectively to obtain (2.31).

(ii) If $\alpha = 0$ then $u_0 = 0$ so the first row and column of the matrices P, Q, W can be deleted. Similarly, if $\beta = \pi$ then $\tilde{u}_n = 0$ so the last row and column can be deleted. $\square$

Corollary 2.2. (i) *Assume $\alpha, \beta \in (0, \pi)$ and $g_{12} = 0$. Define an $(m + n + 1) \times (m + n + 1)$ tridiagonal matrix*

$$P_{\alpha\beta} = \begin{bmatrix} p_1 + \cot\alpha & -p_1 & & & & & & & & \\ -p_1 & p_1 + p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & \\ & & & -g_{22}p_m & \substack{g_{22}p_m \\ +g_{11}\tilde{p}_1+g_{21}} & -\tilde{p}_1 & & & & \\ & & & & -g_{11}\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & & \\ & & & & & -\tilde{p}_2 & \tilde{p}_2 + \tilde{p}_3 & -\tilde{p}_3 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n & -\tilde{p}_n \\ & & & & & & & & -\tilde{p}_n & \tilde{p}_n - \cot\beta \end{bmatrix}, \tag{2.32}$$

and diagonal matrices

$$\begin{aligned} Q_{\alpha\beta} &= diag(q_0, q_1, \ldots, q_{m-1}, g_{22}q_m + g_{11}\tilde{q}_0, \tilde{q}_1, \ldots, \tilde{q}_n), \\ W_{\alpha\beta} &= diag(w_0, w_1, \ldots, w_{m-1}, g_{22}w_m + g_{11}\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_n). \end{aligned} \tag{2.33}$$

Then SLP with transmission condition (1.2), (1.4), (1.8) is equivalent to matrix eigenvalue problem

$$(P_{\alpha\beta} + Q_{\alpha\beta})U = \lambda W_{\alpha\beta}U, \tag{2.34}$$

where $U = [u_0, u_1, \ldots, u_m, \tilde{u}_1, \ldots, \tilde{u}_n]^T$.

(ii) *If $\alpha = 0$ and/or $\beta = \pi$ then a similar statement holds with matrices P, Q, W obtained from matrices (2.32), (2.33) by deleting their first row and column if $\alpha = 0$ and/or the last row and column if $\beta = \pi$.*

Proof. The proof is similar with Corollary 2.1 by operating on Theorem 2.2. $\square$

Theorem 2.1, Theorem 2.2 and their Corollaries show that all SLPs with transmission conditions of Atkinson type with a self-adjoint separated BC have representations by tridiagonal matrix eigenvalue problems. Next we show that all SLPs with transmission conditions of Atkinson type with a real couple self-adjoint BC also have matrix representations. In this case, the matrix P is "almost tridiagonal" in the sense that the entries in the upper right and lower left corners are nonzero.

Theorem 2.3. *Consider the BC (1.9) with $k_{12} = 0$ and assume that $g_{12} \neq 0$. Define an $(m+n+1) \times (m+n+1)$ matrix which is tridiagonal except for the $(1,\ m+n+1)$ and $(m+n+1,\ 1)$ entries*

$$P_0 = \begin{bmatrix} \substack{-k_{11}k_{21}+p_1 \\ +k_{11}^2\tilde{p}_n} & -p_1 & & & & & & & & -k_{11}\tilde{p}_n \\ -p_1 & p_1 + p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & \\ & & & -p_m & p_m + \frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} & & & & \\ & & & & -\frac{\rho}{\theta}\frac{1}{g_{12}} & \tilde{p}_1 + \frac{g_{22}}{g_{12}} & -\tilde{p}_1 & & & \\ & & & & & -\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-2} & \tilde{p}_{n-2} + \tilde{p}_{n-1} & -\tilde{p}_{n-1} \\ -k_{11}\tilde{p}_n & & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n \end{bmatrix}, \tag{2.35}$$

and diagonal matrices

$$\begin{aligned} Q_0 &= diag(q_0 + k_{11}^2\tilde{q}_n, q_1, \ldots, q_m, \tilde{q}_0, \ldots, \tilde{q}_{n-1}), \\ W_0 &= diag(w_0 + k_{11}^2\tilde{w}_n, w_1, \ldots, w_m, \tilde{w}_0, \ldots, \tilde{w}_{n-1}). \end{aligned} \tag{2.36}$$

Then SLP with transmission condition (1.2), (1.4), (1.9) is equivalent to matrix eigenvalue problem

$$(P_0 + Q_0)U = \lambda W_0 U, \tag{2.37}$$

where $U = [u_0, u_1, \ldots, u_m, \tilde{u}_0, \ldots, \tilde{u}_{n-1}]^T$.

Proof. Since $k_{12} = 0$, BC (1.9) is the same as

$$\tilde{u}_n = k_{11}u_0, \qquad \tilde{v}_{n+1} = k_{21}u_0 + k_{22}v_0, \tag{2.38}$$

where $k_{11}k_{22} = 1$, and since $g_{12} \neq 0$, the transmission condition (1.4) has the form of (2.22). We claim that there is a one-to-one correspondence between the solutions consisting of system (2.7)–(2.10), BC (2.38) and transmission condition (2.22) and the solutions of the following system:

$$[-k_{11}k_{21} + (p_1 + q_0 - \lambda w_0) - k_{11}^2(\tilde{p}_n + \tilde{q}_n - \lambda \tilde{w}_n)]u_0 - p_1 u_1 - k_{11}\tilde{p}_n \tilde{u}_{n-1} = 0, \tag{2.39}$$

$$p_{k+1}(u_{k+1} - u_k) - p_k(u_k - u_{k-1}) = u_k(q_k - \lambda w_k), \quad k = 1, 2, \ldots, m-1, \tag{2.40}$$

$$\frac{1}{g_{12}}\tilde{u}_0 - \frac{g_{11}}{g_{12}}u_m - p_m(u_m - u_{m-1}) = u_m(q_m - \lambda w_m), \tag{2.41}$$

$$\tilde{p}_1(\tilde{u}_1 - \tilde{u}_0) - \left(\frac{g_{22}}{g_{12}}\tilde{u}_0 - \frac{\rho}{\theta}\frac{1}{g_{12}}u_m\right) = \tilde{u}_0(\tilde{q}_0 - \lambda \tilde{w}_0), \tag{2.42}$$

$$\tilde{p}_{i+1}(\tilde{u}_{i+1} - \tilde{u}_i) - \tilde{p}_i(\tilde{u}_i - \tilde{u}_{i-1}) = \tilde{u}_i(\tilde{q}_i - \lambda \tilde{w}_i), \quad i = 1, 2, \ldots, n-2, \tag{2.43}$$

$$\tilde{p}_n(k_{11}u_0 - \tilde{u}_{n-1}) - \tilde{p}_{n-1}(\tilde{u}_{n-1} - \tilde{u}_{n-2}) = \tilde{u}_{n-1}(\tilde{q}_{n-1} - \lambda \tilde{w}_{n-1}). \tag{2.44}$$

In fact, assume that $u_k, k = 0, 1, 2, \ldots, m, \tilde{u}_i, i = 0, 1, 2, \ldots, n$, and $v_k, k = 0, 1, 2, \ldots, m+1, \tilde{v}_i, i = 0, 1, 2, \ldots, n+1$, is a solution of system (2.7)–(2.10), (2.38) and (2.22). Then (2.40)–(2.43) follow from (2.7)–(2.10) and (2.22) easily. From (2.7) with $k = 1$ and (2.8) with $k = 0$ we have

$$v_0 = p_1(u_1 - u_0) - u_0(q_0 - \lambda w_0). \tag{2.45}$$

From (2.9) and (2.10) with $i = n$ we have

$$\tilde{v}_{n+1} = \tilde{p}_n(\tilde{u}_n - \tilde{u}_{n-1}) + \tilde{u}_n(\tilde{q}_n - \lambda \tilde{w}_n). \tag{2.46}$$

Combining (2.38), (2.45), and (2.46) we obtain that

$$\tilde{p}_n(k_{11}u_0 - \tilde{u}_{n-1}) + k_{11}u_0(\tilde{q}_n - \lambda \tilde{w}_n) = k_{21}u_0 + k_{22}[p_1(u_1 - u_0) - u_0(q_0 - \lambda w_0)]. \tag{2.47}$$

Note that $k_{11}k_{22} = 1$. Then (2.47) becomes (2.39). From (2.43) with $i = n-1$ and (2.38) we have

$$\tilde{p}_n(k_{11}u_0 - \tilde{u}_{n-1}) - \tilde{p}_{n-1}(\tilde{u}_{n-1} - \tilde{u}_{n-2}) = \tilde{u}_{n-1}(\tilde{q}_{n-1} - \lambda \tilde{w}_{n-1}). \tag{2.48}$$

On the other hand, assume $u_k, k = 0, 1, 2, \ldots, m, \tilde{u}_i, i = 0, 1, 2, \ldots, n$, is a solution of system (2.39)–(2.44). Then $\tilde{u}_n$, v_0 and v_{n+1} are determined by (2.38), (2.45), and (2.46), respectively. Let $v_k, k = 0, 1, 2, \ldots, m$, be defined by (2.7) and $\tilde{v}_i, i = 0, 1, 2, \ldots, n$, be defined by (2.9). Then using (2.45), (2.22) and by induction on (2.46) we obtain (2.8) and (2.10). From (2.45)-(2.47) we see that $\tilde{v}_{n+1} = k_{21}u_0 + k_{22}v_0$. Hence BC (2.38) is satisfied.

Therefore, by Lemma 2.1, any solution of SLP with transmission condition (1.10), (1.9) and (1.4), hence of SLP with transmission condition (1.2), (1.9) and (1.4), is uniquely determined by a solution of system (2.39)–(2.44). $\square$

Theorem 2.4. *Consider the BC (1.9) with $k_{12} \neq 0$ and assume that $g_{12} \neq 0$. Define an $(m+n+2) \times (m+n+2)$ matrix which is tridiagonal except for the $(1, m+n+2)$ and $(m+n+2, 1)$ entries*

$$P_1 = \begin{bmatrix}
p_1 - \frac{k_{11}}{k_{12}} & -p_1 & & & & & & & & \frac{1}{k_{12}} \\
-p_1 & p_1 + p_2 & -p_2 & & & & & & & \\
 & \cdots & \cdots & \cdots & & & & & & \\
 & & -p_{m-1} & p_{m-1} + p_m & -p_m & & & & & \\
 & & & -p_m & p_m + \frac{g_{11}}{g_{12}} & -\frac{1}{g_{12}} & & & & \\
 & & & & -\frac{\rho}{\theta}\frac{1}{g_{12}} & \tilde{p}_1 + \frac{g_{22}}{g_{12}} & -\tilde{p}_1 & & & \\
 & & & & & -\tilde{p}_1 & \tilde{p}_1 + \tilde{p}_2 & -\tilde{p}_2 & & \\
 & & & & & & \cdots & \cdots & \cdots & \\
 & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1} + \tilde{p}_n & -\tilde{p}_n \\
\frac{1}{k_{12}} & & & & & & & & -\tilde{p}_n & \tilde{p}_n - \frac{k_{22}}{k_{12}}
\end{bmatrix}, \tag{2.49}$$

and diagonal matrices

$$Q_1 = diag(q_0, q_1, \ldots, q_m, \tilde{q}_0, \ldots, \tilde{q}_n), \qquad W_1 = diag(w_0, w_1, \ldots, w_m, \tilde{w}_0, \ldots, \tilde{w}_n). \tag{2.50}$$

Then SLP with transmission condition (1.2), (1.4), (1.9) is equivalent to matrix eigenvalue problem

$$(P_1 + Q_1)U = \lambda W_1 U, \tag{2.51}$$

where $U = [u_0, u_1, \ldots, u_m, \tilde{u}_0, \ldots, \tilde{u}_n]^T$.

Proof. BC (1.9) is the same as

$$\tilde{u}_n = k_{11}u_0 + k_{12}v_0, \qquad \tilde{v}_{n+1} = k_{21}u_0 + k_{22}v_0.$$

Since $k_{11}k_{22} - k_{12}k_{21} = 1$, this can be written as

$$v_0 = -\frac{k_{11}}{k_{12}}u_0 + \frac{1}{k_{12}}\tilde{u}_n, \qquad \tilde{v}_{n+1} = -\frac{1}{k_{12}}u_0 + \frac{k_{22}}{k_{12}}\tilde{u}_n. \tag{2.52}$$

Note that the transmission condition has the form (2.22), then the proof is similar with Theorem 2.3. □

Theorem 2.5. *Consider the BC (1.9) with $k_{12} = 0$ and assume that $g_{12} = 0$. Define an $(m+n) \times (m+n)$ matrix which is tridiagonal except for the $(1, m+n)$ and $(m+n, 1)$ entries*

$$P_2 = \begin{bmatrix} {\scriptstyle -k_{11}k_{21}+p_1 \atop +k_{11}^2\tilde{p}_n} & -p_1 & & & & & & & & -k_{11}\tilde{p}_n \\ -p_1 & p_1+p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1}+p_m & -p_m & & & & & \\ & & & -g_{22}p_m & {\scriptstyle g_{22}p_m \atop +g_{11}\tilde{p}_1+g_{21}} & -\tilde{p}_1 & & & & \\ & & & & -g_{11}\tilde{p}_1 & \tilde{p}_1+\tilde{p}_2 & -\tilde{p}_2 & & & \\ & & & & & -\tilde{p}_2 & \tilde{p}_2+\tilde{p}_3 & -\tilde{p}_3 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-2} & \tilde{p}_{n-2}+\tilde{p}_{n-1} & -\tilde{p}_{n-1} \\ -k_{11}\tilde{p}_n & & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1}+\tilde{p}_n \end{bmatrix}, \tag{2.53}$$

and diagonal matrices

$$\begin{aligned} Q_2 &= diag(q_0 + k_{11}^2\tilde{q}_n, q_1, \dots, q_{m-1}, g_{22}q_m + g_{11}\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_{n-1}), \\ W_2 &= diag(w_0 + k_{11}^2\tilde{w}_n, w_1, \dots, w_{m-1}, g_{22}q_m + g_{11}\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_{n-1}). \end{aligned} \tag{2.54}$$

Then SLP with transmission condition (1.2), (1.4), (1.9) is equivalent to matrix eigenvalue problem

$$(P_2 + Q_2)U = \lambda W_2 U, \tag{2.55}$$

where $U = [u_0, u_1, \dots, u_m, \tilde{u}_1, \dots, \tilde{u}_{n-1}]^T$.

Proof. Note that BC is with the form (2.38) and the transmission condition is with the form (2.26), then the proof is similar with Theorem 2.3. □

Theorem 2.6. *Consider the BC (1.9) with $k_{12} \neq 0$ and assume that $g_{12} = 0$. Define an $(m+n+1) \times (m+n+1)$ matrix which is tridiagonal except for the $(1, m+n+1)$ and $(m+n+1, 1)$ entries*

$$P_3 = \begin{bmatrix} p_1 - \frac{k_{11}}{k_{12}} & -p_1 & & & & & & & & \frac{1}{k_{12}} \\ -p_1 & p_1+p_2 & -p_2 & & & & & & & \\ & \cdots & \cdots & \cdots & & & & & & \\ & & -p_{m-1} & p_{m-1}+p_m & -p_m & & & & & \\ & & & -g_{22}p_m & {\scriptstyle g_{22}p_m \atop +g_{11}\tilde{p}_1+g_{21}} & -\tilde{p}_1 & & & & \\ & & & & -g_{11}\tilde{p}_1 & \tilde{p}_1+\tilde{p}_2 & -\tilde{p}_2 & & & \\ & & & & & -\tilde{p}_2 & \tilde{p}_2+\tilde{p}_3 & -\tilde{p}_3 & & \\ & & & & & & \cdots & \cdots & \cdots & \\ & & & & & & & -\tilde{p}_{n-1} & \tilde{p}_{n-1}+\tilde{p}_n & -\tilde{p}_n \\ \frac{1}{k_{12}} & & & & & & & & -\tilde{p}_n & \tilde{p}_n - \frac{k_{22}}{k_{12}} \end{bmatrix}, \tag{2.56}$$

and diagonal matrices

$$\begin{aligned} Q_3 &= diag(q_0, q_1, \dots, q_{m-1}, g_{22}q_m + g_{11}\tilde{q}_0, \tilde{q}_1, \dots, \tilde{q}_n), \\ W_3 &= diag(w_0, w_1, \dots, w_{m-1}, g_{22}q_m + g_{11}\tilde{w}_0, \tilde{w}_1, \dots, \tilde{w}_n). \end{aligned} \tag{2.57}$$

Then SLP with transmission condition (1.2), (1.4), (1.9) is equivalent to matrix eigenvalue problem

$$(P_3 + Q_3)U = \lambda W_3 U, \tag{2.58}$$

where $U = [u_0, u_1, \dots, u_m, \tilde{u}_1, \dots, \tilde{u}_n]^T$.

Proof. Note that BC is with the form (2.52) and the transmission condition is with the form (2.22), then the proof is also similar with Theorem 2.3. □

Remark 2.1. When comparing Theorems 2.3–2.6 we note that the dimensions of the matrix systems are different. The reason for this is that when $k_{12} = 0$, the condition $\tilde{u}_n = k_{11}u_0$ is used to express $\tilde{u}_n$ in terms of u_0 thus eliminating the need for $\tilde{u}_n$ in (2.51), and/or when $g_{12} = 0$, the condition $\tilde{u}_0 = g_{11}u_m$ is used to express $\tilde{u}_0$ in terms of u_m thus eliminating the need for $\tilde{u}_0$ in (2.51). Thus there are exactly $m+n+1$ eigenvalues, counting multiplicity, in Theorem 2.3, exactly $m+n+2$ eigenvalues in Theorem 2.4, exactly $m+n$ eigenvalues in Theorem 2.5, and exactly $m+n+1$ eigenvalues in Theorem 2.6 respectively.

The next result highlights the fact that every SLP with transmission condition of Atkinson type is equivalent to a SLP with the same BC and transmission condition with piecewise constant coefficients.

Theorem 2.7. *Assume Eq. (1.2) is of Atkinson type, and let $p_k, k = 1, 2, \dots, m, \tilde{p}_i, i = 1, 2, \dots, n$, and $q_k, w_k, k = 0, 1, 2, \dots, m, \tilde{q}_i, \tilde{w}_i, i = 0, 1, \dots, n$ be given by (2.4). Define piecewise constant functions $\bar{p}$, $\bar{q}$ and $\bar{w}$ on J by*

$$\bar{p}(x) = \begin{cases} p_k(a_{2k} - a_{2k-1}), & x \in [a_{2k-1}, a_{2k}], \quad k = 1, 2, \dots, m, \\ \infty, & x \in [a_{2k}, a_{2k+1}), \quad k = 0, 1, \dots, m, \\ \tilde{p}_i(a_{2i} - a_{2i-1}), & x \in [b_{2i-1}, b_{2i}], \quad i = 1, 2, \dots, n, \\ \infty, & x \in (b_{2i}, b_{2i+1}], \quad i = 0, 1, \dots, n, \end{cases}$$

$$\bar{q}(x) = \begin{cases} \dfrac{q_k}{(a_{2k+1} - a_{2k})}, & x \in [a_{2k}, a_{2k+1}), \quad k = 0, 1, \dots, m, \\ 0, & x \in [a_{2k-1}, a_{2k}], \quad k = 1, 2, \dots, m, \\ \dfrac{\tilde{q}_i}{(b_{2i+1} - b_{2i})}, & x \in (b_{2i}, b_{2i+1}], \quad i = 0, 1, \dots, n, \\ 0, & x \in [b_{2i-1}, b_{2i}], \quad i = 1, 2, \dots, n, \end{cases} \tag{2.59}$$

$$\bar{w}(x) = \begin{cases} \dfrac{w_k}{(a_{2k+1} - a_{2k})}, & x \in [a_{2k}, a_{2k+1}), \quad k = 0, 1, \dots, m, \\ 0, & x \in [a_{2k-1}, a_{2k}], \quad k = 1, 2, \dots, m, \\ \dfrac{\tilde{w}_i}{(b_{2i+1} - b_{2i})}, & x \in (b_{2i}, b_{2i+1}], \quad i = 0, 1, \dots, n, \\ 0, & x \in [b_{2i-1}, b_{2i}], \quad i = 1, 2, \dots, n. \end{cases}$$

Suppose the self-adjoint BC (1.3) is either separated or real coupled. Then SLP with transmission condition (1.2)–(1.4) has exactly the same eigenvalues as the SLP consisting of the equation with piecewise constant coefficients

$$-(\bar{p}y')' + \bar{q}y = \lambda \bar{w}y, \quad \text{on } J = (a, c) \cup (c, b), \tag{2.60}$$

and the same BC (1.3) and transmission condition (1.4).

Proof. We observe that both SLPs with transmission condition (1.2)–(1.4) and (2.60), (1.3), (1.4) determine the same $p_k, k = 1, 2, \dots, m, \tilde{p}_i, i = 1, 2, \dots, n$ and $q_k, w_k, k = 0, 1, \dots, m, \tilde{q}_i, \tilde{w}_i, i = 0, 1, \dots, n$. Thus by one of Theorems 2.1–2.6, depending which BC (1.3) and transmission condition (1.4) are involved, they are equivalent to the same matrix eigenvalue problem, and hence they have the same eigenvalues. □

By Theorem 2.7 we see that for a fixed BC (1.3) and transmission condition (1.4) on a given interval J, there is a family of SLPs with transmission conditions of Atkinson type which have exactly the same eigenvalues as SLP with transmission condition (2.60), (1.3), (1.4). Such a family is called the equivalent family of SLP with transmission condition (2.60), (1.3), (1.4).

3. Sturm–Liouville representations of matrix eigenvalue problems

In this section we show that matrix eigenvalue problems of the form

$$FX = \lambda HX, \tag{3.1}$$

where $F = (f_{ij})$ is an $l \times l$ real tridiagonal or "almost tridiagonal" matrix with $f_{i,i+1} \neq 0, i = 1, \dots, l-1$, which is "almost symmetric" (here "almost symmetric" means that the matrix is symmetric except for one or two elements in the middle of the matrix are different) and $H = diag(h_{11}, \dots, h_{ll})$ with $h_{kk} \neq 0, k = 1, 2, \dots, l$, have representations as SLPs with transmission conditions of Atkinson type. By Theorem 2.7, such representations are also not unique. Here we characterize all Sturm–Liouville representations of the matrix problem (3.1) using SLP with transmission condition (2.60), (1.3), (1.4) and their equivalent families.

First we consider the case of separated BCs (1.8) and find a kind of converse to Theorems 2.1 and 2.2 (in fact, Corollaries 2.1 and 2.2).

Theorem 3.1. *Let $l > 3$, C, D be any matrices associated with the transmission condition (1.4) which satisfying $det(C) = \rho > 0$, $det(D) = \theta > 0$, let $G = (g_{ij}) = -D^{-1}C$, suppose $g_{12} \neq 0$, and let F be an $l \times l$ "almost symmetric" tridiagonal matrix*

$$F = \begin{bmatrix} f_{11} & f_{12} & & & & & & & \\ f_{12} & f_{22} & f_{23} & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & f_{k,k-1} & f_{k,k} & f_{k,k+1} & & & & \\ & & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} & & & \\ & & & & f_{k+1,k+2} & \cdots & \cdots & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\ & & & & & & & f_{l-1,l} & f_{ll} \end{bmatrix}, \tag{3.2}$$

where $2 \leq k \leq l-2$, $f_{ij} \in \mathbb{R}$, $1 \leq i, j \leq l$, $f_{j,j+1} \neq 0$, $j = 1, \ldots, l-1$, $f_{k+1,k} = \frac{\rho}{\theta} f_{k,k+1}$ and let

$$H = diag(h_{11}, \ldots, h_{ll}), \quad 0 \neq h_{jj} \in \mathbb{R},\ 1 \leq j \leq l. \tag{3.3}$$

Then, given any separated self-adjoint BC (1.8), the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.8). Moreover, given a fixed partition (2.1) of J, it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.8) provided, with the notation in (2.4), one of the following holds:

(i) $\alpha, \beta \in (0, \pi)$;
(ii) $\alpha = 0$ *and* $\beta \in (0, \pi)$, *and* p_1, q_0, *and* w_0 *are fixed;*
(iii) $\alpha \in (0, \pi)$ *and* $\beta = \pi$, *and* $\tilde{p}_n, \tilde{q}_n$, *and* $\tilde{w}_n$ *are fixed;*
(iv) $\alpha = 0$ *and* $\beta = \pi$, *and* p_1, q_0, w_0 *and* $\tilde{p}_n, \tilde{q}_n, \tilde{w}_n$ *are fixed.*

In each of these cases, all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.8) with all possible choices of the parameters; for example, with all possible choices of p_1, q_0, w_0 in case $\alpha = 0$ and $\beta \in (0, \pi)$.

Proof. First consider the case when $\alpha, \beta \in (0, \pi)$. Note that we can normalize the matrices F and H such that $f_{k,k+1} = -\frac{1}{g_{12}}$ by multiplying Eq. (3.1) by $-\frac{1}{g_{12} f_{k,k+1}}$. This operation does not change the eigenvalues of problem (3.1). Let $m = k-1$, $n = l-k-1$, $J = (a, c) \cup (c, b)$, $-\infty < a < b < \infty$. Define a partition of J by (2.1). We construct piecewise constant functions $\bar{p}, \bar{q}, \bar{w}$ on $[a, c) \cup (c, b]$ satisfying (1.5), (2.2) and (2.3). We need to define the values of such functions on those subintervals of $[a, c) \cup (c, b]$ where they are not defined as zero in (2.2), (2.3). To do this, we let

$$p_i = -f_{i,i+1}, \quad i = 1, 2, \ldots, m, \qquad \tilde{p}_j = -f_{j+m+1,j+m+2}, \quad j = 1, 2, \ldots, n;$$
$$w_i = h_{i+1,i+1}, \quad i = 0, 1, \ldots, m, \qquad \tilde{w}_j = h_{j+m+2,j+m+2}, \quad j = 0, 1, \ldots, n;$$

and

$$q_0 = h_{11} - p_1 - \cot\alpha, \qquad q_i = h_{i+1,i+1} - p_i - p_{i+1}, \quad i = 1, 2, \ldots, m-1,$$
$$q_m = h_{m+1,m+1} - p_m - \frac{g_{11}}{g_{12}}, \qquad \tilde{q}_0 = h_{m+2,m+2} - \tilde{p}_1 - \frac{g_{22}}{g_{12}},$$
$$\tilde{q}_j = h_{j+m+2,j+m+2} - \tilde{p}_j - \tilde{p}_{j+1}, \quad j = 1, 2, \ldots, n-1, \qquad \tilde{q}_n = h_{m+n+2,m+n+2} - \tilde{p}_n + \cot\beta.$$

Then define $\bar{p}(x), \bar{q}(x)$, and $\bar{w}(x)$ by (2.59). Such $\bar{p}, \bar{q}, \bar{w}$ are piecewise constant functions on J satisfying (1.3), (2.2), and (2.3). Eq. (2.60) is of Atkinson type, and (2.4) is satisfied. It is easy to see that problem (3.1) is of the same form as problem (2.31). Therefore, by Corollary 2.1, problem (3.1) is equivalent to SLP with transmission condition (1.2), (1.4), (1.8). The last part follows from Theorem 2.7.

The cases $\alpha = 0$ and/or $\beta = \pi$ are treated similarly. □

Theorem 3.2. *Let $l > 4$, C, D be any matrices associated with the transmission condition (1.4) which satisfying $det(C) = \rho > 0$, $det(D) = \theta > 0$, let $G = (g_{ij}) = -D^{-1}C$, suppose $g_{12} = 0$, and let F be an $l \times l$ "almost symmetric" tridiagonal matrix*

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{12} & f_{22} & f_{23} \\ & \cdots & \cdots & \cdots \\ & & f_{k,k-1} & f_{k,k} & f_{k,k+1} \\ & & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} \\ & & & & f_{k+2,k+1} & \cdots & \cdots \\ & & & & & \cdots & \cdots & \cdots \\ & & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\ & & & & & & & f_{l-1,l} & f_{ll} \end{bmatrix}, \tag{3.4}$$

where $2 \le k \le l-3, f_{ij} \in \mathbb{R}, 1 \le i,j \le l, f_{j,j+1} \neq 0, j = 1, \ldots, l-1, f_{k+1,k} = g_{22}f_{k,k+1}, f_{k+2,k+1} = g_{11}f_{k+1,k+2}$, *and let*

$$H = diag(h_{11}, \ldots, h_{ll}), \quad 0 \neq h_{jj} \in \mathbb{R},\ 1 \le j \le l. \tag{3.5}$$

Then, given any separated self-adjoint BC (1.8), the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.8). Moreover, given a fixed partition (2.1) of J, it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.8) provided, with the notation in (2.4), one of the following holds:

(i) $\alpha, \beta \in (0, \pi)$, *and* q_m, *and* w_m *are fixed;*
(ii) $\alpha = 0$ *and* $\beta \in (0, \pi)$, *and* q_m, w_m *and* p_1, q_0, w_0 *are fixed;*
(iii) $\alpha \in (0, \pi)$ *and* $\beta = \pi$, *and* q_m, w_m *and* $\tilde{p}_n, \tilde{q}_n, \tilde{w}_n$ *are fixed;*
(iv) $\alpha = 0$ *and* $\beta = \pi$, *and* q_m, w_m *and* p_1, q_0, w_0 *and* $\tilde{p}_n, \tilde{q}_n, \tilde{w}_n$ *are fixed.*

In each of these cases, all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.8) with all possible choices of the parameters; for example, with all possible choices of q_m, w_m *in case* $\alpha, \beta \in (0, \pi)$.

Proof. Still consider the case when $\alpha, \beta \in (0, \pi)$. Let $m = k$, $n = l-k-1$, $J = (a, c) \cup (c, b)$, $-\infty < a < b < \infty$. Define a partition of J by (2.1). Let

$$p_i = -f_{i,i+1}, \quad i = 1, 2, \ldots, m, \qquad \tilde{p}_j = -f_{j+m,j+m+1}, \quad j = 1, 2, \ldots, n;$$
$$w_i = h_{i+1,i+1}, \quad i = 0, 1, \ldots, m-1, \qquad \tilde{w}_j = h_{j+m+1,j+m+1}, \quad j = 1, 2, \ldots, n,$$
$$\tilde{w}_0 = \frac{1}{g_{11}}(h_{m+1,m+1} + g_{22}w_m), \quad \text{where } w_m \text{ is fixed;}$$

and

$$q_0 = h_{11} - p_1 - \cot\alpha, \qquad q_i = h_{i+1,i+1} - p_i - p_{i+1}, \quad i = 1, 2, \ldots, m-1,$$
$$\tilde{q}_0 = \frac{1}{g_{11}}(h_{m+1,m+1} + g_{22}q_m - g_{22}p_m - g_{11}\tilde{p}_1 - g_{21}), \quad \text{where } q_m \text{ is fixed,}$$
$$\tilde{q}_j = h_{j+m+2,j+m+2} - \tilde{p}_j - \tilde{p}_{j+1}, \quad j = 1, 2, \ldots, n-1, \qquad \tilde{q}_n = h_{m+n+1,m+n+1} - \tilde{p}_n + \cot\beta.$$

Then define $\bar{p}(x)$, $\bar{q}(x)$, and $\bar{w}(x)$ by (2.59). Similar to the proof of Theorem 3.1 we see that problem (3.1) is of the same form as problem (2.34). Therefore, by Corollary 2.2, problem (3.1) is equivalent to SLP with transmission condition (1.2), (1.4), (1.8). The last part also follows from Theorem 2.7.

The cases $\alpha = 0$ and/or $\beta = \pi$ are treated similarly. $\square$

Next we consider the case of coupled BCs (1.9) and find a form of converse of Theorems 2.3–2.6.

Theorem 3.3. *Let $l > 3$, C, D be any matrices associated with the transmission condition (1.4) which satisfying $det(C) = \rho > 0$, $det(D) = \theta > 0$, let $G = (g_{ij}) = -D^{-1}C$, suppose $g_{12} \neq 0$, and let F be an $l \times l$ "almost symmetric" matrix which is tridiagonal except for nonzero entries $f_{1l} = f_{l1}$*

$$F = \begin{bmatrix} f_{11} & f_{12} & & & & & & & f_{1l} \\ f_{12} & f_{22} & f_{23} \\ & \cdots & \cdots & \cdots \\ & & f_{k,k-1} & f_{k,k} & f_{k,k+1} \\ & & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} \\ & & & & f_{k+1,k+2} & \cdots & \cdots \\ & & & & & \cdots & \cdots & \cdots \\ & & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\ f_{1l} & & & & & & & f_{l-1,l} & f_{ll} \end{bmatrix}, \tag{3.6}$$

where $2 \le k \le l-2, f_{ij} \in \mathbb{R}, 1 \le i, j \le l, f_{j,j+1} \ne 0, j = 1, \dots, l-1, f_{k+1,k} = \frac{\rho}{\theta} f_{k,k+1}, f_{1l} = -\frac{g_{12}}{k_{12}} f_{k,k+1}$, *and let*

$$H = diag(h_{11}, \dots, h_{ll}), \quad 0 \ne h_{jj} \in \mathbb{R}, 1 \le j \le l. \tag{3.7}$$

Then, given any real coupled self-adjoint BC (1.9) satisfying $k_{12} \ne 0$, *the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.9). Moreover, given a fixed partition (2.1) of J, it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.9) provided, with the notation in (2.4). And all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.9) with all possible choices of the parameters.*

Proof. Note that we can normalize the matrices F and H such that $f_{k,k+1} = -\frac{1}{g_{12}}$ and $f_{1l} = \frac{1}{k_{12}}$ by multiplying Eq. (3.1) by $(k_{12} f_{1l})^{-1}$. This operation does not change the eigenvalues of problem (3.1). Choose $m = k-1, n = l-k-1, J = (a, c) \cup (c, b), -\infty < a < b < \infty$. Define a partition of J by (2.1). Let

$$\begin{aligned}
&p_i = -f_{i,i+1}, \quad i = 1, 2, \dots, m, \qquad \tilde{p}_j = -f_{j+m+1,j+m+2}, \quad j = 1, 2, \dots, n; \\
&w_i = h_{i+1,i+1}, \quad i = 0, 1, \dots, m, \qquad \tilde{w}_j = h_{j+m+2,j+m+2}, \quad j = 0, 1, \dots, n;
\end{aligned}$$

and

$$\begin{aligned}
&q_0 = h_{11} - p_1 + \frac{k_{11}}{k_{12}}, \qquad q_i = h_{i+1,i+1} - p_i - p_{i+1}, \quad i = 1, 2, \dots, m-1, \\
&q_m = h_{m+1,m+1} - p_m - \frac{g_{11}}{g_{12}}, \qquad \tilde{q}_0 = h_{m+2,m+2} - \tilde{p}_1 - \frac{g_{22}}{g_{12}}, \\
&\tilde{q}_j = h_{j+m+2,j+m+2} - \tilde{p}_j - \tilde{p}_{j+1}, \quad j = 1, 2, \dots, n-1, \qquad \tilde{q}_n = h_{m+n+2,m+n+2} - \tilde{p}_n + \frac{k_{11}}{k_{12}}.
\end{aligned}$$

Then define $\bar{p}(x), \bar{q}(x)$, and $\bar{w}(x)$ by (2.59). Similar to the proof of Theorem 3.1 we see that problem (3.1) is the same as problem (2.51). Therefore, by Theorem 2.4, problem (3.1) is equivalent to SLP with transmission condition (1.2), (1.4), (1.9). □

Theorem 3.4. *Let* $l > 3$, C, D *be any matrices associated with the transmission condition (1.4) which satisfying* $det(C) = \rho > 0$, $det(D) = \theta > 0$, *let* $G = (g_{ij}) = -D^{-1}C$, *suppose* $g_{12} \ne 0$, *and let F be an* $l \times l$ *"almost symmetric" matrix which is tridiagonal except for nonzero entries* $f_{1l} = f_{l1}$

$$F = \begin{bmatrix}
f_{11} & f_{12} & & & & & & & & f_{1l} \\
f_{12} & f_{22} & f_{23} & & & & & & & \\
& \cdots & \cdots & \cdots & & & & & & \\
& & f_{k,k-1} & f_{k,k} & f_{k,k+1} & & & & & \\
& & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} & & & & \\
& & & & f_{k+1,k+2} & \cdots & \cdots & & & \\
& & & & & \cdots & \cdots & \cdots & & \\
& & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\
f_{1l} & & & & & & & f_{l-1,l} & f_{ll}
\end{bmatrix}, \tag{3.8}$$

where $2 \le k \le l-2, f_{ij} \in \mathbb{R}, 1 \le i, j \le l, f_{j,j+1} \ne 0, j = 1, \dots, l-1, f_{k+1,k} = \frac{\rho}{\theta} f_{k,k+1}$, *and let*

$$H = diag(h_{11}, \dots, h_{ll}), \quad 0 \ne h_{jj} \in \mathbb{R}, \ 1 \le j \le l. \tag{3.9}$$

Then, given any real coupled self-adjoint BC (1.9) satisfying $k_{12} = 0$, *the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.9). Moreover, given a fixed partition (2.1) of J, it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.9) provided, with the notation in (2.4). And all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.9) with all possible choices of the parameters.*

Proof. We choose $m = k-1, n = l-k$ and fix q_0 and w_0, then proceed similarly as Theorem 3.3. □

Theorem 3.5. *Let* $l > 4$, C, D *be any matrices associated with the transmission condition (1.4) which satisfying* $det(C) = \rho > 0$, $det(D) = \theta > 0$, *let* $G = (g_{ij}) = -D^{-1}C$, *suppose* $g_{12} = 0$, *and let F be an* $l \times l$ *"almost symmetric" matrix which is tridiagonal* *except for nonzero entries* $f_{1l} = f_{l1}$

$$F = \begin{bmatrix} f_{11} & f_{12} & & & & & & & f_{1l} \\ f_{12} & f_{22} & f_{23} & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & f_{k,k-1} & f_{k,k} & f_{k,k+1} & & & & \\ & & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} & & & \\ & & & & f_{k+2,k+1} & \cdots & \cdots & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\ f_{1l} & & & & & & & f_{l-1,l} & f_{ll} \end{bmatrix}, \tag{3.10}$$

where $2 \le k \le l-3$, $f_{ij} \in \mathbb{R}$, $1 \le i, j \le l$, $f_{j,j+1} \ne 0$, $j = 1, \ldots, l-1$, $f_{k+1,k} = g_{22} f_{k,k+1}$, $f_{k+2,k+1} = g_{11} f_{k+1,k+2}$, *and let*

$$H = diag(h_{11}, \ldots, h_{ll}), \quad 0 \ne h_{jj} \in \mathbb{R}, \ 1 \le j \le l. \tag{3.11}$$

Then, given any real coupled self-adjoint BC (1.9) satisfying $k_{12} \ne 0$, *the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.9). Moreover, given a fixed partition (2.1) of* J, *it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.9) provided, with the notation in (2.4). And all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.9) with all possible choices of the parameters.*

Proof. We choose $m = k$, $n = l - k - 1$ and fix q_m and w_m, then proceed similarly as Theorem 3.3. □

Theorem 3.6. *Let* $l > 4$, C, D *be any matrices associated with the transmission condition (1.4) which satisfying* $det(C) = \rho > 0$, $det(D) = \theta > 0$, *let* $G = (g_{ij}) = -D^{-1}C$, *suppose* $g_{12} = 0$, *and let F be an* $l \times l$ *"almost symmetric" matrix which is tridiagonal except for nonzero entries* $f_{1l} = f_{l1}$

$$F = \begin{bmatrix} f_{11} & f_{12} & & & & & & & f_{1l} \\ f_{12} & f_{22} & f_{23} & & & & & & \\ & \cdots & \cdots & \cdots & & & & & \\ & & f_{k,k-1} & f_{k,k} & f_{k,k+1} & & & & \\ & & & f_{k+1,k} & f_{k+1,k+1} & f_{k+1,k+2} & & & \\ & & & & f_{k+2,k+1} & \cdots & \cdots & & \\ & & & & & \cdots & \cdots & \cdots & \\ & & & & & & f_{l-1,l-2} & f_{l-1,l-1} & f_{l-1,l} \\ f_{1l} & & & & & & & f_{l-1,l} & f_{ll} \end{bmatrix}, \tag{3.12}$$

where $2 \le k \le l-3$, $f_{ij} \in \mathbb{R}$, $1 \le i, j \le l$, $f_{j,j+1} \ne 0$, $j = 1, \ldots, l-1$, $f_{k+1,k} = g_{22} f_{k,k+1}$, $f_{k+2,k+1} = g_{11} f_{k+1,k+2}$ *and let*

$$H = diag(h_{11}, \ldots, h_{ll}), \quad 0 \ne h_{jj} \in \mathbb{R}, \ 1 \le j \le l. \tag{3.13}$$

Then, given any real coupled self-adjoint BC (1.9) satisfying $k_{12} = 0$, *the matrix eigenvalue problem (3.1) has representations as SLPs with transmission conditions of Atkinson type in the form of SLP (1.2), (1.4), (1.9). Moreover, given a fixed partition (2.1) of* J, *it has a unique representation in the form of SLP with transmission condition (2.60), (1.4), (1.9) provided, with the notation in (2.4). And all Sturm–Liouville representations of problem (3.1) are given by the corresponding equivalent families of SLP with transmission condition (2.60), (1.4), (1.9) with all possible choices of the parameters.*

Proof. We choose $m = k$, $n = l - k$ and fix q_0, w_0 and q_m, w_m, then proceed similarly as Theorem 3.3. □

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