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EXPONENTIALLY AND LOGARITHMICALLY CLOSED HARDY FIELDS IN SEVERAL VARIABLES

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We define a notion of (C,P) system of Pfaff type of analytic submanifolds of $\mathbb{R}^n$. Then, we study the existence of Hardy fields in several variables, which are closed by exponentiation and logarithm.

1. (C,P) systems

We denote by C any collection of analytic submanifolds of $\mathbb{R}^n$ such that $\mathbb{R}^n \in C$, $n \in \omega$. Moreover, let P be an operator defined over C such that, if $X \in C$, $P(X)$ is a subring of the ring $\mathscr{H}(X)$ of all real analytic functions over X, containing the ring of polynomials.

Definition 1.1. A system of real analytic manifolds is, for us, a pair (C,P), where C and P are defined as above.

Definition 1.2. A map $\varphi: X_1 \to X_2$, X_1, $X_2 \in C$ is a (C,P) map if $f \circ \varphi \in P(X_1)$ for every $f \in P(X_2)$.

Definition 1.3. If $X_1, X_2 \in C$, $X_1 \subseteq X_2$, X_1 is said to be a (C,P) submanifold of X_2 if $i: X_1 \to X_2$ is a (C,P) map.

Definition 1.4. A system (C,P) is said to be of Pfaff type if, denoting by X any element of C, we have:

(1) The set $\{x \in X: f(x) > 0\}$ is a (C,P) submanifold of X, for every $f \in P(X)$.

(2) If $X_i \in C$, $i = 1, \ldots, n$, then $\prod_{i=1}^n X_i \in C$ and the canonical projections $\pi_i: \prod_{i=1}^n X_i \to X_i$ are (C,P) maps.

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(3) If $f \in P(X)$ and $f(x) \neq 0$ for every $x \in X$, then $1/f \in P(X)$.

(4) If X is a domain of $\mathbb{R}^n$, then $P(X)$ is a differential ring.

(5) If $X_1, X_2 \in C$, $X_1 \subseteq \mathbb{R}^n$, $X_2 \subseteq \mathbb{R}^{n+1}$ and $f \in P(X_1)$, $g \in P(X_2)$ and $\mathrm{Graf}(f) \subseteq X_2$, then $g(x, f(x)) \in P(X_1)$, $x \in X_1$. So by (2) the system is closed under general superposition.

(6) If $X \subseteq \mathbb{R}^{n+k}$ and $F_i(x, y_1, \dots, y_k) \in P(X)$, $i = 1, \dots, k$, and if $(y_1(x), \dots, y_k(x))$ is a solution of the non-degenerate set of equations $F_i(x, y_1(x), \dots, y_k(x)) = 0$, $i = 1, \dots, k$, over D, $D \in C$, then $y_i \in P(D)$, $i = 1, \dots, k$.

(7) If $f(x_1, \dots, x_n)$ is an analytic function, solution over X of the equation $\mathrm{d}f = \sum_{i=1}^n F_i(x_1, \dots, x_n, f)\,\mathrm{d}x_i$ where the F_i's $\in P(A)$, $A \subseteq X \times \mathbb{R}$, then $f \in P(X)$.

(8) If $f \in P(X)$, then $f^{-1}(c)$, $c \in \mathbb{R}$, has finitely many connected components.

All Pfaff manifolds (see [2–4, 7, 9] and 'Liouville manifolds' (see [8]) provide examples of (C, P) systems of Pfaff type.

2. (C, P) Hardy fields in several variables

We recall the definition of a $\mathscr{C}$-Hardy field in several variables [5], where $\mathscr{C}$ denotes any smoothness category of real valued functions of real variables. We denote by $\overline{\mathbb{R}^n}$ the one-point compactification of the euclidean space $\mathbb{R}^n$ to a point $\alpha \notin \mathbb{R}^n$. Moreover, if $\mathscr{B}$ is any filter basis converging to $p \in \overline{\mathbb{R}^n}$, formed by open connected subsets of $\mathbb{R}^n$, $\mathscr{C}(\mathscr{B})$ denotes the ring of germs in p following $\mathscr{B}$ of $\mathscr{C}$-functions.

Definition 2.1. A subring K of $\mathscr{C}(\mathscr{B})$ is said to be an n-variable $\mathscr{C}$-Hardy field in p for $\mathscr{B}$ if

(a) K is a subfield of $\mathscr{C}(\mathscr{B})$;

(b) $f \in K \Rightarrow \partial f / \partial x_i \in K$, $i = 1, \dots, n$.

From now on we denote by (C, P) any system of Pfaff type.

Let $\mathscr{B}_1$ be a filter basis formed by open intervals $(0, a)$, $a \in \mathbb{R}$, converging to 0 in the usual topology of $\mathbb{R}$. By property (1) $\mathscr{B}_1 \subseteq C$ and we denote by K_1 the ring of germs in 0 following $\mathscr{B}_1$ of 1-variable functions $f \in P(I)$ with $I \in \mathscr{B}_1$. We use the same symbol f for the germ $[f]$ and the function $f \in [f]$.

K_1 is a Hardy field by properties (8), (3), (4); it is real closed by property (6) (see also [5]) and it is exponentially and logarithmically closed by property (7). So we can state the following theorem known for the special case of 1-variable Pfaff functions:

Theorem 2.2. *K_1 is a real closed analytic Hardy field exponentially and logarithmically closed.* □

Definition 2.3. For any $I \in \mathscr{B}_1$ and any $f \in P(I)$ such that $f(x) > 0$ $\forall x \in I$ and $\lim_{x \to 0} f(x) = 0$, we define the sets $C_2(f, I) = \{(x, y): x \in I,\ 0 < y < f(x)\}$.

Proposition 2.4. *For f and I as above, we have:*

(a) $C_2(f, I) \in C$.

(b) *If* $J \in \mathscr{B}_1$ *and* $J \subseteq I$, *then* $C_2(f|_J, J)$ *is a* (C, P) *submanifold of* $C_2(f, I)$.

(c) *Given* $C_2(f, I)$ *and* $C_2(g, I)$ *there exists* $J \in \mathscr{B}_1$, $J \subseteq I$, *such that either* $C_2(f|_J, I)$ *is a* (C, P) *submanifold of* $C_2(g|_J, J)$ *or viceversa.*

Proof. Claims (a) and (b) follow from properties (1) and (2); to prove (c) we need also property (8). □

Proposition 2.5. *The collection of sets* $\mathscr{B}_2 = \{C_2(f, I): I \in \mathscr{B}_1, f \in P(I)\}$ *is an open connected filter basis in* $\mathbb{R}^2$ *converging to* 0 *in the usual topology.*

Proof. The proof follows from Proposition 2.4.

Definition 2.6. *Any* 2-*variable Hardy field of germs over* $\mathscr{B}_2$ *of functions* $f \in P(A)$, $A \in \mathscr{B}_2$ *is said to be a* (C, P) *Hardy field.*

Proposition 2.7. *If K is a* (C, P) *Hardy field, then its real closure* $\bar{K}$ *is also a* (C, P) *Hardy field.*

Proof. K is an analytic Hardy field [5] and the proof follows by property (6). □

Theorem 2.8. *Let K be a* (C, P) *Hardy field and* $\bar{K}$ *its real closure. If* $y: A \to \mathbb{R}$ *with* $A \in \mathscr{B}_2$ *is an analytic function such that* $\partial y / \partial x_i = F_i(x_1, x_2, y)$, $F_i(x_1, x_2, Y) \in \bar{K}[Y]$, $i = 1, 2$, *then* $\bar{K}(y)$ *is a* (C, P) *Hardy field.*

Proof. We prove that if $p(Y) \in \bar{K}[Y]$, then $p(y)$ has constant sign (>0, <0, =0) over some set of $\mathscr{B}_2$. In fact,

$$p(Y) = \alpha \prod_{l=1}^{q} (Y - \gamma_l) \prod_{r=1}^{s} [(Y + \beta_r)^2 + \delta_r^2]$$

with $\alpha, \gamma_l, \beta_r, \delta_r \in \bar{K}$ and $\delta_r \neq 0$ in $\bar{K}$, $l = 1, \ldots, q$, $r = 1, \ldots, s$. Obviously we can suppose $\alpha \neq 0$ in $\bar{K}$. So there exists $\mathscr{V} \in \mathscr{B}_2$ where y is defined and $\alpha, \gamma_l, \beta_r, \delta_r$ are defined and have constant sign. Hence,

$$Z(p(x_1, x_2, y(x_1, x_2))) \cap \mathscr{V} = Z\left(\prod_{l=1}^{q} (y(x_1, x_2) - \gamma_l(x_1, x_2)) \right)$$

where $Z(f)$ denotes the zeroset of f. We consider the analytic functions $t_l(x_1, x_2) = y(x_1, x_2) - \gamma_l(x_1, x_2)$, then: $\partial t_l / \partial x_i = G_i(x_1, x_2, t_l)$ with $G_i(x_1, x_2, Y) \in \bar{K}[Y]$, $i = 1, 2$,

$l=1,\dots,q$. By property (7), $t_l \in P(\mathscr{V})$. Therefore, by property (8), $Z(t_l(x_1,x_2)) \cap \mathscr{V}$ has a finite number of connected components.

If the origin is not a cluster point of $Z(t_l(x_1,x_2)) \cap \mathscr{V}$, then t_l has constant sign over some set of $\mathscr{B}_2$. Otherwise, writing

$$G_i(x_1,x_2,t_l) = \sum_{j=0}^{m_i} g_{ij}(x_1,x_2)t_l^j, \quad i=1,2,$$

we can consider the following three cases:

(1) $g_{20} \neq 0$ in $\bar{K}$;

(2) $g_{10} \neq 0$ and $g_{20}=0$ in $\bar{K}$;

(3) $g_{10}=g_{20}=0$ in $\bar{K}$,

taking care to choose $\mathscr{V}$ such that also the g_{i0}'s, $i=1,2$, are defined and have constant sign.

Case 1. Let Σ be a connected component of $Z(t_l(x_1,x_2)) \cap \mathscr{V}$ with the origin as a cluster point. Σ is the graph of an analytic function φ defined on the interval $I=\Pi_1(\Sigma)$ where Π_1 is the canonical projection. For every $a \in I$, $t_l(a,x_2)$ is analytic, so, by the assumption over g_{20}, it has at most a finite number of zeros $b_1 < b_2 < \cdots < b_n$. On the other hand, $t_l(a,b_j)=t_l(a,b_{j+1})=0$ contradicts the fact that $\operatorname{sign}(\partial t_l/\partial x_2)(a,b_j) = \operatorname{sign}(\partial t_l/\partial x_2)(a,b_{j+1})$. So $n=1$. The function φ turns out to be analytic by the implicit function theorem. $t_l(x_1,\varphi(x_1))=0$, hence, by property (6), $\varphi \in P(I)$. So $C_2(\varphi,I) \in \mathscr{B}_2$.

Case 2. In this case the origin is not a cluster point of a connected component of $Z(t_l(x_1,x_2)) \cap \mathscr{V}$. In fact the equation of the tangent line to the curve $t_l(x_1,x_2)=0$ in any point (a,b) such that $(a,b) \in Z(t_l(x_1,x_2)) \cap \mathscr{V}$ is $x_1=a$. So, by the properties of analytic functions and the assumption over g_{10}, $Z(t_l(x_1,x_2)) \cap \mathscr{V}$ is formed by a finite number of points or by a vertical segment.

Case 3. If $(a,b) \in Z(t_l(x_1,x_2)) \cap \mathscr{V}$, using the Taylor series with center (a,b) for the analytic function $t_l(x_1,x_2)$, we obtain that $t_l(x_1,x_2)=0$ in a neighbourhood of (a,b). So, by the analytic continuation principle, $t_l(x_1,x_2)=0$ over $\mathscr{V}$.

So $\bar{K}(y)$ turns out to be an ordered field over $\mathscr{B}_2$. Moreover, $\bar{K}(y)$ is closed under the operator $\partial/\partial x_i$, $i=1,2$, that is, $\bar{K}(y)$ is a 2-variable Hardy field. By properties (7), (4) and (3), $\bar{K}(y)$ is a (C,P) Hardy field. □

Corollary 2.9. *Let K be any (C,P) Hardy field over $\mathscr{B}_2$. There exist (C,P) Hardy fields L over $\mathscr{B}_2$ such that $k \subseteq L$ and if $y: A \to \mathbb{R}$, $A \in \mathscr{B}_2$ is an analytic solution of the system $\partial y/\partial x_i = F_i(x_1,x_2,y)$, $F_i(x_1,x_2,Y) \in L[Y]$, $i=1,2$, then $y \in L$.*

Proof. The proof follows by Zorn's lemma for the inductive class of (C,P) Hardy fields containing K, using Theorem 2.8. □

Remark 2.10. L is exponentially and logarithmically closed.

3. The field of functions H_2 over $\mathscr{B}_2$

Let K_0 be any (C, P) Hardy field. Following Hardy's work [1] and using Theorem 2.8 we construct the extension $H(K_0)$ of K_0. $H(K_0)$ is the class of the analytic functions $f: A \to \mathbb{R}$, $A \in \mathscr{B}_2$ such that there exist (C, P) Hardy fields $K_1, \dots, K_n$ with $K_{i+1} = K_i(a_{i+1})$ where $a_{i+1} = \log |z_i|$ or $a_{i+1} = \exp(z_i)$, $z_i \in K_i$, $0 \le i \le n-1$, for which $f \in K_n$.

Proposition 3.1. *$H(K_0)$ is a (C, P) Hardy field.*

Proof. The proof follows directly from the construction. We note that $H(K_0)$ is the smallest (C, P) Hardy field exponentially and logarithmically closed extending K_0. □

Definition 3.2. The field of functions H_2 over $\mathscr{B}_2$ *is the extension* $H(\mathbb{R}_2)$ of the (C, P) Hardy field $\mathbb{R}_2$ of 2-variable rational functions.

Denoting by $\mathscr{L}$ the special (C, P) system, defined by Van Den Dries, and using his decomposition theorem for the zeroset of 2-variable functions [9, Section IX], we can state that the ring of germs of all 2-variable functions following the filter basis $\mathscr{B}_2$ is a 2-variable real closed $\mathscr{L}$-Hardy field. This field is exponentially and logarithmically closed. Then, working as in Section 2, see also [6], we define the filter basis $\mathscr{B}_3$ and following the same pattern we can prove Theorem 2.8 for 3-variable $\mathscr{L}$-Hardy fields over $\mathscr{B}_3$.

We observe that the constructions and results of Section 2, obtained working in the point $p = 0$, above the graph of the 'reference' function $y = 0$, can be generalized if p is any point of $\overline{\mathbb{R}^2}$, the 'reference' function $y = f(x)$ is any function of K_1, 'converging' to p, taking $\mathscr{B}_2$ above or below the graph of $y = f(x)$.

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