Journal of Algebra 322 (2009) 2566–2579

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Betti numbers of Springer fibers in type A

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ARTICLE INFO

Article history:
Received 27 January 2009
Available online 29 July 2009
Communicated by Peter Littelmann

Keywords:
Flags
Schubert cells
Coxeter length
Springer fiber
Young diagrams

ABSTRACT

We determine the Betti numbers of the Springer fibers in type A. To do this, we construct a cell decomposition of the Springer fibers. The codimension of the cells is given by an analogue of the Coxeter length. This makes our cell decomposition well suited for the calculation of Betti numbers.

1. Introduction

1.1. Let V be a $\mathbb{C}$-vector space of dimension $n \geqslant 0$ and let $u : V \to V$ be a nilpotent endomorphism. We denote by $\mathcal{F}$ the (algebraic) variety of complete flags of V and by $\mathcal{F}_u$ the subset of u-stable complete flags, i.e. flags $(V_0, \ldots, V_n)$ such that $u(V_i) \subset V_i$ for all i. The variety $\mathcal{F}$ is projective, and $\mathcal{F}_u$ is a projective subvariety of it. The variety $\mathcal{F}_u$ is called *Springer fiber* since it can be seen as the fiber over u of the Springer resolution of singularities of the cone of nilpotent endomorphisms of V (see for example [9]).

Springer constructed representations of the symmetric group S_n on the cohomology spaces $H^*(\mathcal{F}_u, \mathbb{Q})$ (see [12]). The characters of these representations were determined by Lusztig in [6]. More explicitly he connected the multiplicities of irreducible summands of $H^*(\mathcal{F}_u, \mathbb{Q})$ with the coefficients of Kostka polynomials. This allows to calculate the Betti numbers $b_m = \dim H^{2m}(\mathcal{F}_u, \mathbb{Q})$. The aim of this article is to give a more direct calculation of them.

1.2. Following [3], a finite partition of a variety X is said to be an α-partition if the subsets in the partition can be indexed $X_1, \ldots, X_k$ so that $X_1 \cup \cdots \cup X_l$ is closed in X for $l = 1, 2, \ldots, k$. Thus each subset in the partition is a locally closed subvariety of X. An α-partition into subsets which

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0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2009.07.008

are isomorphic to affine spaces is called a cell decomposition. If X is a projective variety with a cell decomposition, then the cohomology of X vanishes in odd degrees and $\dim H^{2m}(X,\mathbb{Q})$ is the number of m-dimensional cells (see 4.1).

It is known from [8] and [10] that $\mathcal{F}_u$ admits a cell decomposition, and there are also many references proving the existence for other types (see [11] or [13]) or more general contexts (Springer fibers of any type in [3], partial u-stable flags in [7]). A simple manner to construct a cell decomposition of $\mathcal{F}_u$ is to take the intersection with the Schubert cells of the flag variety, then we obtain a cell decomposition provided that the Schubert cells are defined according to appropriate conventions (see [8] or 3.9). However the dimension of the cells is given by a complicated formula, it makes this cell decomposition not practical to compute Betti numbers. We construct a different cell decomposition which is better suited for the calculation of Betti numbers.

1.3. Let $\lambda(u) = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ be the lengths of the Jordan blocks of u and let $Y(u)$ be the Young diagram of rows of these lengths. If $\mu_1, \ldots, \mu_s$ are the lengths of the columns of $Y(u)$, recall from [10, §II.5.5] that $\dim \mathcal{F}_u = \sum_{q=1}^{s} \mu_q(\mu_q - 1)/2$.

A standard tableau of shape $Y(u)$ is a numbering of the boxes of $Y(u)$ by $1, \ldots, n$ such that numbers in the rows increase to the right and numbers in the columns increase to the bottom.

We call *row-standard tableau of shape* $Y(u)$ a numbering of the boxes of $Y(u)$ by $1, \ldots, n$ such that numbers in the rows increase to the right. Let τ be a row-standard tableau. We call *inversion* a pair of numbers $i < j$ in the same column of τ and such that one of the following conditions is satisfied:

- i or j has no box on its right and i is below j,
- i, j have respective entries i', j' on their right, and $i' > j'$.

For example $\tau = \begin{array}{|c|c|c|}\hline 2 & 4 & 8 \\ \hline 3 & 6 & 7 \\ \hline 1 & 5 \\ \cline{1-2}\end{array}$ has four inversions: the pairs $(1,2)$, $(4,6)$, $(5,6)$, $(7,8)$.

Let $n_{\text{inv}}(\tau)$ be the number of inversions of τ. We see that $n_{\text{inv}}(\tau) = 0$ if and only if τ is a standard tableau. For $u = 0$ the diagram $Y(u)$ has only one column, hence τ is equivalent to a permutation ($\sigma \in S_n$ corresponds to the tableau numbered by $\sigma_1, \ldots, \sigma_n$ from top to bottom) and $n_{\text{inv}}(\tau)$ is the usual inversion number for permutations.

Our main result is the following

Theorem. *The variety $\mathcal{F}_u$ has a cell decomposition $\mathcal{F}_u = \bigcup_\tau C(\tau)$ parameterized by the row-standard tableaux of shape $Y(u)$, and such that* $\operatorname{codim}_{\mathcal{F}_u} C(\tau) = n_{\text{inv}}(\tau)$.

And we deduce:

Corollary. *Let $d = \dim \mathcal{F}_u$. For $m \geqslant 0$, the Betti number $b_m := \dim H^{2m}(\mathcal{F}_u, \mathbb{Q})$ is the number of row-standard tableaux τ of shape $Y(u)$ such that $n_{\text{inv}}(\tau) = d - m$.*

If $u = 0$, then $\mathcal{F}_u$ is the whole flag variety $\mathcal{F}$, and we get the classical formula giving the Betti numbers of the flag variety. In general, we find that the dimension of the cohomology space of maximal degree is the number of standard tableaux of shape $Y(u)$. This is also classical, since the Springer representation of S_n on the cohomology space in maximal degree is irreducible and isomorphic to the Specht module corresponding to the Young diagram $Y(u)$, whose dimension is precisely the number of standard tableaux of shape $Y(u)$ (see [12]). We also recall in 1.5 that $\mathcal{F}_u$ is equidimensional and that its components are parameterized by standard tableaux.

1.4. Let us make precise the relation between standard and row-standard tableaux. If T is standard, then the shape of its subtableau $T[1, \ldots, i]$ of entries $1, \ldots, i$ is a subdiagram $Y_i(T) \subset Y(u)$. In that way a standard tableau T is equivalent to the data of a complete chain of subdiagrams $\emptyset = Y_0(T) \subset Y_1(T) \subset \cdots \subset Y_n(T) = Y(u)$. We call (ordered) partition of n a decreasing sequence of nonnegative integers whose sum is n. The lengths of the rows of $Y_i(T)$ form a partition of i, $\lambda^{(i)} = (\lambda_1^{(i)} \geqslant \cdots \geqslant \lambda_r^{(i)})$.

In that way, T is also equivalent to a maximal chain of partitions $\emptyset = \lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(n)} = \lambda(u)$ from $\emptyset = (0, 0, \ldots)$ to $\lambda(u)$ (we denote by $\subset$ the partial inclusion order on partitions, which means $\lambda_p^{(i)} \leqslant \lambda_p^{(i+1)}$ for every p).

If τ is row-standard, then the lengths of the rows of its subtableau of entries $1, \ldots, i$ form a sequence of nonnegative integers $\pi^{(i)} = (\pi_1^{(i)}, \ldots, \pi_r^{(i)})$ of sum i (not necessarily decreasing). In that way τ is equivalent to the data of a maximal chain of finite sequences of nonnegative integers $\emptyset = \pi^{(0)} \subset \pi^{(1)} \subset \cdots \subset \pi^{(n)} = \lambda(u)$ from $\emptyset$ to $\lambda(u)$ (where $\subset$ means $\pi_p^{(i)} \leqslant \pi_p^{(i+1)}$ for every p).

If we arrange the entries in each column of τ in increasing order to the bottom, then we get a standard tableau that we denote by $\mathrm{st}(\tau)$. We will call it the standardization of τ. Similarly if we arrange the terms of each sequence $\pi^{(i)}$ in decreasing order, then we get a partition $\mathrm{ord}(\pi^{(i)})$, the partitions $(\mathrm{ord}(\pi^{(i)}))_{i=0,\ldots,n}$ form a maximal chain from $\emptyset$ to $\lambda(u)$ and $\mathrm{st}(\tau)$ is the standard tableau which corresponds to it. As we show in 2.2, the inversion number of τ can be interpreted as a minimal number of elementary operations which allow to transform τ into its standardization $\mathrm{st}(\tau)$.

1.5. Our construction relies on an α-partition of $\mathcal{F}_u$ into subsets parameterized by standard tableaux. Let us recall the construction, due to Spaltenstein, of such an α-partition. Let $(V_0, \ldots, V_n) \in \mathcal{F}_u$. For each i consider the Young diagram $Y(u_{|V_i})$ associated to the restriction $u_{|V_i}$ in the sense of 1.3. Let T be a standard tableau of shape $Y(u)$. The shape of the subtableau of entries $1, \ldots, i$ is a subdiagram $Y_i(T) \subset Y(u)$ (cf. 1.4). Define $\mathcal{F}_u^T$ as the set of u-stable flags such that $Y(u_{|V_i}) = Y_i(T)$ for every i. By [10, §II.5], the $\mathcal{F}_u^T$'s form an α-partition of $\mathcal{F}_u$ into irreducible, nonsingular subsets of same dimension as $\mathcal{F}_u$. Therefore, the components of $\mathcal{F}_u$ are exactly the closures of the $\mathcal{F}_u^T$'s.

We generalize this construction. Let $\mathcal{R}_n$ denote the set of double sequences of integers $(i_k, j_k)_{k=0,\ldots,n}$ with $(i_k)_k$ weakly decreasing, $(j_k)_k$ weakly increasing, $0 \leqslant i_k \leqslant j_k \leqslant n$ and $j_k - i_k = k$ for every k. Let $\rho = (i_k, j_k)_k \in \mathcal{R}_n$. Instead of considering the restrictions of u to the subspaces of the flag, we consider the maximal chain of subquotients

$$0 = V_{j_0}/V_{i_0} \subset V_{j_1}/V_{i_1} \subset \cdots \subset V_{j_n}/V_{i_n} = V.$$

For any k we consider the Young diagram $Y(u_{|V_{j_k}/V_{i_k}})$ associated to the nilpotent endomorphism of the subquotient V_{j_k}/V_{i_k} induced by u. We denote by $\mathcal{F}_{u,T}^{\rho}$ the set of u-stable flags $(V_0, \ldots, V_n)$ such that $Y(u_{|V_{j_k}/V_{i_k}}) = Y_k(T)$. The double sequence ρ being fixed, we prove that the $\mathcal{F}_{u,T}^{\rho}$'s form an α-partition of $\mathcal{F}_u$ (see 3.1) into irreducible, nonsingular subsets of same dimension as $\mathcal{F}_u$ (see Theorem 3.2).

For each T, we construct a cell decomposition $\mathcal{F}_{u,T}^{\rho} = \bigcup C^{\rho}(\tau)$ indexed on row-standard tableaux with $\mathrm{st}(\tau) = T$, and such that the codimension of $C^{\rho}(\tau)$ in $\mathcal{F}_{u,T}^{\rho}$ is $n_{\mathrm{inv}}(\tau)$ (see Theorem 3.3).

Finally, by collecting together the cell decompositions of the $\mathcal{F}_{u,T}^{\rho}$'s for T running over the set of standard tableaux of shape $Y(u)$ (and fixing ρ), we get a cell decomposition $\mathcal{F}_u = \bigcup_{\tau} C^{\rho}(\tau)$. It is not unique, since it depends on the parameter ρ.

1.6. Observe that in the cell decomposition of $\mathcal{F}_{u,T}^{\rho}$ mentioned above, the dimension of the cells does not depend on ρ. Therefore the cohomology with compact support of $\mathcal{F}_{u,T}^{\rho}$ only depends on T (see 4.2). If $T_{\min}$ is the minimal standard tableau of shape $Y(u)$ for the dominance order (see 3.1), then we prove that $\mathcal{F}_{u,T_{\min}}^{\rho}$ is a closed subset of $\mathcal{F}_u$, thus it is a nonsingular irreducible component of $\mathcal{F}_u$. Then, the cell decomposition allows to compute its Betti numbers. When ρ is changing, the subset $\mathcal{F}_{u,T_{\min}}^{\rho} \subset \mathcal{F}_u$ is changing too, and we get thus a family of components of $\mathcal{F}_u$ which are all nonsingular and have the same Betti numbers.

1.7. This article contains four parts. In part 2, we establish some properties of the inversion number $n_{\mathrm{inv}}(\tau)$. In geometric part 3, we prove the results announced in 1.5. Part 3 is independent from part 2 before. In part 4, we apply results of the two parts before to the calculation of Betti numbers.

Fix some conventional notation. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$ be the set of nonnegative integers. Let $\mathbb{C}$ be the field of complex numbers. Let $\mathbb{Q}$ be the field of rational numbers. Let S_n be the group of permutations of $\{1, \ldots, n\}$. We denote by $\#A$ the number of elements in a finite set A. Other pieces of notation will be introduced in what follows.

2. Inversion number of row-standard tableaux

2.1. First we define an elementary operation on row-standard tableaux. For $i = 1, \ldots, n$ let D_i denote the set of row-standard tableaux τ which satisfy the following properties:

1. i is not in the first row of τ. Then let j be the entry in the neighbor box above i.
2. If i has an entry i' on its right, then $j < i'$. If j has an entry j' on its right, then $i < j'$.
3. For every k in the same column as i, j and such that $\min(i, j) < k < \max(i, j)$, either $(\min(i, j), k)$ or $(k, \max(i, j))$ is an inversion (but not both).

Let $\tau \in D_i$. Let $i_1 \leqslant \cdots \leqslant i_q = i$ be the entries until i of the row containing i, and let $j_1 \leqslant \cdots \leqslant j_q = j$ be the entries until j of the row containing j. Then define $\delta_i(\tau)$ as the tableau obtained by switching i_k and j_k for every $k = 1, \ldots, q$. The tableau $\delta_i(\tau)$ remains row-standard. Observe that $\delta_i(\tau) \in D_j$ and that we have $\tau = \delta_j \delta_i(\tau)$.

Lemma. *Let $\tau \in D_i$ and let j be the neighbor entry above i. Then we have $n_{\mathrm{inv}}(\delta_i(\tau)) = n_{\mathrm{inv}}(\tau) - 1$ if $(\min(i, j), \max(i, j))$ is an inversion of τ, and $n_{\mathrm{inv}}(\delta_i(\tau)) = n_{\mathrm{inv}}(\tau) + 1$ otherwise.*

Proof. Let $\mathrm{Inv}(\tau)$ (resp. $\mathrm{Inv}(\delta_i(\tau))$) be the set of inversions of τ (resp. of $\delta_i(\tau)$). For $\{k, l\} \cap \{i, j\} = \emptyset$, it is clear that $(k, l) \in \mathrm{Inv}(\tau) \Leftrightarrow (k, l) \in \mathrm{Inv}(\delta_i(\tau))$. Let k be in the same column as i, j. Observe that i, j are neighbor in τ as in $\delta_i(\tau)$, hence k is above i if and only if it is above j, and (k, i) have the same relative position in τ and $\delta_i(\tau)$. If $k < \min(i, j)$ then it is clear that $(k, i) \in \mathrm{Inv}(\tau) \Leftrightarrow (k, j) \in \mathrm{Inv}(\delta_i(\tau))$ and $(k, j) \in \mathrm{Inv}(\tau) \Leftrightarrow (k, i) \in \mathrm{Inv}(\delta_i(\tau))$. Likewise if $k > \max(i, j)$ then we have $(i, k) \in \mathrm{Inv}(\tau) \Leftrightarrow (j, k) \in \mathrm{Inv}(\delta_i(\tau))$ and $(j, k) \in \mathrm{Inv}(\tau) \Leftrightarrow (i, k) \in \mathrm{Inv}(\delta_i(\tau))$. Now suppose $\min(i, j) < k < \max(i, j)$. Say $i < j$ (the other case is treated similarly). It follows from the definition of inversion that (i, k) is an inversion of τ if and only if (k, j) is not an inversion of $\delta_i(\tau)$. Likewise (k, j) is an inversion of τ if and only if (i, k) is not an inversion of $\delta_i(\tau)$. By applying condition 3 above, we get $(i, k) \in \mathrm{Inv}(\tau) \Leftrightarrow (k, j) \notin \mathrm{Inv}(\tau) \Leftrightarrow (i, k) \in \mathrm{Inv}(\delta_i(\tau))$ and similarly $(k, j) \in \mathrm{Inv}(\tau) \Leftrightarrow (k, j) \in \mathrm{Inv}(\delta_i(\tau))$.

Finally we get that the number of inversions (k, l) with $\{k, l\} \neq \{i, j\}$ is the same for τ and $\delta_i(\tau)$. Now observe that, as the right neighbors of i and j are switched from τ to $\delta_i(\tau)$, we have $(i, j) \in \mathrm{Inv}(\tau) \Leftrightarrow (i, j) \notin \mathrm{Inv}(\delta_i(\tau))$ (resp. $(j, i) \in \mathrm{Inv}(\tau) \Leftrightarrow (j, i) \notin \mathrm{Inv}(\delta_i(\tau))$ if $j < i$). The lemma follows. $\square$

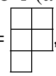
2.2. Next we show that $n_{\mathrm{inv}}(\tau)$ is the minimal number of operations to transform τ into its standardization $\mathrm{st}(\tau)$. We need the following

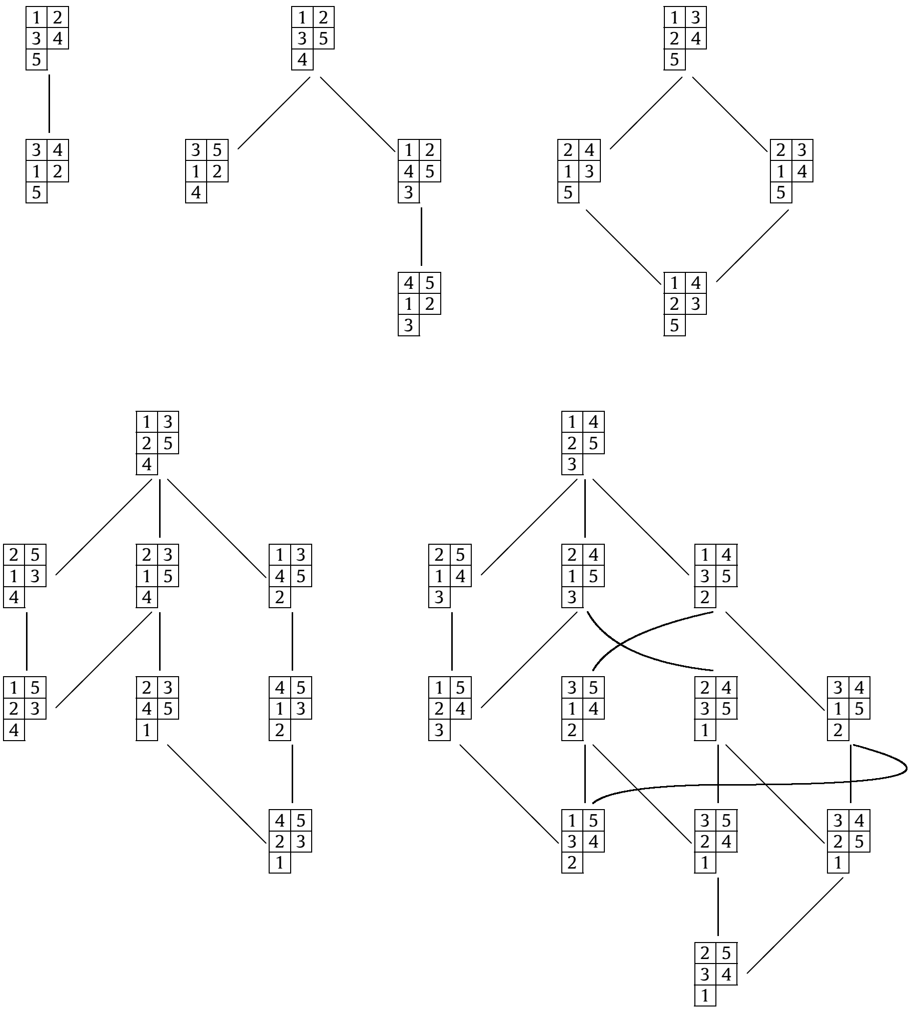
Lemma. *Suppose $\tau \neq \mathrm{st}(\tau)$. Let m be the maximal entry which is not at the same place in τ and $\mathrm{st}(\tau)$. Then m has a below-neighbor entry i, which satisfies $i < m$, and we have $\tau \in D_i$ and $n_{\mathrm{inv}}(\delta_i(\tau)) = n_{\mathrm{inv}}(\tau) - 1$.*

Proof. By maximality of m, there is $i' < m$ below m, and all $j > m$ of the same column as m are below i'. In particular m has a below-neighbor i and we have $i < m$. If m has a right neighbor m', then we have $i < m < m'$. If i has a right neighbor i', then m' also exists, and by maximality of m, the entries in the column of m' are in good order from m' to the bottom, in particular we have $m < m' < i'$. For $k = i + 1, \ldots, m - 1$ in the same column as i, m, either k is above i, m, then (i, k) is an inversion and (k, m) is not one, or k is below i, m, then (i, k) is not an inversion and (k, m) is one. Therefore $\tau \in D_i$ and (i, m) is an inversion. By Lemma 2.1 it follows $n_{\mathrm{inv}}(\delta_i(\tau)) = n_{\mathrm{inv}}(\tau) - 1$. $\square$

Proposition. *Let τ be row-standard, then there is a sequence of integers $i_1, \ldots, i_m$ such that $\mathrm{st}(\tau) = \delta_{i_1} \cdots \delta_{i_m}(\tau)$. The inversion number $n_{\mathrm{inv}(\tau)}$ is the minimal length of such a sequence.*

Proof. If there are $i_1, \ldots, i_m$ with $\mathrm{st}(\tau) = \delta_{i_1} \cdots \delta_{i_m}(\tau)$, then we get $m \geqslant n_{\mathrm{inv}}(\tau)$ by Lemma 2.1. We prove by induction on $n_{\mathrm{inv}}(\tau)$ that there are $i_1, \ldots, i_m$ with $m = n_{\mathrm{inv}}(\tau)$ such that $\mathrm{st}(\tau) = \delta_{i_1} \cdots \delta_{i_m}(\tau)$. If $n_{\mathrm{inv}}(\tau) = 0$ then $\tau = \mathrm{st}(\tau)$ and it is immediate. Suppose $n_{\mathrm{inv}}(\tau) > 0$. By the lemma above there is i such that $\tau \in D_i$ and $n_{\mathrm{inv}}(\delta_i(\tau)) = n_{\mathrm{inv}}(\tau) - 1$. The property follows by induction hypothesis applied to $\delta_i(\tau)$. $\square$

We construct a graph whose vertices are row-standard tableaux of shape $Y(u)$ and with one edge between τ and τ' if there is i such that $\tau' = \delta_i(\tau)$. For the diagram $Y(u) =$ , we get for example the following graph.



Each connected component contains a unique standard tableau. Two tableaux τ and τ' are in the same connected component if we have $\mathrm{st}(\tau) = \mathrm{st}(\tau')$. By the proposition, the number of inversions $n_{\mathrm{inv}}(\tau)$ is the length between τ and $\mathrm{st}(\tau)$ in the graph.

2.3. Let T be standard. For each i, let q_i be the index of the column of T containing i and let p_i be the number of rows of length q_i in the subtableau $T[1,\dots,i]$. The next proposition allows to describe the distribution of inversion numbers.

Proposition.

(a) *Let $\kappa_1,\dots,\kappa_n$ be integers with $0\leqslant\kappa_i\leqslant p_i-1$ for any i. Then $(\delta_n)^{\kappa_n}\cdots(\delta_1)^{\kappa_1}(T)$ is well defined.*
(b) *For every τ row-standard such that $\mathrm{st}(\tau)=T$, there are unique integers $\kappa_1,\dots,\kappa_n$ with $0\leqslant\kappa_i\leqslant p_i-1$ such that we have $\tau=(\delta_n)^{\kappa_n}\cdots(\delta_1)^{\kappa_1}(T)$. Moreover $n_{\mathrm{inv}}(\tau)=\kappa_1+\cdots+\kappa_n$.*

By the proposition, we obtain the formula $\#\{\tau\colon\ \mathrm{st}(\tau)=T,\ n_{\mathrm{inv}}(\tau)=m\}=\#\{(\kappa_1,\dots,\kappa_n)\colon\ 0\leqslant\kappa_i\leqslant p_i-1\ \forall i,\ \sum_{i=1}^n\kappa_i=m\}$.

Proof. First, observe that, if $\tau\in D_i$ and if $i+1,\dots,n$ have the same place in τ and $\mathrm{st}(\tau)$, then $i+1,\dots,n$ remain at the same place in $\delta_i(\tau)$ and $\mathrm{st}(\tau)$.

(a) We reason by induction on n with immediate initialization in 1. Let us prove the property for $n\geqslant 2$. By induction hypothesis (relying on the subtableau of entries $1,\dots,n-1$) the tableau $\tau'=(\delta_{n-1})^{\kappa_{n-1}}\cdots(\delta_1)^{\kappa_1}(T)$ is well defined and n has the same place in τ' and T. Then we reason by induction on $\kappa_n\geqslant 0$ with immediate initialization for $\kappa_n=0$. Let us prove the property for $\kappa_n\geqslant 1$. By induction hypothesis, $\tau''=(\delta_n)^{\kappa_n-1}(\tau')$ is well defined. The entry n has been moved by κ_n-1 ranks to the up from τ' to τ''. As $\kappa_n<p_n$, there is an entry j just above n in τ'', and j is the last box of its own row. As in the proof of Lemma 2.2, each $k=j+1,\dots,n-1$ in the same column as j,n is such that either (j,k) or (k,n) is an inversion (but not both), therefore $\tau''\in D_n$, and $\delta_n(\tau'')$ is well defined.

(b) Suppose $\tau=(\delta_n)^{\kappa_n}\cdots(\delta_1)^{\kappa_1}(T)=(\delta_n)^{\kappa'_n}\cdots(\delta_1)^{\kappa'_1}(T)$. Then κ_n (and similarly κ'_n) is the number of boxes below n in τ. Thus $\kappa_n=\kappa'_n$. As δ_n is injective we get $(\delta_{n-1})^{\kappa_{n-1}}\cdots(\delta_1)^{\kappa_1}(T)=(\delta_{n-1})^{\kappa'_{n-1}}\cdots(\delta_1)^{\kappa'_1}(T)$. Then κ_{n-1} (and similarly κ'_{n-1}) is the number of entries $j<n$ below $n-1$ in this new tableau. Thus $\kappa_{n-1}=\kappa'_{n-1}$. And so on. We deduce $\kappa_i=\kappa'_i$ for any i. Moreover we see that, if $m+1,\dots,n$ have the same place in τ and T, then we must have $\kappa_n=\cdots=\kappa_{m+1}=0$.

We prove the existence by induction on $n_{\mathrm{inv}}(\tau)$ with trivial initialization for $n_{\mathrm{inv}}(\tau)=0$. Suppose $n_{\mathrm{inv}}(\tau)>0$. Then there is m maximal which has not the same place in τ and T. By Lemma 2.2 there is $i<m$ just below m and we have $\tau\in D_i$ and $n_{\mathrm{inv}}(\delta_i(\tau))=n_{\mathrm{inv}}(\tau)-1$. Let $\tau'=\delta_i(\tau)$. Then $\tau'\in D_m$ and $\tau=\delta_m(\tau')$. By induction hypothesis we have $\tau'=(\delta_m)^{\kappa'_m}\cdots(\delta_1)^{\kappa'_1}(T)$ with $\kappa'_1+\cdots+\kappa'_m=n_{\mathrm{inv}}(\tau')$. We get $\tau=(\delta_m)^{\kappa'_m+1}\cdots(\delta_1)^{\kappa'_1}(T)$ and we have $\kappa'_1+\cdots+(\kappa'_m+1)=n_{\mathrm{inv}}(\tau)$. $\square$

3. Geometric constructions

3.1. We deal with the partition $\mathcal{F}_u=\bigsqcup_T\mathcal{F}_{u,T}^{\rho}$ introduced in 1.5. Let us recall the dominance order on standard tableaux. For T standard, let $c_{\leqslant q}T[1,\dots,i]$ be the number of boxes in the first q columns of the subtableau of entries $1,\dots,i$. We write $T\preccurlyeq T'$ if $c_{\leqslant q}T[1,\dots,i]\geqslant c_{\leqslant q}T'[1,\dots,i]$ for any i and any q. First we prove the following

Proposition. *Fix $\rho\in\mathcal{R}_n$. Let T be standard. Then we have $\overline{\mathcal{F}_{u,T}^{\rho}}\subset\bigcup_{S\preccurlyeq T}\mathcal{F}_{u,S}^{\rho}$ where the union is taken over standard tableaux S such that $S\preccurlyeq T$.*

It follows from this proposition that the $\mathcal{F}_{u,T}^{\rho}$ for ρ fixed and T running over the set of standard tableaux of shape $Y(u)$ form an α-partition of $\mathcal{F}_u$. Indeed, take a total order on standard tableaux completing the dominance order. Then, the $\mathcal{F}_{u,T}^{\rho}$'s, arranged according to this order, form a sequence whose first k terms always have a closed union.

There is a unique tableau $T_{\min}$ of shape $Y(u)$ which is minimal for the dominance order. Let $\mu_1,\dots,\mu_s$ be the lengths of the columns of $Y(u)$. Then $T_{\min}$ is the standard tableau with numbers $1,\dots,\mu_1$ in the first column, $\mu_1+1,\dots,\mu_1+\mu_2$ in the second column, etc. For example for

$\lambda(u) = (3, 2, 2)$ we have

$$T_{\min} = \begin{array}{|c|c|c|} \hline 1 & 4 & 7 \\ \hline 2 & 5 \\ \cline{1-2} 3 & 6 \\ \cline{1-2} \end{array}$$

By the proposition we get that $\mathcal{F}^{\rho}_{u,T_{\min}}$ is a closed subset of $\mathcal{F}_u$.

Proof of the proposition. By definition, for $(V_0, \ldots, V_n) \in \mathcal{F}^{\rho}_{u,T}$ and $(i_k, j_k) \in \rho$ the Young diagram $Y(u_{|V_{j_k}/V_{i_k}})$ associated to the nilpotent map induced by u on the subquotient V_{j_k}/V_{i_k} is the shape of the subtableau $T[1, \ldots, k]$. Thus the number of boxes in the first q columns of both coincide for any $q \geqslant 1$. Thus $\dim\ker u^q_{|V_{j_k}/V_{i_k}} = c_{\leqslant q} T[1, \ldots, k]$.

Suppose that $\mathcal{F}^{\rho}_{u,S} \cap \overline{\mathcal{F}^{\rho}_{u,T}}$ is nonempty and take $(V_0, \ldots, V_n) \in \mathcal{F}^{\rho}_{u,S} \cap \overline{\mathcal{F}^{\rho}_{u,T}}$. Then we have $\dim\ker u^q_{|V_{j_k}/V_{i_k}} \geqslant c_{\leqslant q} T[1, \ldots, k]$ by the following lemma. It follows $c_{\leqslant q} S[1, \ldots, k] \geqslant c_{\leqslant q} T[1, \ldots, k]$ for any $q \geqslant 1$ and $k = 1, \ldots, n$, therefore $S \preccurlyeq T$. $\square$

Lemma. *The set* $\{(V_0, \ldots, V_n) \in \mathcal{F}_u\colon \dim\ker u^q_{|V_j/V_i} \geqslant c\}$ *is closed for any* $c \in \mathbb{N}$.

Proof. We show the lemma for $q = 1$. For the general case, replace u by u^q. We prove that $\dim\ker u_{|V_j/V_i} = j + i - \dim(V_i + u(V_j))$. Then the lemma follows from the lower semicontinuity of the map $(V_i, V_j) \mapsto \dim(V_i + u(V_j))$ defined on the product of Grassmannians.

We have $\ker u_{|V_j/V_i} = u^{-1}(V_i) \cap V_j$. By the rank formula applied to the restriction of u to $u^{-1}(V_i) \cap V_j$ we get $\dim u^{-1}(V_i) \cap V_j = \dim V_j \cap \ker u + \dim V_i \cap u(V_j)$. On one hand, we have $\dim V_i \cap u(V_j) = i + \dim u(V_j) - \dim(V_i + u(V_j))$. On the other hand, the rank formula gives $\dim V_j \cap \ker u = j - \dim u(V_j)$. The desired formula follows. $\square$

3.2. The following theorem generalizes [10, §II.5.5].

Theorem. *Fix $\rho \in \mathcal{R}_n$. Let T be standard. The set $\mathcal{F}^{\rho}_{u,T}$ is an irreducible, nonsingular subvariety of $\mathcal{F}_u$ and we have* $\dim\mathcal{F}^{\rho}_{u,T} = \dim\mathcal{F}_u$.

The theorem is proved by induction in Sections 3.4–3.8. From the theorem and Proposition 3.1, we deduce the following

Corollary. *Fix $\rho \in \mathcal{R}_n$. For every T, the closure $\overline{\mathcal{F}^{\rho}_{u,T}}$ is an irreducible component of $\mathcal{F}_u$ and every irreducible component is obtained in that way. Moreover we have $\overline{\mathcal{F}^{\rho}_{u,T_{\min}}} = \mathcal{F}^{\rho}_{u,T_{\min}}$ and in particular $\overline{\mathcal{F}^{\rho}_{u,T_{\min}}}$ is a nonsingular component.*

For each ρ, we obtain a different parameterization of the components of $\mathcal{F}_u$ by the standard tableaux, and the $\mathcal{F}^{\rho}_{u,T_{\min}}$'s for ρ running over the set $\mathcal{R}_n$ form a family of nonsingular components.

Remark. Let us describe the link between the different parameterizations of the components. Take as reference the component $\mathcal{K}^T = \overline{\mathcal{F}^T_u}$ obtained as the closure of the Spaltenstein set (see 1.5), and let us describe S such that $\mathcal{K}^T = \overline{\mathcal{F}^{\rho}_{u,S}}$. By [5, Theorem 3.3], for $(V_0, \ldots, V_n) \in \mathcal{K}^T$ generic and any $1 \leqslant i < j \leqslant n$ the Young diagram $Y(u_{|V_j/V_i})$ is the shape of the tableau obtained as rectification by jeu de taquin of the subtableau $T[i+1, \ldots, j]$. Write $\rho = (i_k, j_k)_k$. For each k let $Y^{(k)}$ be the Young diagram forming the shape of the rectification by jeu de taquin of the subtableau $T[i_k + 1, \ldots, j_k]$. We get a chain of subdiagrams $\emptyset = Y^{(0)} \subset Y^{(1)} \subset \cdots \subset Y^{(n)} = Y(u)$ and S is the standard tableau which corresponds to this chain in the sense of 1.4.

3.3. We fix $\rho \in \mathcal{R}_n$. The main result of this section states the existence of a cell decomposition for each $\mathcal{F}_{u,T}^{\rho}$.

Theorem. *Let T be standard. The set $\mathcal{F}_{u,T}^{\rho}$ has a cell decomposition $\mathcal{F}_{u,T}^{\rho} = \bigsqcup C^{\rho}(\tau)$ parameterized by the row-standard tableaux τ of standardization $\mathrm{st}(\tau) = T$, such that the codimension of the cell $C^{\rho}(\tau)$ in $\mathcal{F}_{u,T}^{\rho}$ is equal to the inversion number $n_{\mathrm{inv}}(\tau)$.*

As said in 3.1 the subsets $\mathcal{F}_{u,T}^{\rho}$ form an α-partition of $\mathcal{F}_u$. Therefore by collecting together the cell decompositions of the $\mathcal{F}_{u,T}^{\rho}$'s for T running over the set of standard tableaux, we obtain a cell decomposition of $\mathcal{F}_u$. This proves Theorem 1.3.

We prove both theorems simultaneously, by induction on $n = \dim V$.

Proof of Theorems 3.2 and 3.3.

3.4. First, we point out a duality in the family parameterized by $\rho \in \mathcal{R}_n$ of partitions of $\mathcal{F}_u$. It will allow us to suppose that the sequence $\rho = (i_k, j_k)_k$ is such that $(i_{n-1}, j_{n-1}) = (0, n-1)$.

Let V^* be the dual vector space of V. The dual map $u^* : V^* \to V^*$ is also nilpotent. Let $\mathcal{F}_{u^*}$ be the Springer fiber relative to u^*. The maps u^* and u are transposes of each other, in particular they have the same Jordan form. For a subspace $W \subset V$ let $W^{\perp} = \{\phi \in V^*:\ \phi(w) = 0\ \forall w \in W\}$. The map

$$\Psi : \mathcal{F}_u \to \mathcal{F}_{u^*}, \quad (V_0, \ldots, V_n) \mapsto \left(V_n^{\perp}, \ldots, V_0^{\perp}\right)$$

is well defined and is an isomorphism of algebraic varieties. Writing $\rho = (i_k, j_k)_{k=0,\ldots,n}$, let us define $\rho^* = (i_k^*, j_k^*)_{k=0,\ldots,n} \in \mathcal{R}_n$ by $i_k^* = n - j_k$ and $j_k^* = n - i_k$ for every k. The map Ψ restricts to an isomorphism of algebraic varieties between $\mathcal{F}_{u,T}^{\rho}$ and $\mathcal{F}_{u^*,T}^{\rho^*}$, for every standard tableau T. Indeed, for $F = (V_0, \ldots, V_n) \in \mathcal{F}_u$ and any $k = 1, \ldots, n$, the quotient $V_{i_k}^{\perp}/V_{j_k}^{\perp}$ is naturally isomorphic to the dual space $(V_{j_k}/V_{i_k})^*$, and the endomorphism $(u^*)_{|V_{i_k}^{\perp}/V_{j_k}^{\perp}}$ induced by u^* coincides with the dual map of $u_{|V_{j_k}/V_{i_k}}$. It follows that the linear maps $(u^*)_{|V_{i_k}^{\perp}/V_{j_k}^{\perp}}$ and $u_{|V_{j_k}/V_{i_k}}$ are conjugated, thus they have the same Jordan form. Therefore, we have $\Psi(\mathcal{F}_{u,T}^{\rho}) = \mathcal{F}_{u^*,T}^{\rho^*}$ for every T.

In what follows, we may thus assume that $\rho = (i_k, j_k)_k$ is such that $(i_{n-1}, j_{n-1}) = (0, n-1)$, since otherwise we can deal with (u^*, ρ^*) instead of (u, ρ).

3.5. Let $\mathcal{H}_u$ be the set of u-stable hyperplanes $H \subset V$. Let $Z(u) \subset GL(V)$ be the (closed) subgroup of elements g such that $gu = ug$. The group $Z(u)$ is connected since it is an open subset of the vector space of endomorphisms which commute with u. The action of $Z(u)$ on hyperplanes leaves $\mathcal{H}_u$ invariant. The action of $Z(u)$ on flags leaves the Springer fiber $\mathcal{F}_u$ invariant. The map

$$\Phi : \mathcal{F}_u \to \mathcal{H}_u, \quad (V_0, \ldots, V_n) \mapsto V_{n-1}$$

is algebraic and $Z(u)$-equivariant.

Now we fix a standard tableau T. It is easy to see that the action of $Z(u)$ on flags leaves $\mathcal{F}_{u,T}^{\rho}$ invariant. We consider the restriction of Φ to $\mathcal{F}_{u,T}^{\rho}$

$$\Phi_T : \mathcal{F}_{u,T}^{\rho} \to \mathcal{H}_u, \quad (V_0, \ldots, V_n) \mapsto V_{n-1}$$

which is algebraic and $Z(u)$-equivariant. Let T' be the subtableau obtained from T by deleting the box number n. Let Y' be the shape of T', which is the subdiagram of $Y(u)$ obtained by deleting the same box. Write $\rho' = (i_k, j_k)_{k=1,\ldots,n-1}$. The image of Φ_T is the subset of u-stable hyperplanes H such that the Young diagram $Y(u_{|H})$ associated to the restriction of u to H is equal to Y'. Let $H \in \mathcal{H}_u$ be such a hyperplane. Then we have

$$\Phi_T^{-1}(H) = \{(V_0, \ldots, V_n) \in \mathcal{F}_{u,T}^{\rho}\colon V_{n-1} = H\} = \mathcal{F}_{u_{|H},T'}^{\rho'}$$

where $\mathcal{F}_{u_{|H},T'}^{\rho'}$ is the subset which corresponds to T' in the Springer fiber $\mathcal{F}_{u_{|H}}$ associated to the nilpotent map $u_{|H} : H \to H$.

We prove Theorems 3.2 and 3.3 by induction on $n = \dim V$. For Theorem 3.2 we show that Φ_T is locally trivial. For Theorem 3.3, using the local triviality of Φ_T, we construct a cell decomposition of $\mathcal{F}_{u,T}^{\rho}$ over a cell decomposition of the image of Φ_T.

3.6. First, we study the action of $Z(u)$ on $\mathcal{H}_u$. Note that a hyperplane H is u-stable if and only if $H \supset \operatorname{Im} u$. Let $W = V/\operatorname{Im} u$ and let $\zeta : V \to W$ be the surjective linear map. Then the variety $\mathcal{H}_u$ is isomorphic to the variety $\mathcal{H}(W)$ of hyperplanes of W. Each $g \in Z(u)$ defines a quotient map in $GL(W)$. We get a morphism of algebraic groups $\varphi : Z(u) \to GL(W)$. Then $Z(u)$ acts linearly on $\mathcal{H}(W)$ and the isomorphism $\mathcal{H}_u \cong \mathcal{H}(W)$ is $Z(u)$-equivariant.

The iterated kernels form a partial flag $0 \subset \ker u \subset \ker u^2 \subset \cdots \subset \ker u^s = V$. We get a partial flag of W:

$$0 \subset \zeta(\ker u) \subset \zeta(\ker u^2) \subset \cdots \subset \zeta(\ker u^s) = W.$$

Let $W_q = \zeta(\ker u^q)$. Let

$$P = \{g \in GL(W)\colon g(W_q) = W_q \ \forall q\}.$$

This is a parabolic subgroup. Each kernel $\ker u^q$ is invariant by $g \in Z(u)$, hence the image of φ is contained in P. We prove the following

Lemma. *There is a morphism of algebraic groups $\psi : P \to Z(u)$ such that $\varphi \circ \psi = \mathrm{id}_P$.*

Proof. We fix a linear embedding $\xi : W \hookrightarrow V$ such that $\zeta \circ \xi = \mathrm{id}_W$ and such that in addition $\xi(W_q) \subset \ker u^q$. Hence $\xi(W_q) = \xi(W) \cap \ker u^q$. Any $g \in P$ induces a linear map $\xi g \xi^{-1} : \xi(W) \to V$. Let $W' = \xi(W)$ and $g' = \xi g \xi^{-1}$. Let us prove that there is a unique linear map $\bar{g} : V \to V$ commuting with u which extends g'. We have $V = \bigoplus_{q=0}^{s-1} u^q(W')$. For $v = u^q(w) \in u^q(W')$, we must have $\bar{g}(v) = u^q(g'(w))$. Thus the extension is unique. Let us show that $\bar{g}$ defined in that way on $u^q(W')$ is well defined. If $v = u^q(w) = u^q(w')$ with $w, w' \in W'$ then $w - w' \in \ker u^q$. As g leaves W_q invariant and as $\xi(W_q) = W' \cap \ker u^q$, we get $g'(w - w') \in \ker u^q$ hence $u^q g'(w - w') = 0$. Thus $\bar{g}(v) = u^q(g'(w)) = u^q(g'(w'))$ is well defined. It is straightforward to show that the map so obtained is linear on $u^q(W')$. By collecting together these maps on the $u^q(W')$'s we get a linear map $\bar{g} : V \to V$ which commutes with u. By construction, the map $g \mapsto \bar{g}$ is algebraic. Moreover, by uniqueness, we have $\overline{h \circ g} = \bar{h} \circ \bar{g}$ for $g, h \in P$. Therefore, the map $\psi : P \to Z(u)$ defined by $\psi(g) = \bar{g}$ is a morphism of algebraic groups. $\square$

By the lemma the orbits of $\mathcal{H}(W)$ for the action of $Z(u)$ are the orbits for the action of P, which are the nonempty subsets among the $\mathcal{H}(W)_q$'s, defined for $q = 1, \ldots, s$ by

$$\mathcal{H}(W)_q = \{H \in \mathcal{H}(W)\colon H \supset W_{q-1} \text{ and } H \not\supset W_q\}.$$

The orbits of $\mathcal{H}_u$ for the action of $Z(u)$ are the corresponding subsets $\mathcal{H}_{u,q}$ defined for $q = 1, \ldots, s$ by

$$\mathcal{H}_{u,q} = \{H \in \mathcal{H}_u\colon H \supset \ker u^{q-1} \text{ and } H \not\supset \ker u^q\}.$$

Recall that $\lambda(u) = (\lambda_1 \geqslant \cdots \geqslant \lambda_r)$ is the partition of n formed by the sizes of the Jordan blocks of u, and $Y(u)$ is the Young diagram of rows of lengths $\lambda_1, \ldots, \lambda_r$. Let $\mu = (\mu_1 \geqslant \cdots \geqslant \mu_s)$ be the conjugated partition, i.e., μ_q is the length of the q-th column of the diagram. In particular $\mu_1 = r$ is the length of the first column. Let $\mu'_q = \mu_q - \mu_{q+1}$ for $q < s$ and $\mu'_s = \mu_s$. Thus μ'_q is the number of rows of length q in the diagram. We have $\dim W_q = \mu'_1 + \cdots + \mu'_q$. Observe that $\mathcal{H}(W)_q$ is nonempty if and only if $\mu'_q \neq 0$, and then it is isomorphic to an open subset of $\mathcal{H}(W/W_{q-1})$, the variety of hyperplanes of W/W_{q-1}, hence $\dim \mathcal{H}(W)_q = \mu_q - 1$.

Let $B \subset P$ be a Borel subgroup. The orbits of $\mathcal{H}_u$ for the action of B form a cell decomposition of $\mathcal{H}_u$, which can be written $\mathcal{H}_u = \bigsqcup_{l=1}^{r} \mathcal{C}(l)$ so that $\dim \mathcal{C}(l) = l - 1$ and $\mathcal{H}_{u,q} = \mathcal{C}(\mu_{q+1} + 1) \sqcup \cdots \sqcup \mathcal{C}(\mu_q)$. For each l choose a representative $H_l \in \mathcal{C}(l)$. There is a unipotent subgroup $U(l) \subset B$ such that the map $\phi_l : U(l) \to C(l)$, $g \mapsto gH_l$ is an isomorphism of algebraic varieties. (We use the isomorphism $\mathcal{H}_u \cong \mathcal{H}(W)$ and we consider the Schubert cell decomposition of $\mathcal{H}(W)$, see [2, §1.1].) In particular, if $\mu'_q \neq 0$, then $\mathcal{C}(\mu_q)$ is an open subset of the orbit $\mathcal{H}_{u,q}$.

3.7. We come back to the map $\Phi_T : \mathcal{F}^{\rho}_{u,T} \to \mathcal{H}_u$ of 3.5. Let q be the index of the column of T containing n. Let us show that the image of Φ_T is the $Z(u)$-orbit $\mathcal{H}_{u,q}$.

We use the same notation as in 3.5. A hyperplane H in the image of Φ_T is such that the Young diagram $Y(u_{|H})$ associated to the restriction $u_{|H}$ is equal to Y'. As Y' is obtained from $Y(u)$ by removing one box in the q-th column, it follows $\ker u^{q-1} \subset H$ and $\ker u^q \not\subset H$. Thus $\Phi_T(\mathcal{F}^{\rho}_{u,T}) \subset \mathcal{H}_{u,q}$. As $\mathcal{H}_{u,q}$ is a $Z(u)$-orbit and as Φ_T is $Z(u)$-equivariant, we get $\Phi_T(\mathcal{F}^{\rho}_{u,T}) = \mathcal{H}_{u,q}$.

3.8. Let $H' = H_{\mu_q}$ be one representative of the open cell $\mathcal{C}(\mu_q) \subset \mathcal{H}_{u,q}$. Let $u' = u_{|H'}$ be the restriction of u. Then $\Phi_T^{-1}(H') = \mathcal{F}^{\rho'}_{u',T'}$. By induction hypothesis $\mathcal{F}^{\rho'}_{u',T'}$ is irreducible, nonsingular and $\dim \mathcal{F}^{\rho'}_{u',T'} = \dim \mathcal{F}_{u'}$. Moreover there is a cell decomposition $\mathcal{F}^{\rho'}_{u',T'} = \bigsqcup_{\tau'} C^{\rho'}(\tau')$ parameterized by the row-standard tableaux τ' of shape Y' with standardization $\mathrm{st}(\tau') = T'$, and such that $\dim C^{\rho'}(\tau') = \dim \mathcal{F}_{u'} - n_{\mathrm{inv}}(\tau')$.

As $Z(u)$ acts transitively on the image of Φ_T, it follows that the algebraic map

$$\Xi : Z(u) \times \mathcal{F}^{\rho'}_{u',T'} \to \mathcal{F}^{\rho}_{u,T}, \quad \big(g, (V_0, \ldots, V_{n-1})\big) \mapsto (gV_0, \ldots, gV_{n-1}, V)$$

is surjective. Moreover the restriction of Ξ to $U(\mu_q) \times \mathcal{F}^{\rho'}_{u',T'}$ is an isomorphism of algebraic varieties onto $\Phi_T^{-1}(\mathcal{C}(\mu_q))$, the inverse image of the open cell of $\mathcal{H}_{u,q}$. As $Z(u)$ and $\mathcal{F}^{\rho'}_{u',T'}$ are irreducible, the surjectivity of Ξ implies that $\mathcal{F}^{\rho}_{u,T}$ is irreducible. The subsets $g\Phi_T^{-1}(\mathcal{C}(\mu_q))$ for $g \in Z(u)$ form a covering of $\mathcal{F}^{\rho}_{u,T}$ by nonsingular open subsets, hence $\mathcal{F}^{\rho}_{u,T}$ is nonsingular. Moreover we have

$$\dim \mathcal{F}^{\rho}_{u,T} = \dim \Phi_T^{-1}\big(\mathcal{C}(\mu_q)\big) = \mu_q - 1 + \dim \mathcal{F}_{u'} = \dim \mathcal{F}_u$$

(by the formula in 1.3). The proof of Theorem 3.2 is complete.

For $l = \mu_{q+1} + 1, \ldots, \mu_q$ we can find $g_l \in Z(u)$ such that $H_l = g_l H'$. Then the restriction of Ξ to $U(l)g_l \times \mathcal{F}^{\rho'}_{u',T'}$ is an isomorphism of algebraic varieties onto $\Phi_T^{-1}(\mathcal{C}(l))$. For τ row-standard with $\mathrm{st}(\tau) = T$, the entry n is in the q-th column of τ, at the end of some row. Thus there is $l \in \{\mu_{q+1} + 1, \ldots, \mu_q\}$ such that n is at the end of the l-th row of n. Let τ' be the row-standard tableau obtained from τ by putting the l-th row at the place of the μ_q-th row and moving by one rank to the up each row among the $(l+1)$-th, $\ldots$, μ_q-th ones. Then n is at the same place in τ' and T, we denote by τ'' the subtableau of τ' obtained by deleting n. This is a row-standard tableau of shape Y' and standardization $\mathrm{st}(\tau'') = T'$. We define

$$C^{\rho}(\tau) = \Xi\big(U(l)g_l \times C^{\rho'}(\tau'')\big).$$

We get thus a partition $\mathcal{F}_{u,T}^{\rho} = \bigsqcup_{\tau} C^{\rho}(\tau)$ parameterized by row-standard tableaux τ of standardization $\mathrm{st}(\tau) = T$. This partition is a cell decomposition since it is the product of two cell decompositions. It follows from the definition of inversions that $n_{\mathrm{inv}}(\tau) = n_{\mathrm{inv}}(\tau'') + (\mu_q - l)$. We deduce

$$\dim C^{\rho}(\tau) = \dim \mathcal{C}(l) + \dim C^{\rho'}(\tau'') = l - 1 + \dim \mathcal{F}_{u'} - n_{\mathrm{inv}}(\tau'') = \dim \mathcal{F}_u - n_{\mathrm{inv}}(\tau).$$

Therefore this cell decomposition satisfies the required properties. The proof of Theorem 3.3 is complete. □

3.9. Remark. Another cell decomposition of $\mathcal{F}_u$

The construction of our cell decomposition relies on the Schubert cell decomposition of the Grassmannian of hyperplanes of $\mathcal{H}(V/\mathrm{Im}\,u)$, and an inductive argument. A construction of a different cell decomposition relies on the Schubert cell decomposition of the flag variety $\mathcal{F}$. Recall that, if $B \subset GL(V)$ is a Borel subgroup, then the B-orbits of $\mathcal{F}$ form a cell decomposition $\mathcal{F} = \bigsqcup_{\sigma \in S_n} S(\sigma)$ parameterized by the permutations, and the cells are called Schubert cells. We show that the intersection of the Schubert cells with $\mathcal{F}_u$ gives a cell decomposition of $\mathcal{F}_u$ provided that the Borel subgroup B is well chosen. Our proof is different than in [8].

We consider a Jordan basis of u. Recall that $\lambda_1 \geqslant \cdots \geqslant \lambda_r$ denote the lengths of the Jordan blocks of u. Let us index the basis $(e_{p,q})$ for $p = 1, \ldots, r$ and $q = 1, \ldots, \lambda_p$ so that $(e_{p,q})_{q=1,\ldots,\lambda_p}$ is the subbasis corresponding to the p-th Jordan block and we have

$$u(e_{p,1}) = 0 \quad \text{and} \quad u(e_{p,q}) = e_{p,q-1} \quad \text{for } q = 2, \ldots, \lambda_p.$$

Such a pair (p,q) with $1 \leqslant q \leqslant \lambda_p$ forms the coordinates of some box in the diagram $Y(u)$. The Jordan basis is thus indexed on the boxes of $Y(u)$.

We associate a particular flag $F_\tau \in \mathcal{F}_u$ to each row-standard tableau τ of shape $Y(u)$. For $p = 1, \ldots, r$ and $i = 1, \ldots, n$ let $\pi_p^{(i)}$ be the number of entries among $1, \ldots, i$ in the p-th row of τ. For $i = 1, \ldots, n$ we define the subspace

$$V_i = \langle e_{p,q}\colon\ p = 1, \ldots, r,\ q = 1, \ldots, \pi_p^{(i)} \rangle.$$

It is immediate that this subspace is stable by u. Finally let $F_\tau = (V_0, \ldots, V_n)$.

The basis being considered, as above, as indexed on the boxes of the diagram $Y(u)$, we see that V_i is generated by the vectors associated to the boxes of numbers $1, \ldots, i$ in τ.

Let $H \subset GL(V)$ be the subgroup of diagonal automorphisms in the basis. The flags F_τ are exactly the elements of $\mathcal{F}_u$ which are fixed by H for its natural action on flags. However H does not leave $\mathcal{F}_u$ invariant. We introduce a subtorus $H' \subset H$ with the same fixed points, which leaves $\mathcal{F}_u$ invariant. To do this, set $\epsilon_{p,q} = nq - p$. For $t \in \mathbb{C}^*$ let $h_t \in GL(V)$ be defined by $h_t(e_{p,q}) = t^{\epsilon_{p,q}} e_{p,q}$ for $p = 1, \ldots, r$ and $q = 1, \ldots, \lambda_p$. Let $H' = (h_t)_{t \in \mathbb{C}^*}$ be the subtorus so-obtained. The $\epsilon_{p,q}$'s are pairwise distinct (since $1 \leqslant p \leqslant n$) hence H' has the same fixed points as H. Moreover we have $h_t^{-1} u h_t = t^n u$ for any t. As t^n acts trivially on flags, it follows that h_t leaves $\mathcal{F}_u$ invariant.

For any $F \in \mathcal{F}_u$, as $\mathcal{F}_u$ is a projective variety, the map $t \mapsto h_t F$ admits a limit when $t \to \infty$, and this limit is a fixed point for the action of H'. For τ row-standard, write

$$S(\tau) = \Big\{ F \in \mathcal{F}_u\colon\ \lim_{t \to \infty} h_t F = F_\tau \Big\}.$$

We get a partition $\mathcal{F}_u = \bigsqcup_{\tau} S(\tau)$ parameterized by row-standard tableaux.

Write $\{(p,q)\colon\ p = 1, \ldots, r,\ q = 1, \ldots, \lambda_p\} = \{(p_i, q_i)\colon\ i = 1, \ldots, n\}$ such that we have $\epsilon_{p_1,q_1} > \cdots > \epsilon_{p_n,q_n}$. Write $e_i = e_{p_i,q_i}$. Let $B \subset GL(V)$ be the Borel subgroup of lower triangular automorphisms in the basis $(e_1, \dots, e_n)$. Then the set $S(\tau)$ in the partition is the intersection between $\mathcal{F}_u$ and the Schubert cell $BF_\tau \subset \mathcal{F}$.

We see that the flag F_τ belongs to the Spaltenstein subset $\mathcal{F}_u^T$ for $T = \mathrm{st}(\tau)$. Let $P = \{g \in GL(V)\colon\ g(\ker u^q) = \ker u^q\}$. This is a parabolic subgroup of $GL(V)$. We can see that each $\mathcal{F}_u^T$ in the Spaltenstein partition of $\mathcal{F}_u$ is the intersection between $\mathcal{F}_u$ and some P-orbit of the flag variety. Observe that $B \subset P$. Then we obtain $S(\tau) \subset \mathcal{F}_u^T$.

In particular we have $S(\tau) = \{F \in \mathcal{F}_u^T\colon\ \lim_{t\to\infty} h_t F = F_\tau\}$. The subvariety $\mathcal{F}_u^T \subset \mathcal{F}_u$ is nonsingular. By [1, §4] it follows that $S(\tau)$ is isomorphic to an affine space.

Therefore the $S(\tau)$'s form a cell decomposition of $\mathcal{F}_u$ parameterized by row-standard tableaux. Moreover $\mathcal{F}_u^T = \bigsqcup_\tau S(\tau)$ where the union is taken over tableaux τ of standardization $\mathrm{st}(\tau) = T$. This cell decomposition is different than the decomposition of Theorem 3.3. Indeed for $\tau = \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline\end{array}$ we see that $S(\tau) = \{F_\tau\}$ is a cell of codimension 2, whereas $n_{\mathrm{inv}}(\tau) = 1$. The dimension of cells in this decomposition is given in [7, §5.10].

4. Calculation of Betti numbers

4.1. Let X be an algebraic variety. We consider the classical cohomology of sheafs (see [4] for example). Let $H^*(X, \mathbb{Q})$ denote the cohomology space with rational coefficients and let $H_c^*(X, \mathbb{Q})$ denote the rational cohomology with compact support (both coincide when X is projective). The following proposition recalls that the knowledge of a cell decomposition of X allows to compute the Betti numbers of X (see [4, §4.6]).

Proposition. *Let X be an algebraic variety on $\mathbb{C}$ which admits a cell decomposition $X = \bigsqcup_{i\in I} Z_i$. For $m \in \mathbb{N}$ let r_m be the number of m-dimensional cells.*

(a) *We have $H_c^l(X, \mathbb{Q}) = 0$ for l odd and $\dim H_c^{2m}(X, \mathbb{Q}) = r_m$ for any $m \in \mathbb{N}$.*
(b) *If X is projective, then we have $H^l(X, \mathbb{Q}) = 0$ for l odd and $\dim H^{2m}(X, \mathbb{Q}) = r_m$ for any $m \in \mathbb{N}$.*

4.2. Let $d = \dim \mathcal{F}_u$ (see 1.3). We fix $\rho \in \mathcal{R}_n$ and T a standard tableau. By Theorem 3.3 and Proposition 4.1, for any $m \in \mathbb{N}$, we get the formula

$$\dim H_c^{2m}\big(\mathcal{F}_{u,T}^\rho, \mathbb{Q}\big) = \#\big\{\tau \text{ row-standard: } \mathrm{st}(\tau) = T,\ n_{\mathrm{inv}}(\tau) = d - m\big\}.$$

Let $b_m^T = \dim H_c^{2m}(\mathcal{F}_{u,T}^\rho, \mathbb{Q})$. For $i = 1, \dots, n$, let q_i be the index of the column of T containing i and let p_i be the number of rows of length q_i in the subtableau $T[1, \dots, i]$. By Proposition 2.3 we have

$$b_{d-m}^T = \#\big\{(\kappa_1, \dots, \kappa_n)\colon\ 0 \leqslant \kappa_i \leqslant p_i - 1,\ \kappa_1 + \dots + \kappa_n = m\big\} \quad \forall m = 0, \dots, d.$$

Let $\chi^T(x) = \sum_{m=0}^d b_{d-m}^T x^m$. For $p \in \mathbb{N}$ we write $[p]_x = 1 + x + \dots + x^{p-1}$. We get:

Proposition. *We have $\chi^T(x) = \prod_{i=1}^n [p_i]_x$.*

4.3. We deduce the Betti numbers of certain components of $\mathcal{F}_u$. Let $T_{\min}$ be the minimal standard tableau of shape $Y(u)$ for the dominance relation (see 3.1). By Corollary 3.2 the subset $\mathcal{F}_{u,T_{\min}}^\rho \subset \mathcal{F}_u$ is a nonsingular irreducible component. The polynomial $\chi^{T_{\min}}(x)$ is its Poincaré polynomial. Let $\mu_1, \dots, \mu_s$ be the lengths of the columns of $Y(u)$. Then $\chi^{T_{\min}}(x)$ is

$$\chi^{T_{\min}}(x) = \prod_{q=1}^s [\mu_q]_x!$$

where, for $m \in \mathbb{N}$, we write $[m]_x! = \prod_{p=1}^m [p]_x$.

Example. Suppose $Y(u) = \begin{array}{|c|c|}\hline & \\ \hline & \\ \hline \\ \cline{1-1}\end{array}$, thus $T_{\min} = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 & 5 \\ \hline 3 \\ \cline{1-1}\end{array}$. We get $\chi^{T_{\min}}(x) = [2]_x^2 \cdot [3]_x = 1 + 3x + 4x^2 + 3x^3 + x^4$.

4.4. Let $d = \dim \mathcal{F}_u$. Let $b_m = \dim H^{2m}(\mathcal{F}_u, \mathbb{Q})$. Set $\chi(x) = \sum_{m=0}^{d} b_{d-m} x^m$. By Theorem 3.3 and Proposition 4.1, we get the following

Proposition. *We have $\chi(x) = \sum_T \chi^T(x)$, where the sum is taken for T running over the set of standard tableaux of shape $Y(u)$.*

4.5. Inductive formula

If Y is a Young diagram, then write $\chi(Y)(x) := \chi(x) = \sum_{m=0}^{d} \dim H^{2(d-m)}(\mathcal{F}_u, \mathbb{Q}) x^m$ the above polynomial for $Y = Y(u)$. A box of Y is said to be a corner if it has no neighbor on the right or below. Let $C(Y) \subset Y$ be the set of corners of Y. Removing a corner c, we get a subdiagram $Y \setminus c \subset Y$. For $c \in C(Y)$, let q_c be the index of the column of Y containing c and let p_c be the number of rows of Y of length q_c. We have the following inductive formula for the polynomial $\chi(Y)(x)$.

Proposition. *We have $\chi(Y)(x) = \sum_{c \in C(Y)} [p_c]_x \chi(Y \setminus c)(x)$.*

Proof. Let $\chi(Y)_c(x) = \sum_{T \in \mathrm{Tab}_c(Y)} \chi^T(Y)(x)$, where $\mathrm{Tab}_c(Y)$ denotes the set of standard tableaux of shape Y such that c contains the entry n. We have thus $\chi(Y)(x) = \sum_{c \in C(Y)} \chi(Y)_c(x)$. The set $\mathrm{Tab}_c(Y)$ is in bijection with the set of standard tableaux of shape $Y \setminus c$, and, by Proposition 4.2, we see that $\chi(Y)_c(x) = [p_c]_x \chi(Y \setminus c)(x)$. The proof is complete. □

Example. We have computed $\chi(Y)(x)$ by induction, for several forms of Y:

$$\overbrace{\square\square\cdots\square}^{n}\ \chi(Y)(x) = 1 \qquad \begin{array}{l}\overbrace{\square\square\cdots\square}^{s}\\ \square\end{array}\ \chi(Y)(x) = s + x$$

$$n\left\{\begin{array}{l}\square\\ \square\\ \vdots\\ \square\end{array}\right.\ \chi(Y)(x) = [n]_x! \qquad r\left\{\begin{array}{l}\square\square\\ \square\\ \vdots\\ \square\end{array}\right.\ \chi(Y)(x) = [r-1]_x! \sum_{p=0}^{r-1} (r-p) x^p$$

and more generally:

$$r\left\{\begin{array}{l}\overbrace{\square\square\cdots\square}^{s}\\ \square\\ \vdots\\ \square\end{array}\right.\ \chi(Y)(x) = [r-1]_x! \sum_{p=0}^{r-1} \binom{s+p-2}{p} [r-p]_x \quad (\text{for } s \geqslant 2)$$

$$\begin{array}{l}\overbrace{\square\square\cdots\square\square}^{s}\\ \underbrace{\square\cdots\square\square}_{t}\end{array}\ \chi(Y)(x) = [2]_x^t + \sum_{p=1}^{t} \binom{s+p-1}{p-1} \frac{s-p}{p} [2]_x^{t-p} \quad (\text{for } t \leqslant s)$$

and also:

$$\overbrace{\begin{array}{l}\square\square\square\cdots\square\\ \square\square\\ \square\end{array}}^{s}\quad \chi(Y)(x) = \frac{s+3}{3}\binom{s-1}{2} + \left(1+\frac{s-2}{3}\binom{s+3}{2}\right)[2]_x + \binom{s+1}{2}[2]_x^2$$
$$+ (s-2)[2]_x[3]_x + [2]_x^4.$$

References

[1] A. Bialynicki-Birula, Some theorems on actions of algebraic groups, Ann. of Math. 98 (1973) 480–497.

[2] M. Brion, Lectures on the geometry of flag varieties, in: Topics in Cohomological Studies of Algebraic Varieties, in: Trends Math., Birkhäuser, Basel, 2005, pp. 33–85.

[3] C. de Concini, G. Lusztig, C. Procesi, Homology of the zero-set of a nilpotent vector field on a flag manifold, J. Amer. Math. Soc. 1 (1988) 15–34.

[4] M. Kashiwara, P. Schapira, Sheaves on Manifolds, Grundlehren Math. Wiss., vol. 292, Springer-Verlag, Berlin, 1990.

[5] M.A.A. van Leeuwen, Flag varieties and interpretations of Young tableau algorithms, J. Algebra 224 (2000) 397–426.

[6] G. Lusztig, Green polynomials and singularities of unipotent classes, Adv. in Math. 42 (1981) 169–178.

[7] H. Nakajima, Homology of moduli spaces of instantons on ALE spaces I, J. Differential Geom. 40 (1994) 105–127.

[8] N. Shimomura, The fixed point subvarieties of unipotent transformations on the flag varieties, J. Math. Soc. Japan 37 (1985) 537–556.

[9] P. Slodowy, Four Lectures on Simple Groups and Singularities, Communications of the Mathematical Institute, Rijksuniversiteit Utrecht, vol. 11, 1980.

[10] N. Spaltenstein, Classes Unipotentes et Sous-Groupes de Borel, Lecture Notes in Math., vol. 946, Springer-Verlag, Berlin, New York, 1982.

[11] N. Spaltenstein, On unipotent and nilpotent elements of groups of type E_6, J. London Math. Soc. 27 (1983) 413–420.

[12] T.A. Springer, Trigonometric sums, Green functions of finite groups and representations of Weyl groups, Invent. Math. 36 (1976) 173–207.

[13] N. Xi, A partition of the Springer fibers $\mathcal{B}_N$ for type A_{n-1}, B_2, G_2 and some applications, Indag. Math. 10 (1999) 307–320.