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Approximation Theory on $SU(2)$

DAVID L. RAGOZIN[1]

Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

INTRODUCTION

In the present paper we prove direct and inverse approximation theorems of the Jackson and Bernstein type for functions defined on the compact group $G = SU(2)$. The main results are Theorems 3.6, 4.1, and 4.2. In distinction to the methods of [*8*], where we proved Jackson type theorems for functions on any compact manifold embedded in Euclidean space, our present proofs of the direct theorems are constructive. More specifically, we exhibit, for each k, a sequence of kernel functions, $\{{}_kK_n\}$, such that the convolution, ${}_kK_n * f$, for any $f \in C^k(G)$, is a polynomial of degree at most n, and $\|f - {}_kK_n * f\|_\infty = O(n^{-k}\,\omega(1/n))$, where ω is an appropriate modulus of smoothness of the kth derivatives of f.

Our techniques for the case $k = 0$ derive from Korovkin's work on trigonometric approximation. They are the same, although not necessarily recognizably so, as those used by Newman and Shapiro in [*7*].

One interesting outcome of the present work is the fact that for constructive proofs of Jackson type theorems on groups, the case $k = 0$ is crucial, since the case of $f \in C^k(G)$, $k > 0$, can be reduced to it by a simple method. (See the development following Theorem 3.5.) In the case of functions on the unit circle, this was known, e.g. [*1*], p. 146, but appeared to depend on the fact that every trigonometric polynomial without a constant term was the derivative of another such polynomial. On multi-dimensional spaces, without a preferred derivative operator, this approach would seem impossible. However, a careful examination of the induction arguments in the proofs for the circle revealed that they could be rearranged to yield a direct development, independent of the fact about derivatives.

In [*8*] we proved inverse theorems of Bernstein type for a class of manifolds which includes all compact Lie groups. As in the classical case of the circle group, a key step was an estimate of the norm of some derivative operators acting on polynomials. Here, where the requisite knowledge of Lie theory is minimal, we prove a special case of the relevant inequality (Theorem 4.1).

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[1] Most of the results of this paper are contained in the author's Ph.D. dissertation written at Harvard University under the direction of Professor A. M. Gleason. The author would like to thank Professors Gleason and R. A. Mayer.

Now we give a brief summary of the individual sections. In Section 1, we introduce the relevant notation, and briefly describe two invariant metrics on G. In Section 2, we use one of these metrics to define the first- and second-order moduli of smoothness, and we give some elementary properties of these. Sections 3 and 4 contain the main direct and inverse theorems for polynomial approximation on G.

1. Metrics on $SU(2)$

Let $G = SU(2)$ be the group of 2×2 unitary matrices with determinant 1. An element g of G can be written

$$g = \begin{pmatrix} a(g) & \bar{b}(g) \\ b(g) & \bar{a}(g) \end{pmatrix}, \qquad |a(g)|^2 + |b(g)|^2 = 1,$$

where $a(g)$, $b(g)$, $\bar{a}(g)$, $\bar{b}(g)$ are the coordinate functions of the identity representation of G. As a subset of $\mathbf{C}^4$, G inherits the Euclidean metric. The Euclidean distance between g and $h \in G$, which we shall write as $|g-h|$, is given by each of the following equal expressions:

$$\begin{aligned} |g-h|^2 &= \operatorname{Tr}(g-h)(g-h)^* = 4 - 2\operatorname{Tr}(gh^{-1}) \\ &= 2(|a(g)-a(h)|^2 + |b(g)-b(h)|^2). \end{aligned}$$

Thus, the map $g \mapsto (\sqrt{2}a(g), \sqrt{2}b(g))$ is an isometric map of G with the Euclidean metric onto the 3-sphere of radius $\sqrt{2}$ in $\mathbf{C}^2$.

Another metric on G arises from the Riemannian metric on the Lie algebra, $\mathfrak{g}$, of G, determined by the Killing form. Recall that $\mathfrak{g}$ constists of all 2×2 skew Hermitian matrices with zero trace. A matrix $D \in \mathfrak{g}$ determines a left-invariant vector field according to

$$Df(g) = \frac{d}{dt} f(g \exp tD)|_{t=0}.$$

The Killing form determines the Riemannian metric, $(\ ,\ )$, on G, given by

$$(D_i, D_j) = 4\operatorname{Tr}(D_i D_j^*) = -4\operatorname{Tr}(D_i D_j), \qquad D_i, D_j \in \mathfrak{g}$$

(see [3], p. 269). The associated arclength metric, ρ, is such that if $t \mapsto x \exp tD$, $t \in [0, 1]$, $D \in \mathfrak{g}$, is a minimal geodesic from x to y, then $\rho(x, y) = \|D\| = (D, D)^{1/2}$. Under the map of G onto the 3-sphere of radius $\sqrt{2}$, ρ goes over to a constant times the great circle metric on the sphere. The constant can be found by noting that if

$$D = \begin{pmatrix} i\pi & 0 \\ 0 & -i\pi \end{pmatrix},$$

then for $t \in [0,1]$, $t \mapsto \exp tD$ is a minimal geodesic from e to $-e$. Hence $\rho(e,-e) = \|D\| = 2(\sqrt{2}\pi) = 2$ (great circle distance between poles on a sphere of radius $\sqrt{2}$). Moreover, by comparing the great circle metric with the Euclidean metric on a sphere, it is easy to show that

$$2|g-h| \leqslant \rho(g,h) \leqslant \pi|g-h|.$$

We remark that both these metrics are left and right invariant.

2. Moduli of Smoothness

By use of the metric ρ, we can introduce moduli of smoothness for functions on G. First we recall that the left and right translation operators, $L(g)$ and $R(g)$, are defined by

$$L(g)f(x) = f(g^{-1}x), \qquad R(g)f(x) = f(xg), \qquad f \in C(G).$$

Now we define the first- and second-order *moduli of smoothness*, by

$$\omega_1(f;h) = \sup\{\|f - L(g)f\|_\infty \mid \rho(g,e) \leqslant h\}.$$
$$\omega_2(f;h) = \sup\{\|f - 2L(g)f + L(g^2)f\|_\infty \mid \rho(g,e) \leqslant h\}.$$

These two moduli of smoothness will suffice for our work.

The following properties of ω_j are proved just as in the case of periodic functions on the real line (See [5], pp. 47–48).

Proposition 2.1. *If $f \in C(G)$, then*

(i) *$\omega_j(f;h)$ is increasing,*
(ii) *$\omega_j(f;h) \to 0$ as $h \to 0$,*
(iii) *$\omega_j(f;nh) \leqslant n^j\omega_j(f;h)$, $\quad n = 1,2,\ldots,$*
(iv) *$\omega_j(f;\lambda h) \leqslant (1+\lambda)^j\omega_j(f;h)$, $\quad \lambda > 0,$*

for $j = 1,2$.

To deal with differentiable functions, we fix an orthonormal basis D_1, D_2, D_3 of $\mathfrak{g}$, where

$$\sqrt{8}D_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \sqrt{8}D_2 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad \sqrt{8}D_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Now, by induction on k, we introduce the following notation for functions f in $C^{k+1}(G)$:

$$\|f^{(k+1)}\|_\infty = \sum_1^3 \|(D_i f)^{(k)}\|_\infty,$$

$$\omega_j(f^{(k+1)};h) = \sum_1^3 \omega_j((D_i f)^{(k)};h).$$

The following standard estimates of ω_1 and ω_2 will be quite useful.

PROPOSITION 2.2.

(i) *If $f \in C(G)$, then $\omega^2(f; h) \leqslant 2\omega_1(f; h)$.*
(ii) *If $f \in C^1(G)$, then $\omega_1(f; h) \leqslant \|f^{(1)}\|_\infty h$.*
(iii) *If $f \in C^1(G)$, then $\omega_2(f; h) \leqslant \omega_1(f^{(1)}; h) h$.*
(iv) *If $f \in C^2(G)$, then $\omega_2(f; h) \leqslant \|f^{(2)}\|_\infty h^2$.*

Proof. (i) This follows from the inequalities

$$\begin{aligned}\|f - 2L(g)f + L(g^2)f\|_\infty &\leqslant \|f - L(g)f\|_\infty + \|L(g)(f - L(g)f\|_\infty \\ &\leqslant 2\omega_1(f; \rho(g, e)).\end{aligned}$$

(ii) For any $g, x \in G$, let $D \in \mathfrak{g}$ be chosen so that $t \mapsto x \exp tD$, $t \in [0, 1]$, is a minimal geodesic from x to $g^{-1}x$; in particular $\|D\| = \rho(x, g^{-1}x) = \rho(g, e)$. Then

$$\begin{aligned}|f(x) - f(g^{-1}x)| &= \left|\int_0^1 \frac{d}{dt} f(x \exp tD)\, dt\right| \\ &\leqslant \int_0^1 |Df(x \exp tD)| dt \leqslant \|Df\|_\infty.\end{aligned}$$

Since we can find $c_i \in \mathbf{R}$ with $D = \sum c_i D_i$ and $\rho(g, e) = \|D\| = (\sum c_i{}^2)^{1/2}$, we have

$$\begin{aligned}|f(x) - f(g^{-1}x) &\leqslant \sum |c_i| \|D_i f\|_\infty \leqslant (\sum c_i{}^2)^{1/2} (\sum \|D_i f\|_\infty{}^2)^{1/2} \\ &\leqslant \rho(g, e) \sum_1^3 \|D_i f\|_\infty.\end{aligned}$$

Hence,

$$\|f - L(g)f\|_\infty \leqslant \|f^{(1)}\|_\infty \rho(g, e), \tag{*}$$

which immediately yields (ii).

(iii) If we replace f by $f - L(g)f$ in (*), we get

$$\begin{aligned}\|f - 2L(g)f + L(g^2)f\|_\infty &\leqslant \|(f - L(g)f)^{(1)}\|_\infty \rho(g, e) \\ &= \left(\sum_1^3 \|D_i f - L(g) D_i f\|_\infty\right) \rho(g, e),\end{aligned} \tag{**}$$

since $D_i L(g) f = L(g) D_i f$ by the left invariance of $D_i \in \mathfrak{g}$. Now because $\|D_i f - L(g) D_i f\|_\infty \leqslant \omega_1(D_i f; \rho(g, e))$, (iii) follows immediately from (**).

(iv) (ii) applied to $D_i f$ shows that (**) yields (iv).

3. Direct Approximation Theorems on $SU(2)$

We shall consider the problem of approximating f in $C(G)$ by members of the classes

$$\mathscr{P}_n = \{\text{polynomials of degree at most } n \text{ in } a(g), b(g), \bar{a}(g), \bar{b}(g)\}.$$

These are the analogs of the trigonometric polynomials on the unit circle in $\mathbf{R}^2$. Moreover, if we realize G as a 3-sphere in $\mathbf{R}^4$, then $\mathscr{P}_n$ is just the class of ordinary polynomials of total degree at most n, restricted to G. As usual for $f \in C(G)$, we set

$$E_n(f) = \inf\{\|f - p_n\|_\infty | p_n \in \mathscr{P}_n\}.$$

In order to find efficient and constructive ways of approximating a function by elements of $\mathscr{P}_n$, we need to use some facts about $L^2(G;$ Haar measure) (Haar measure on G coincides with the usual normalized surface measure on the 3-sphere). $L^2(G)$ is a ring under convolution, where

$$f * h(x) = \int f(g) h(g^{-1} x)\, dg.$$

$\mathscr{P}_n$ is invariant under $L(g)$ and $R(g)$, hence it is a twosided ideal. Thus $H_n = \mathscr{P}_n \cap \mathscr{P}_{n-1}^{\perp}$ is an ideal; in fact H_n is the minimal twosided ideal in $L^2(G)$ corresponding to the unique irreducible representation of G of dimension $n+1$ ([6], p. 91). If χ_n is the character of this representation, then $\chi_n \perp \chi_m$, $n \neq m$, and χ_n is real. The orthogonal projection of $L^2(G)$ onto H_n is given by convolution with $(n+1)\chi_n$ (see [4], in particular §40). (We remark that if we consider G to be the 3-sphere, then H_n is just the space of spherical harmonics of degree n.)

We now adapt a technique of Korovkin for approximating on the circle group ([2], pp. 336–9 and [7]). Thus, we first examine some properties of positive kernels K_n in $\mathscr{P}_n$.

LEMMA 3.1. *If* $K_n = 1 + \sum_1^n r_k(k+1)\chi_k$ *and* $K_n \geqslant 0$, *then*

$$\int K_n(g) \rho^2(g, e)\, dg \leqslant 4\pi^2(1 - r_1).$$

Proof. From the inequality $\rho(x, y) \leqslant \pi|x - y|$, and the fact that $|x - y|^2 = 4 - 2\operatorname{Tr}(xy^{-1})$, we have

$$\rho^2(g, e) \leqslant \pi^2(4 - 2\operatorname{Tr}(g)).$$

Since $\chi_1(g) = \operatorname{Tr}(g)$ is the character of the identity representation,

$$\rho^2(g, e) \leqslant \pi^2(4 - 2\chi_1(g)).$$

Thus,

$$\int K_n(g)\rho^2(g,e)\,dg \leqslant \pi^2 \int \left(1+\sum_1^n r_k(k+1)\chi_k(g)\right)(4-2\chi_1(g))\,dg$$
$$= 4\pi^2(1-r_1)$$

by the orthogonality relations for the characters.

This leads to an estimate of $\|K_n * f - f\|_\infty$ for kernels of the given type. Since $\mathscr{P}_n$ is an ideal, $K_n * f$ will be in $\mathscr{P}_n$, and thus we get an upper bound for $E_n(f)$.

PROPOSITION 3.2. *Let $K_n = 1 + \sum_1^n r_{nk}(k+1)\chi_k$, and suppose $K_n \geqslant 0$. Then for any $d > 0$, and $f \in C(G)$,*

(i) $\|K_n * f - f\|_\infty \leqslant (1 + 4\pi^2 d^2(1-r_{n1}))\,\omega_2(f;\, 1/d)$,
(ii) $\|K_n * f - f\|_\infty \leqslant (1 + 2\pi d(1-r_{n1})^{1/2})\,\omega_1(f;\, 1/d)$.

Proof. (i) Since any $g \in G$ is conjugate to g^{-1}, and each χ_n is constant over conjugacy classes, we have $\chi_n(g) = \chi_n(g^{-1})$, and consequently $K_n(g) = K_n(g^{-1})$. Thus,

$$K_n * f(h) = \int K_n(g) f(g^{-1}h)\,dg = \int K_n(g^{-1}) f(g^{-1}h)\,dg = \int K_n(g) f(gh)\,dg,$$

by the invariance of Haar measure under inversion. Since

$$\int K_n(g)\,dg = 1,$$

we have

$$f(h) = \int K_n(g) f(h)\,dg.$$

Thus

$$|K_n * f(h) - f(h)| = \tfrac{1}{2}\left|\int K_n(g)(f(g^{-1}h) - 2f(h) + f(gh))\,dg\right|$$
$$\leqslant \tfrac{1}{2}\int K_n(g)|f(g^{-1}h) - 2f(h) + f(gh)|\,dg$$
$$\leqslant \tfrac{1}{2}\int K_n(g)\,\omega_2(f;\, \rho(g,e))\,dg,$$

as $K_n \geqslant 0$. Now, by Proposition 2.1 (iv), with $h = 1/d$, $\lambda = d\rho(g,e)$, we have

$$\omega_2(f;\rho(g,e)) \leqslant (1 + d\rho(g,e))^2\,\omega_2(f;1/d)$$
$$\leqslant 2(1 + d^2\rho^2(g,e))\,\omega_2(f;1/d).$$

Thus

$$\|K_n * f - f\|_\infty \leqslant \omega_2(f; 1/d) \int K_n(g)(1 + d^2 \rho^2(g, e))\, dg$$

$$= \omega_2(f; 1/d)(1 + d^2 \int K_n(g) \rho^2(g, e)\, dg)$$

$$\leqslant (1 + 4\pi^2 d^2(1 - r_{n1}))\omega_2(f; 1/d),$$

by Lemma 3.1.

(ii) is proved similarly, using the estimate

$$\int K_n(g) \rho(g) e\, dg \leqslant \left(\int K_n(g)\, dg\right)^{1/2} \left(\int K_n(g) \rho^2(g, e)\, dg\right)^{1/2}.$$

To apply this proposition, we shall exhibit a sequence of positive kernels, K_n, for which $1 - r_{n1} \leqslant C\, 1/n^2$.

LEMMA 3.3. *For each n, set*

$$K_{2n} = A_n(a_0 \chi_0 + a_1 \chi_1 + \cdots + a_n \chi_n)^2,$$

where

$$a_k = \sin \frac{k+1}{n+2} \pi,$$

and

$$A_n = (a_0{}^2 + a_1{}^2 + \cdots + a_n{}^2)^{-1}.$$

Then

$$K_{2n} = 1 + \sum_1^{2n} r_{2n,k}(k+1) \chi_k \geqslant 0, \qquad \textit{and} \qquad r_{2n,1} = \cos \frac{\pi}{n+2}.$$

Proof. First write

$$(a_0 \chi_0 + \cdots + a_n \chi_n)_2 = \sum_0^n a_i{}^2 \chi_i{}^2 + 2 \sum_{i<j} a_i a_j \chi_i \chi_j.$$

The Clebsch–Gordan series for G shows that if $i \leqslant j$, then

$$\chi_i \chi_j = \sum_{k=0}^{i} \chi_{j+i-2k}$$

([9], p. 128). Thus, in expressing $(\sum a_i \chi_i)^2$ as a sum of χ_i, we pick up χ_0 only from the $\chi_i{}^2$ terms, and χ_1 only from the terms $\chi_i \chi_{i+1}$, $i \leqslant n-1$. Also, the highest order term that appears is χ_{2n}, from the expansion of $\chi_n{}^2$. Thus,

$$\left(\sum_0^n a_i \chi_i\right)^2 = \left(\sum_0^n a_i{}^2\right) \chi_0 + \left(\sum_0^{n-1} a_i a_{i+1}\right) 2\chi_1 + \cdots + a_n{}^2 \chi_{2n}.$$

Also, since

$$\cos\frac{\pi}{n+2}a_i^2 = \cos\frac{\pi}{n+2}\sin\frac{i+1}{n+2}\pi\sin\frac{i+1}{n+2}\pi$$
$$= \tfrac{1}{2}\left(\sin\frac{i+2}{n+2}\pi + \sin\frac{i}{n+2}\pi\right)\sin\frac{i+1}{n+2}\pi$$
$$= \tfrac{1}{2}(a_i a_{i+1} + a_{i-1} a_i) \qquad (a_{-1} = a_{n+1} = 0),$$

we have

$$\cos\frac{\pi}{n+2}\sum_0^n a_i^2 = \sum_0^{n-1} a_i a_{i+1}.$$

Thus,

$$K_{2n} = 1 + \left(\cos\frac{\pi}{n+2}\right)2\chi_1 + \cdots + A_n \sin^2\frac{n+1}{n+2}\pi\chi_{2n}.$$

We remark that it is not difficult to show that the given choice of a_i maximizes $r_{2n,1} = \sum_1^{n-1} a_i a_{i+1}/\sum_1^n a_1^2$. Moreover, it is a mysterious fact that the same maximization problem occurs in connection with approximation on the circle (see [2], p. 337).

From here on, K_{2n} will stand for the function defined in Lemma 3.3. Also, for odd integers $2n+1$, we define $K_{2n+1} = K_{2n}$. Now we combine the two previous results to prove the analog of Jackson's theorem for continuous functions on $SU(2)$.

THEOREM 3.4. *If $f \in C(G)$, then*

(i) $E_n(f) \leqslant \|K_n * f - f\|_\infty \leqslant (1 + 8\pi^4)\,\omega_2(f; 1/n)$,
(ii) $E_n(f) \leqslant \|K_n * f - f\|_\infty \leqslant (1 + 2\sqrt{2}\pi^2)\,\omega_1(f; 1/n)$.

Proof. As noted earlier, since $K_n \in \mathscr{P}_n$, $E_n(f) \leqslant \|K_n * f - f\|_\infty$. On the other hand, the inequalities on the right of (i) and (ii) follow from the two previous results. We consider the details only for (i) in case $n = 2m$.

We apply Proposition 3.2, with $d = 2m$. Then, since

$$1 - r_{2m,1} = 1 - \cos\frac{\pi}{m+2} = 2\sin^2\frac{\pi}{2(m+2)} \leqslant \frac{\pi^2}{2(m+2)^2},$$

we have

$$\|K_{2m} * f - f\|_\infty \leqslant (1 + 16m^2\pi^2(1 - r_{2m,1}))\,\omega_2(f; 1/2m)$$
$$\leqslant (1 + 8\pi^4)\,\omega_2(f; 1/2m).$$

COROLLARY 3.5. *If $f \in C^2(G)$, then*

$$\|K_n * f - f\|_\infty \leqslant (1 + 8\pi^4)\|f^{(2)}\|_\infty n^{-2}.$$

Proof. Simply use the estimate of $\omega_2(f; 1/n)$ given by Proposition 2.2 (iv), in part (i) of the previous theorem.

We shall use this corollary to prove Jackson-type theorems for f in $C^k(G)$. To do this, we introduce the following kernels:

$$_0K_n = K_n, \qquad {}_{k+1}K_n = K_n + {}_kK_n - K_n * {}_kK_n.$$

(If δ_e is the point mass at e, then we have the explicit formula ${}_kK_n = \delta_e - (\delta_e - K_n)^{*(k+1)}$, where $(\delta_e - K_n)^{*(k+1)} = (\delta_e - K_n) * (\delta_e - K_n)^{*k}$.) Now we can prove the desired

THEOREM 3.6. (i) *If $f \in C^{2k}(G)$, then*

$$E_n(f) \leqslant \|{}_kK_n * f - f\|_\infty \leqslant C_k n^{-2k} \omega_2(f^{(2k)}; 1/n),$$

where $C_k = (1 + 8\pi^4)^{k+1}$.

(ii) *If $f \in C^{2k+1}(G)$, then*

$$E_n(f) \leqslant \|{}_{k+1}K_n * f - f\|_\infty \leqslant (1 + 2\sqrt{2}\pi^2) C_k n^{-2k-1} \omega_2(f^{(2k+1)}; 1/n).$$

Proof. The left-hand inequalities are clear, since ${}_kK_n \in \mathscr{P}_n$. The proof of the right-hand inequality in (i) is by induction on k; $k = 0$ is Theorem 3.4 (i). Now assume (i) is true for k. By the definition of ${}_{k+1}K_n$, we have

$${}_{k+1}K_n * f - f = K_n * (f - {}_kK_n * f) - (f - {}_kK_n * f). \tag{*}$$

Because $k + 1 > 0$, we have $f \in C^2(G)$, so Corollary 3.5 applied to $g = f - {}_kK_n * f$ yields

$$\|K_n * g - g\|_\infty \leqslant (1 + 8\pi^4) \left(\sum_{i,j} \|D_i D_j g\|_\infty \right) n^{-2}. \tag{**}$$

To estimate $\|D_i D_j g\|_\infty$, we note that, since for any $D \in \mathfrak{g}$ and differentiable f, $D(h * f) = h * Df$, we have

$$D_i D_j g = D_i D_j(f - {}_kK_n * f) = D_i D_j f - {}_kK_n * D_i D_j f.$$

Now, the induction hypothesis applied to $D_i D_j f \in C^2(G)$ gives

$$\|D_i D_j g\|_\infty = \|D_i D_j f - {}_kK_n * D_i D_j g\|_\infty \leqslant C_k n^{-2k} \omega_2((D_i D_j f)^{(2k)}; 1/n).$$

If we substitute this in (**), and take into account (*), we get the desired result for $k + 1$.

(ii) This is derived from (i) as follows. Since $f \in C^{2k+1}(G) \subseteq C^1(G)$, we may apply Proposition 2.2 (iii) to $g = {}_kK_n * f - f$, to conclude

$$\omega_1(g; h) \leqslant \left(\sum_1^3 \|{}_kK_n * D_i f - D_i f\|_\infty \right) h.$$

Thus, by Theorem 3.4 (ii), we get

$$\begin{aligned} \|{}_{k+1}K_n * f - f\|_\infty &= \|K_n * g - g\|_\infty \\ &\leqslant (1 + 2\sqrt{2}\pi^2) n^{-1} \left(\sum_1^3 \|{}_kK_n * D_i f - D_i f\|_\infty \right). \end{aligned}$$

The use of part (i) of the present theorem to estimate the last sum, yields (ii).

COROLLARY 3.7. *Theorem* 3.6 *holds, with* $2\omega_1$ *in place of* ω_2.

4. INVERSE THEOREMS ON $SU(2)$

We now consider the inverse problem of inferring differentiability and smoothness properties from the behavior of $E_n(f)$. As for the circle group, the key is the following analog of Bernstein's inequality for the derivative of a trigonometric polynomial.

THEOREM 4.1. *If* $p_n \in \mathscr{P}_n$ *and* $D \in \mathfrak{g}$, *then* $\|Dp_n\|_\infty \leqslant Cn\|D\| \, \|p_n\|_\infty$, *where* $C = 8^{-1/2}$.

Proof. We reduce the result to showing that $|D_1 p_n(e)| \leqslant Cn\|p_n\|_\infty$. The proof of this case is as follows. Since

$$\exp tD_1 = \begin{pmatrix} \exp(itC) & 0 \\ 0 & \exp(-itC) \end{pmatrix},$$

we have $a(\exp tD_1) = \exp(itC)$, $b(\exp tD_1) = 0$. Thus, as p_n is a polynomial in $a, \bar{a}, b, \bar{b}$ of total degree at most n, $p_n(\exp tD_1) = T_n(\theta(t))$, where $T_n(\theta)$ is an ordinary trigonometric polynomial of degree n, and $\theta(t) = tC$.

Now,

$$D_1 p_n(e) = \frac{d}{dt} p_n(\exp tD_1)|_{t=0} = CT_n'(0),$$

and by Bernstein's inequality, $\|T_n'\|_\infty \leqslant n\|T_n\|_\infty$. Thus, $|D_1 p_n(e)| \leqslant Cn\|p_n\|_\infty$, as desired, because $\|T_n\|_\infty \leqslant \|p_n\|_\infty$.

To reduce to the case just considered, we first note that $Dp_n(x) = L(x^{-1}) Dp_n(e) = D(L(x^{-1}) p_n)(e)$, $L(x^{-1}) p_n \in \mathscr{P}_n$, and $\|L(x^{-1}) p_n\|_\infty = \|p_n\|_\infty$. Thus, replacing p_n by $L(x^{-1}) p_n$, we can consider only derivatives at e. Second, for any $D \in \mathfrak{g}$, there exists $g \in G$ with $D = \|D\| \, g^{-1} D_1 g$. (This is just the statement that any skew-hermitian matrix is unitarily equivalent to a diagonal matrix with pure imaginary entries.) Hence,

$$\begin{aligned} Dp_n(e) &= \frac{d}{dt} p_n(\exp tD)|_{t=0} \\ &= \frac{d}{dt} p_n(g^{-1}(\exp t\|D\|\, D_1)\, g)|_{t=0} \\ &= \frac{d}{dt} (L(g)\, R(g)\, p_n)(\exp t\|D\|\, D_1)|_{t=0} \\ &= \|D\|\, D_1(L(g)\, R(g)\, p_n)(e). \end{aligned}$$

Since $L(g)\,R(g)\,p_n \in \mathscr{P}_n$, we have reduced to the case considered.

With the estimate of this theorem, we can proceed exactly as in the case of the circle group. We state the desired theorems with a few remarks and refer to the circle group case in [5], pp. 58–62, for detailed proofs.

THEOREM 4.2. *There exist $C_j > 0, j = 1, 2$, such that for $f \in C(G)$,*

(i) $\|f - L(g)f\|_\infty \leqslant C_1\, \rho(g, e) \sum' E_n(f)$, *and*
(ii) $\|f - 2L(g)f + L(g^2)f\|_\infty \leqslant C_2\, \rho^2(g, e) \sum' (n+1)\, E_n(f)$,

where $\sum'$ is the sum over $n \leqslant 1/\rho(g, e)$.

THEOREM 4.3. *If $\sum_1^\infty j^{k-1} E_j(f) < \infty$, then $f \in C^k(G)$. Moreover, there exists $C > 0$, independent of f, such that if $D^k = D_{i_1} D_{i_2} \ldots D_{i_k}$, $i_m \in \{1, 2, 3\}$, then*

$$E_n(D^k f) \leqslant C \sum_{j > n/2} j^{k-1} E_j(f).$$

Finally, we can use these theorems to characterize those classes of functions for which $E_n(f) = O(n^{-r})$.

THEOREM 4.4. *$f \in C^k(G)$ and $\omega_2(f^{(k)}; h) = O(h^\alpha)$, $0 < \alpha \leqslant 1$, if and only if $E_n(f) = O(n^{-k-\alpha})$. Moreover, if $\alpha < 1$, then $\omega^2(f^{(2)}; h)$ may be replaced by $\omega_1(f^{(1)}; h)$.*

The proofs of these three theorems follow those of [5], pp. 59–62, once we make the following remarks. One, since $\mathscr{P}_n$ is right invariant, and $D \in \mathfrak{g}$ is an infinitesimal right translation, $D\mathscr{P}_n \subseteq \mathscr{P}_n$. Two, Theorem 4.1 replaces Bernstein's inequality. And finally, Proposition 2.2 supplies some needed estimates for $\omega_j(p; h)$ in terms of $\|p^{(j)}\|_\infty$.

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