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Another Approach to the Existence of Value Functions of Stochastic Differential Games

Andrzej Święch

School of Mathematics, Georgia Institute of Technology, Atlanta, Georgia 30332

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The existence of value functions for general two-player, zero-sum stochastic differential games has been obtained by Fleming and Souganidis. In this paper we present a new approach to this problem. We prove optimality inequalities of dynamic programming for viscosity sub- and supersolutions of the associated Bellman–Isaacs equations. These inequalities are well known for deterministic differential games but are new for stochastic differential games. It then easily follows that value functions are the unique viscosity solutions of the Bellman–Isaacs equations and satisfy the principle of dynamic programming. The results presented here are not the same as those of Fleming and Souganidis because we work with different reference spaces and the independence of value functions of the choice of reference spaces is not clear to us. © 1996 Academic Press, Inc.

1. INTRODUCTION AND PRELIMINARIES

Let $(\Omega, \mathscr{F}, \mathscr{F}_t, P)$ be a probability space with a right-continuous filtration of complete σ-fields and let W be an n_1-dimensional $\mathscr{F}_t$-Brownian motion. We consider a stochastic initial value problem

$$\begin{cases} dX_s = b(X_s, Y_x, Z_s)\, ds + \sigma(X_s, Y_s, Z_s)\, dW_s & \text{for } s \in [0,\infty) \\ X_0 = x & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $b : \mathbb{R}^n \times \mathscr{Y} \times \mathscr{Z} \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \times \mathscr{Y} \times \mathscr{Z} \to \mathscr{S}(n \times n_1)$, the set of $n \times n_1$ matrices and $\mathscr{Y}, \mathscr{Z}$ are complete, separable metric spaces. With (1.1) we associate the pay-off functional

$$J(x; Y, Z) = E\left\{ \int_0^\infty e^{-\lambda s} h(X_s, Y_s, Z_s)\, ds \right\}, \tag{1.2}$$

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where $h : \mathbb{R}^n \times \mathscr{Y} \times \mathscr{Z} \to \mathbb{R}$, $\lambda > 0$, and E denotes the expected value. In what follows we will refer to (1.1) and (1.2) as infinite horizon stochastic differential game (SDG) with state variable in $\mathbb{R}^n$.

The existence of value functions of a general two-player, zero-sum stochastic differential game has been proved by W. H. Fleming and P. E. Souganidis in [7] and we refer the reader to this paper and [8] (see also [19]) for the description of earlier results and to [1, 4, 5, 9−11] for more information about differential games. The process employed in [7] was based upon working directly with the value functions (see the definitions below) to prove that they satisfy the dynamic programming principle and then showing that they solve the associated Bellman−Isaacs equations. This turned out to create serious measurability problems and the result was obtained with the help of a discretization procedure. The same method was used by M. E. Katsoulakis in [15] to prove representation formulas for solutions of second order parabolic equations. In this paper we would like to present a different approach to the existence of value functions which in a sense is opposite to that of [7]. We start with solutions of the upper and lower Bellman−Isaacs equations which exist by the general theory and prove that they must satisfy certain optimality inequalities (see [18] for the deterministic case and also [6, 16, 17] for the case of stochastic control) which in turn yield that solutions are equal to the value functions. These so-called sub- and superoptimality inequalities of dynamic programming are interesting for their own. The proofs presented here use some ideas from [21] and the proof of dynamic programming principle for stochastic control given in [6]. We employ general PDE and stochastic methods, in particular approximations of solutions of Bellman−Isaacs equations by inf- and sup-convolutions and stochastic processes by ‘‘non-degenerate’’ ones.

The results presented here are not exactly the same as those in [7]. They are somehow complementary. Fleming and Souganidis in [7] worked with the canonical sample space for the Brownian motion. At the end of the paper they hinted at another approach based on a full discretization in time and space but the independence of value functions of the choice of a sample space is not clear to us. This issue should be resolved. Our results may depend in some sense on the sample space. More precisely, given an initial sample space we embed it into a bigger one for which we have the optimality principles, existence of value functions, and all results are independent of the new reference spaces. The drawback of the approach presented here seems to be the rather slim possibility of extending it to the infinite dimensional setting while the method of [7] works in certain cases (see [22]). Finally we mention that some of the assumptions on the data could be relaxed but we do not attempt to do so.

We will write $|\cdot|$ for the norm in $\mathbb{R}, \mathbb{R}^n$, the space of matrices, and the Lebesgue measure in $\mathbb{R}^n$, the choice being obvious from the context, and $\|\cdot\|_n$ for the L^n norm in $\mathbb{R}^n$. We say that a nondecreasing function $\rho : [0,\infty) \to [0,\infty)$ is a modulus if ρ is continuous, subadditive, nondecreasing, and $\rho(0) = 0$. A continuous function $\rho : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a local modulus if it is nondecreasing in both arguments, subadditive in the first argument, and for every $s \geq 0$, $\rho(0, s) = 0$. We write $B_r(x)$ for the ball of radius r centered at x. For a metric space H we denote by $\mathrm{BUC}(H)$ the space of bounded and uniformly continuous functions on H and by $\mathscr{B}(H)$ the Borel σ-algebra in H.

We assume that b, σ, h are uniformly continuous functions such that there is a constant L such that

$$\begin{gathered} |b(x_1, y, z) - b(x_2, y, z)|, \\ |\sigma(x_1, y, z) - \sigma(x_2, y, z)| \leq L|x_1 - x_2| \\ |h(x_1, y, z) - h(x_2, y, z)| \leq \rho(|x_1 - x_2|) \end{gathered} \tag{1.3}$$

for every $(x_1, x_2, y, z) \in \mathbb{R}^n \times \mathbb{R} \times \mathscr{Y} \times \mathscr{Z}$, and

$$|b(x, y, z)|, |\sigma(x, y, z)|, |h(x, y, z)| \leq L \quad \text{for every } (x, y, z) \in \mathbb{R}^n \times \mathscr{Y} \times \mathscr{Z}, \tag{1.4}$$

where ρ is a modulus.

We need to introduce the sample space we will be working with. We take an n-dimensional Wiener process $\tilde{W}$ independent of $\mathscr{F}_t$ and consider a new $n_1 + n$-dimensional Wiener process $\overline{W} = (W, \tilde{W})$ defined on a product space. $\overline{W}$ is progressively measurable with respect to a new $\overline{\mathscr{F}}_t$ into which $\mathscr{F}_t$ embeds naturally, and therefore W is also an $\overline{\mathscr{F}}_t$-Brownian motion. We refer the reader to [16, 6] for more on the construction. We will be using P to denote probability on a new space. For $\gamma \geq 0$ we define $\sigma^\gamma(x, y, z)$ to be an $n \times (n_1 + n)$-matrix whose first n_1 columns form the matrix $\sigma(x, y, z)$ and columns $n_1 + 1, \ldots, n_1 + n$ form a matrix γI. Matrices σ^γ give rise to the stochastic differential equations associated with (1.1)

$$\begin{cases} dX_s^\gamma = b(X_2^\gamma, Y_s, Z_s)\, ds + \sigma^\gamma(X_s^\gamma, Y_s, Z_s)\, d\overline{W}_s & \text{for } s \in [0,\infty) \\ X_0^\gamma = x_0. \end{cases} \tag{1.5}$$

Solutions of (1.5) are "nondegenerate" processes since as it is easy to see for $a^\gamma = \sigma^\gamma(\sigma^\gamma)^*$ we have

$$\langle a^\gamma \xi, \xi \rangle \geq \gamma^2 |\xi|^2.$$

We also point out that from the uniqueness of solutions of (1.5) it follows that if X solves (1.1) then it also solves (1.5) with $\gamma = 0$.

Admissible controls and strategies, and value functions of our (SGD) are defined in the following way.

DEFINITION 1.1. An admissible control Y (respectively Z) for player I (respectively II) is an $\overline{\mathscr{F}}_t$-progressively measurable process taking its values in $\mathscr{Y}$ (respectively $\mathscr{Z}$). The set of all admissible controls for player I (respectively II) is denoted by M (respectively N).

We say that controls $Y^1, Y^2 \in M$ are equal on $[0, t]$ if $P(Y_s^1 = Y_s^2$ for a.e. $s \in [0, t]) = 1$. Controls in N are identified the same.

DEFINITION 1.2. An admissible strategy α (respectively β) for player I (respectively II) is a mapping $\alpha : N \to M$ (respectively $\beta : M \to N$) such that if $Z = \tilde{Z}$ (respectively $Y = \tilde{Y}$) on $[0, s]$ then $\alpha[Z] = \alpha[\tilde{Z}]$ (respectively $\beta[Y] = \beta[\tilde{Y}]$) on $[0, s]$ for every $s \in [0, \infty)$. The set of all admissible strategies for player I (respectively II) is denoted by Γ (respectively Δ).

DEFINITION 1.3. The lower value of the (SDG) is given by

$$V(x) = \inf_{\beta \in \Delta} \sup_{Y \in M} J(x; Y, \beta[Y]). \tag{1.6}$$

The upper value of the (SDG) is given by

$$U(x) = \sup_{\alpha \in \Gamma} \inf_{Z \in N} J(x; \alpha[Z], Z). \tag{1.7}$$

We are going to prove that the lower and upper value functions are viscosity solutions of the associated Bellman–Isaacs (BI) equations. More precisely, we define the lower value (BI) equation as

$$\lambda u + H^-(x, Du, D^2u) = 0 \qquad \text{for } x \in \mathbb{R}^n, \tag{1.8}$$

and the upper value (BI) equation as

$$\lambda u + H^+(x, Du, D^2u) = 0 \qquad \text{for } x \in \mathbb{R}^n, \tag{1.9}$$

where for a symmetric $n \times n$ matrix A and $p, x \in \mathbb{R}^n$, H^- and H^+ are defined by

$$H^-(x, p, A) = \inf_{y \in \mathscr{Y}} \sup_{z \in \mathscr{Z}} \Big\{ -\frac{1}{2}\operatorname{tr}(a(x, y, z)A) \\ -\langle b(x, y, z), p\rangle - h(x, y, z)\Big\}, \quad (1.10)$$

and

$$H^+(x, p, A) = \sup_{z \in \mathscr{Z}} \inf_{y \in \mathscr{Y}} \Big\{ -\frac{1}{2}\operatorname{tr}(a(x, y, z)A) \\ -\langle b(x, y, z), p\rangle - h(x, y, z)\Big\}, \quad (1.11)$$

where $a = \sigma\sigma^*$. We refer the reader to [3] for the definition and properties of viscosity solutions.

2. SUB- AND SUPEROPTIMALITY PRINCIPLES AND THE EXISTENCE OF VALUE FUNCTIONS

In the theorem below we prove the optimality inequalities of dynamic programming for the upper and lower value functions.

THEOREM 2.1. *Let* (1.3) *and* (1.4) *be satisfied and let* $\lambda \geq 0$. *Let* $x_0 \in \mathbb{R}^n$, $T \geq 0$. *Let* $u \in \mathrm{BUC}(\mathbb{R}^n)$. *Then*:

(i) *If u is a viscosity subsolution of* (1.9) *then*

$$u(x_0) \leq \sup_{\alpha \in \Gamma} \inf_{Z \in N} E\Big\{\int_0^T s^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u(X_T)\Big\}, \quad (2.1)$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Y = \alpha[Z]$ *for* $Z \in N$.

(ii) *If u is a viscosity supersolution of* (1.9) *then*

$$u(x_0) \geq \sup_{\alpha \in \Gamma} \inf_{Z \in N} E\Big\{\int_0^T s^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u(X_T)\Big\}, \quad (2.2)$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Y = \alpha[Z]$ *for* $Z \in N$.

(iii) *If u is a viscosity subsolution of* (1.8) *then*

$$u(x_0) \leq \inf_{\beta \in \Delta} \sup_{Y \in M} E\Big\{\int_0^T e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda T} u(X_T)\Big\}, \quad (2.3)$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Z = \beta[Y]$ *for* $Y \in M$.

(iv) *If u is a viscosity supersolution of* (1.8) *then*

$$u(x_0) \geq \inf_{\beta\in\Delta} \sup_{Y\in M} E\left\{\int_0^T e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda T} u(X_T)\right\}, \quad (2.4)$$

where X is the solution of (1.1) *with $X_0 = x_0$ and $Z = \beta[Y]$ for $Y \in M$.*

As an immediate consequence of the above theorem we obtain that solutions of Bellman–Isaacs equations satisfy the dynamic programming principle and therefore we have the existence of value functions for the (SDG).

COROLLARY 2.2. *Let* (1.3) *and* (1.4) *be satisfied and let* $\lambda \geq 0$. *Let* $x_0 \in \mathbb{R}^n$, $T \geq 0$. *Let* $u \in \mathrm{BUC}(\mathbb{R}^n)$ *be a viscosity solution of* (1.9) *and* $v \in \mathrm{BUC}(\mathbb{R}^n)$ *be a viscosity solution of* (1.8). *Then the dynamic programing principle holds, i.e.,*

$$u(x_0) = \sup_{\alpha\in\Gamma} \inf_{Z\in N} E\left\{\int_0^T e^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u(X_T)\right\}, \quad (2.5)$$

where X is the solution of (1.1) *with $X_0 = x_0$ and $Y = \alpha[Z]$ for $Z \in N$ and*

$$v(x_0) = \inf_{\beta\in\Delta} \sup_{Y\in M} E\left\{\int_0^T e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda T} v(X_T)\right\}, \quad (2.6)$$

where X is the solution of (1.1) *with $X_0 = x_0$ and $Z = \beta[Y]$ for $Y \in M$. In particular, if $\lambda > 0$, U is the unique bounded viscosity solution of* (1.9), *V is the unique bounded viscosity solution of* (1.8), *and if the Isaacs condition $H^+ = H^-$ holds then the* (*SDG*) *has a value.*

We begin the proof of Theorem 2.1 with a lemma. It holds under much more general assumptions but since we only need it in this form we make it as simple as possible for the clarity of argument.

LEMMA 2.3. *Theorem* 2.1 *holds for $u \in C^2(\mathbb{R}^n)$ such that u, Du, D^2u are bounded and Lipschitz continuous.*

Remark 2.4. As it will be obvious from the proof, Lemma 2.3 is independent of the choice of a sample space so we may as well work with the original one.

Proof of Lemma 2.3. We only prove (iii) and (iv) since (i) and (ii) are proved the same. Let $x_0 \in \mathbb{R}^n$, $T \geq 0$. Let K be both the Lipschitz constant and infinity norms of u, Du, D^2u. For a positive integer m denote $t = T/m$.

Proof of (iii). Fix $Z \in N$. Choose $y_1 \in \mathscr{Y}$ such that

$$\sup_{z \in \mathscr{Z}} \Big\{ -\frac{1}{2}\operatorname{tr}\big(a(x_0, y_1, z) D^2 u(x_0)\big) - \langle f(x_0, y_1, z), Du(x_0)\rangle + \lambda u(x_0) - h(x_0, y_1, z) \Big\} \le \frac{1}{m}. \quad (2.7)$$

Denote

$$L^s u(x) = -\frac{1}{2}\operatorname{tr}\big(a(x, y_1, Z_s) D^2 u(x)\big) - \langle f(x, y_1, Z_s), Du(x)\rangle + \lambda u(x)$$

for $0 \le s \le t$. If X is the solution of (1.1) on $[0, t]$ with $X_0 = x_0, Y^m \equiv y_1$, and Z by Ito's formula we have

$$u(x_0) = E\Big\{ \int_0^t e^{-\lambda s} L^s u(X_s)\, ds + e^{-\lambda t} u(X_t) \Big\}. \quad (2.8)$$

A standard martingale inequality gives us that for any solution X of (1.1)

$$P\Big(\sup_{0 \le s \le t} |X_s - x| \ge t^{1/4} \Big) \le K_1 t \quad (2.9)$$

for some constant K_1 independent of $Y \in M$ and $Z \in N$. Combining (2.8), (2.9), using the assumptions on u, and then (2.7) we obtain

$$\begin{aligned} u(x_0) &\le E\Big\{ \int_0^t e^{-\lambda s} L^s u(x_0)\, ds + e^{-\lambda t} u(X_t) \Big\} + K_2 t^{5/4} \\ &\le E\Big\{ \int_0^t e^{-\lambda s} h(x_0, Y_s^m, Z_s)\, ds + e^{-\lambda t} u(X_t) \Big\} + K_2 t \Big(\frac{1}{m} + t^{1/4} \Big) \\ &\le E\Big\{ \int_0^t e^{-\lambda s} h(X_s, Y_s^m, Z_s)\, ds + e^{-\lambda t} u(X_t) \Big\} + K_3 t \Big(\frac{1}{m} + \rho_1(t) \Big), \end{aligned} \quad (2.10)$$

where K_3 is a constant which depends only on L, K, K_1, and ρ_1 is a modulus depending on ρ. The first step has been accomplished and now we need to extend Y^m. To do this we proceed with a kind of construction employed in [5, 7]. Define

$$\Lambda(x, y) = \sup_{z \in \mathscr{Z}} \Big\{ -\frac{1}{2}\operatorname{tr}\big(a(x, y, z) D^2 u(x)\big) - \langle f(x, y, z), Du(x)\rangle + \lambda u(x) - h(x, y, z) \Big\}.$$

We notice that Λ is uniformly continuous on $\mathbb{R}^n \times \mathscr{Y}$. Since $\mathscr{Y}$ is separable we can therefore find a countable sequence $\{y_i\}_{i=1}^{\infty}$ in $\mathscr{Y}$ and a family of balls $\{B_{r_i}(x_i)\}_{i=1}^{\infty}$ covering $\mathbb{R}^n$ such that

$$\Lambda(x, y_i) \le \frac{1}{m} \qquad \text{if } x \in B_{r_i}(x_i).$$

Define a map $\psi : \mathbb{R}^n \to \mathscr{Y}$ by

$$\psi(x) = y_k \qquad \text{if } x \in B_{r_k}(x_k) \setminus \bigcup_{i=1}^{k-1} B_{r_i}(x_i).$$

This is a $(\mathscr{B}(\mathbb{R}^n), \mathscr{B}(\mathscr{Y}))$ measurable map. Moreover

$$\Lambda(x, \psi(x)) \le \frac{1}{m} \qquad \text{for every } x \in \mathbb{R}^n. \tag{2.11}$$

Define a new control Y^m on $[0, 2t)$ by

$$Y_s^m = \begin{cases} y_1 & \text{if } s \in [0, t) \\ \psi(X_t) & \text{if } s \in [t, 2t) \end{cases}. \tag{2.12}$$

Then Y^m is $\mathscr{F}_t$-progressively measurable and if $\tilde{X}$ is the solution of (1.1) on $[t, 2t]$ with Z, Y^m, and $\tilde{X}_t = X_t$, arguing as before we have

$$Ee^{-\lambda t} u\big(\tilde{X}_t\big) \le E\bigg\{\int_t^{2t} e^{-\lambda s} h\big(\tilde{X}_s, Y_s^m, Z_s\big)\, ds + e^{-2\lambda t} u\big(\tilde{X}_t\big)\bigg\}$$
$$+ K_3 t\left(\frac{1}{m} + \rho_1(t)\right). \tag{2.13}$$

Using the uniqueness of solutions of (1.1) and combining (2.10) and (2.13) we obtain

$$u(x_0) \le E\bigg\{\int_0^{2t} e^{-s} h(X_s, Y_s^m, Z_s)\, ds + e^{-2t} u(X_t)\bigg\} + 2K_3 t\left(\frac{1}{m} + \rho_1(t)\right),$$

where X is the solution of (1.1) on $[0, 2t]$ with Z, Y^m, and $X_0 = x_0$. Repeating the process m times yields us a piecewise linear random process $Y^m \in M$ such that if X is the solution of (1.1) with Z, Y^m, and $X_0 = x_0$ then

$$u(x_0) \le E\bigg\{\int_0^{T} e^{-\lambda s} h(X_s, Y_s^m, Z_s)\, ds + e^{-\lambda T} u(X_T)\bigg\}$$
$$+ K_3 T\left(\frac{1}{m} + \rho_1\left(\frac{T}{m}\right)\right). \tag{2.14}$$

Define a strategy $\alpha^m \in \Gamma$ by $\alpha^m[Z] = Y^m$. We notice that from the construction of α^m it follows that $\alpha^m[Z]_{|[it,(i+i)t)}$ depends only on $Z_{|[0,it)}$. Therefore by a rather routine construction, for every $\beta \in \Delta$ we can find $\tilde{Y} \in M$ and $\tilde{Z} \in N$ such that

$$\alpha^m[\tilde{Z}] = \tilde{Y}, \qquad \text{and} \qquad \tilde{Z} = \beta[\tilde{Y}]. \tag{2.15}$$

To find such controls we proceed inductively. $\tilde{Y}_{|[0,t)} = y_1$ and then let $\tilde{Z}_{|[0,t)} = \beta[\tilde{Y}]_{|[0,t)}$ (the value of $\beta[\tilde{Y}]$ on $[0,t)$ only depends on $\tilde{Y}_{|[0,t)}$). Having defined $\tilde{Z}$ and $\tilde{Y}$ on $[0,it)$ we known what $\tilde{Y}_{|[0,(i+1)t)}$ is (see (2.12)) and then we set $\tilde{Z}_{|[0,(i+1)t)} = \beta[\tilde{Y}]_{|[0,(i+1)t)}$. One easily checks that such constructed $\tilde{Y}$ and $\tilde{Z}$ satisfy (2.15). Using this fact in (2.14) and then letting $m \to \infty$ we therefore obtain that for every $\beta \in \Delta$

$$u(x_0) \le \sup_{Y\in M} E\left\{\int_0^T e^{-\lambda s} h(X_s, Y, \beta[Y])\, ds + e^{-\lambda T} u(X_T)\right\} \tag{2.16}$$

and (2.3) follows.

Proof of (iv). The proof is similar to the proof of (iii). Fix $Y \in M$ and denote

$$\Lambda(x,y,z) = -\frac{1}{2}\operatorname{tr}\big(a(x,y,z)D^2u(x)\big) - \langle f(x,y,z), Du(x)\rangle \\ + \lambda u(x) - h(x,y,z).$$

Again Λ is uniformly continuous on $\mathbb{R}^n \times \mathscr{Y} \times \mathscr{Z}$ and since

$$\inf_{y\in\mathscr{Y}} \sup_{z\in\mathscr{Z}} \Lambda(x,y,z) \ge 0$$

for every $x \in \mathbb{R}^n$ we can therefore find a countable sequence $\{z_i\}_{i=1}^\infty$ in $\mathscr{Z}$ and a family $\{B_{r_i}(x_i) \times B_{\tilde{r}_i}(y_i)\}_{i=1}^\infty$ covering $\mathbb{R}^n \times \mathscr{Y}$ such that

$$\Lambda(x,y,z_i) \ge -\frac{1}{m} \qquad \text{if } (x,y) \in B_{r_i}(x_i) \times B_{\tilde{r}_i}(y_i).$$

Define a map $\psi : \mathbb{R}^n \times \mathscr{Y} \to \mathscr{Z}$ by

$$\psi(x,y) = z_k \qquad \text{if } (x,y) \in B_{r_k}(x_k) \times B_{\tilde{r}_k}(y_k) \setminus \bigcup_{i=1}^{k-1} B_{r_i}(x_i) \times B_{\tilde{r}_i}(y_i).$$

This is a $(\mathscr{B}(\mathbb{R}^n \times \mathscr{Y}), \mathscr{B}(\mathscr{Z}))$ measurable map and

$$\Lambda(x, y, \psi(x, y)) \geq -\frac{1}{m} \qquad \text{for every } (x, y) \in \mathbb{R}^n \times \mathscr{Y}.$$

Define a control Z^m on $[0, t)$ by

$$Z_s^m = \psi(x_0, Y_s). \tag{2.17}$$

Z^m is obviously $\mathscr{F}_s$-progressively measurable. We argue as around (2.8)–(2.10) in the proof of (iii) to obtain

$$u(x_0) \geq E\left\{\int_0^t e^{-\lambda s} h(X_s, Y_s, Z_s^m)\, ds + e^{-\lambda t} u(X_t)\right\} - t\rho_1(t) \tag{2.18}$$

for some independent modulus ρ_1, where X is the solution of (1.1) on $[0, t]$ with $X_0 = x_0$, Y, and Z^m. This allows us to extend Z^m on $[0, 2t)$ by setting

$$Z_s^m = \psi(X_t, Y_s) \qquad \text{if } s \in [t, 2t)$$

and we continue the process. Therefore we can construct a control $Z^m \in N$ such that

$$u(x_0) \geq E\left\{\int_0^T e^{-\lambda s} h(X_s, Y_s, Z_s^m)\, ds + e^{-\lambda T} u(X_T)\right\} - \rho_2(t)$$

for some modulus ρ_2, where X is the solution of (1.1) on $[0, T]$ with $X_0 = x_0$, Y, and Z^m. Inequality (2.4) follows by setting $\beta^m[Y] = Z^m$ and letting $m \to \infty$. It is obvious from the construction that $\beta^m \in \Delta$.

Remark 2.5. We point out the fundamental difference between strategies α^m constructed in the proof of (iii) and β^m constructed in the proof of (iv). The $\alpha^m[Z]$ were piecewise constant processes and $\alpha^m[Z]_{|[0,(i+1)t)}$ depended only on $Z_{|[0, it)}$ for $i = 1, \dots, m-1$, while β^m could be any element of Δ.

Proof of Theorem 2.1. We only prove (i), the arguments for (ii), (iii), and (iv) being similar. For $\epsilon > 0$ let u_ϵ be the sup-convolution of u, i.e.,

$$u_\epsilon(x) = \sup_{\xi \in \mathbb{R}^n} \left\{u(\xi) - \frac{|\xi - x|^2}{2\epsilon}\right\}.$$

It is now rather standard to notice (see [3, 13]) that $u_\epsilon \to u$ uniformly in $\mathbb{R}^n$ as $\epsilon \to 0$, u_ϵ are bounded, Lipschitz continuous, semiconvex, and satisfy a.e. on $\mathbb{R}^n$ (see [12] for the precise argument, see also [13])

$$\lambda u_\epsilon(x) + H^+\big(x, Du_\epsilon(x), D^2u_\epsilon(x)\big) \le \rho_0(\epsilon), \tag{2.19}$$

for some modulus ρ_0. Given $\delta > 0$ let u_ϵ^δ denote the standard mollification of u_ϵ. Functions u_ϵ^δ are smooth, $u_\epsilon^\delta, Du_\epsilon^\delta, D^2u_\epsilon^\delta$ are bounded and Lipschitz continuous, $u_\epsilon^\delta \to u_\epsilon$ uniformly in $\mathbb{R}^n$, and $Du_\epsilon^\delta(x) \to Du_\epsilon(x)$, $D^2u_\epsilon^\delta(x) \to D^2u_\epsilon(x)$ for a.e. $x \in \mathbb{R}^n$ (see [14], also [2]). Moreover the u_ϵ^δ have the same Lipschitz and semiconvexity constants as u_ϵ (the semiconvexity constant is $1/2\epsilon$) and for any $\gamma > 0$ satisfy on $\mathbb{R}^n$

$$\lambda u_\epsilon^\delta(x) - \frac{\gamma^2}{2}\operatorname{tr} D^2u_\epsilon^\delta(x) + H^+\big(x, Du_\epsilon^\delta(x), D^2u_\epsilon^\delta(x)\big)$$
$$\le \rho_0(\epsilon) + g_\delta(x) + \frac{\gamma^2 n}{\epsilon}, \tag{2.20}$$

where the g_δ are uniformly continuous (with moduli of continuity possibly depending on δ), bounded, uniformly in δ, and $g_\delta(x) \to 0$ for a.e. $x \in \mathbb{R}^n$. The uniform boundedness of g_δ is a consequence of the uniform semiconvexity and uniform Lipschitz continuity of u_ϵ^δ. Denote $h_\delta(x,y,z) = h(x,y,z) + \rho_0(\epsilon) + g_\delta(x) + \gamma^2 n/\epsilon$. Applying Lemma 2.3 to u_ϵ^δ and (2.20) we have

$$u_\epsilon^\delta(x_0) \le \sup_{\alpha\in\Gamma}\inf_{Z\in N} E\left\{\int_0^T e^{-\lambda s}h_\delta(X_s^\gamma, \alpha[Z]_s, Z_s)\,ds + e^{-\lambda T}u_\epsilon^\delta(X_T^\gamma)\right\}, \tag{2.21}$$

where X^γ is the solution of (1.5) with $X_0^\gamma = x_0$ and $Y = \alpha[Z]$ for $Z \in N$, i.e.,

$$X_s^\gamma = x_0 + \int_0^s b(X_t^\gamma, \alpha[Z]_t, Z_t)\,dt + \int_0^s \sigma^\gamma(X_t^\gamma, \alpha[Z]_t, Z_t)\,d\overline{W}_t. \tag{2.22}$$

We will now pass to limits. Standard martingale inequalities give us that for every $\theta > 0$ we can choose a constant R_θ (independent of controls and strategies) such that

$$P\Big(\sup_{0\le s\le T}|X_s^\gamma| \ge R_\theta\Big) \le \theta. \tag{2.23}$$

We then take a set Ω_θ such that $|\Omega_\theta| \le \theta$ and $g_\delta \to 0$ uniformly on $B_{R\theta}(0)\setminus\Omega_\theta$. From [16, Theorem 2.3.4] (see also [20])

$$E\int_0^T \chi_{\Omega_\theta}(X_s^\gamma)\,ds \le N\|\chi_{\Omega_\theta}\|_n$$

for some independent constant $N = N(n, \gamma, L)$. Therefore, from the above and (2.23),

$$\left| E\int_0^T g_\delta(X_s^\gamma)\, ds \right| \le \rho_1(\delta, \gamma),$$

for some local modulus ρ_1 which does not depend on the controls and strategies. Using this and (2.21) we thus obtain

$$u_\epsilon(x_0) \le \sup_{\alpha \in \Gamma} \inf_{Z \in N} E\left\{ \int_0^T e^{-\lambda s} h(X_s^\gamma, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u_\epsilon(X_T^\gamma) \right\}$$
$$+ \rho_2(\delta, \gamma) + T\left(\rho_0(\epsilon) + \frac{\gamma^2 n}{\epsilon} \right) \tag{2.24}$$

for some local modulus ρ_2. Moment estimates (see [16, Theorem 2.5.9]) yield

$$E\left(\max_{0 \le s \le T} |X_s - X_s^\gamma|^2 \right) \le C_1 \gamma^2 E\int_0^T |X_t^\gamma|^2\, dt \le C_2 \gamma^2, \tag{2.25}$$

where X is the solution of (1.1). (We remind that we do not indicate the dependence of constants on T and x_0 since they are fixed.) Hence (2.24) and (2.25) finally yield

$$u_\epsilon(x_0) \le \sup_{\alpha \in \Gamma} \inf_{Z \in N} E\left\{ \int_0^T e^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u_\epsilon(X_T) \right\}$$
$$+ \rho_3(\delta, \gamma) + T\left(\rho_0(\epsilon) + \frac{\gamma^2 n}{\epsilon} \right) \tag{2.26}$$

for some new local modulus ρ_3. We obtain (2.1) upon passing to limits in (2.26) as δ, γ, and then $\epsilon \to 0$. ∎

We point out that we have actually proved stronger statements of Theorem 2.1. Careful examination of the proof of Lemma 2.3(iii) shows that (2.16) can be rephrased as follows. For every $\epsilon > 0$ and $\beta \in \Delta$ there exists $Y \in M$ such that

$$u(x_0) \le E\left\{ \int_0^T e^{-\lambda s} h(X_s, Y, \beta[Y])\, ds + e^{-\lambda T} u(X_T) \right\} + \rho(\epsilon)$$

for some modulus ρ which does not depend on β. Moreover, the control Y and ρ are also good for all times $0 \le \tau \le T$, i.e.,

$$u(x_0) \le \inf_{0 \le \tau \le T} E\left\{ \int_0^\tau e^{-\lambda s} h(X_s, Y, \beta[Y])\, ds + e^{-\lambda \tau} u(X_\tau) \right\} + \rho(\epsilon).$$

Therefore for every $T \geq 0$ we obtain

$$u(x_0) \leq \inf_{\beta \in \Delta} \sup_{Y \in M} \inf_{0 \leq r \leq T} E\left\{ \int_0^\tau e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda\tau} u(X_\tau) \right\},$$

and choosing $\tau = 0$ yields the equality above. Similar observation can be made about the proof of Lemma 2.3(iv). Since the approximation procedure in the proof of Theorem 2.1 is independent of the control and strategies for bounded T we obtain the following corollary.

COROLLARY 2.6. *Let* (1.3) *and* (1.4) *be satisfied and let* $\lambda > 0$. *Let* $x_0 \in \mathbb{R}^n$, *and* $u \in \mathrm{BUC}(\mathbb{R}^n)$. *Then*:

(i) *If u is a viscosity subsolution of* (1.9) *then*

$$u(x_0) = \sup_{\alpha \in \Gamma} \inf_{Z \in N} \inf_{0 \leq T < \infty} E\left\{ \int_0^T s^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u(X_T) \right\},$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Y = \alpha[Z]$ *for* $Z \in N$.

(ii) *If u is a viscosity supersolution of* (1.9) *then*

$$u(x_0) = \sup_{\alpha \in \Gamma} \inf_{Z \in N} \sup_{0 \leq T < \infty} E\left\{ \int_0^T s^{-\lambda s} h(X_s, \alpha[Z]_s, Z_s)\, ds + e^{-\lambda T} u(X_T) \right\},$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Y = \alpha[Z]$ *for* $Z \in N$.

(iii) *If u is a viscosity subsolution of* (1.8) *then*

$$u(x_0) = \inf_{\beta \in \Delta} \sup_{Y \in M} \inf_{0 \leq T < \infty} E\left\{ \int_0^T e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda T} u(X_T) \right\},$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Z = \beta[Y]$ *for* $Y \in M$.

(iv) *If u is a viscosity supersolution of* (1.8) *then*

$$u(x_0) = \inf_{\beta \in \Delta} \sup_{Y \in M} \sup_{0 \leq T < \infty} E\left\{ \int_0^T e^{-\lambda s} h(X_s, Y_s, \beta[Y]_s)\, ds + e^{-\lambda T} u(X_T) \right\},$$

where X is the solution of (1.1) *with* $X_0 = x_0$ *and* $Z = \beta[Y]$ *for* $Y \in M$.

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