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Exact Embedding Functors for Module Categories and Submodule Lattice Quasivarieties*

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1. INTRODUCTION

Given rings with unit R and S, we will write $R \precsim S$ if there exists an exact embedding functor F: R-Mod $\to$ S-Mod. Many equivalent or sufficient conditions for the existence of such F are known. Let $\mathscr{L}(R)$ denote the quasivariety of lattices generated by the family of all submodule lattices $\mathrm{Su}({}_R M)$, ${}_R M$ a left R-module. A lattice L is in $\mathscr{L}(R)$ if and only if it is isomorphic to a sublattice of some $\mathrm{Su}({}_R M)$. The inclusion $\mathscr{L}(R) \subseteq \mathscr{L}(S)$ is known to be equivalent to $R \precsim S$. The theory of quasivarieties $\mathscr{L}(R)$ lies on the border of lattice theory and abelian category theory. The previous investigations in this field include [4–7, 11–22].

Let $\mathscr{R}$ denote the class (and category) of all rings with unit. The ring homomorphisms of $\mathscr{R}$ will preserve ring units. In the following discussion, rings will always be assumed to have 1; i.e., they will be objects in $\mathscr{R}$. The relation $R \precsim S$ is a reflexive and transitive relation on $\mathscr{R}$. So, we can define an induced equivalence: $R \sim S$ if and only if $R \precsim S$ and $S \precsim R$. Every ring is equivalent to some denumerable ring. There are continuously many different equivalence classes of rings, even if we restrict consideration to all rings with a fixed characteristic p^k, p prime and $k \geq 2$.

Let $\mathscr{W}$ denote the set of all quasivarieties of lattices, which is a complete lattice under inclusion. If $\mathscr{R}'$ is any nonempty class of rings, let $\mathscr{W}(\mathscr{R}')$ denote the subset of $\mathscr{W}$ consisting of all quasivarieties equal to $\mathscr{L}(R)$ for some R in $\mathscr{R}'$. $\mathscr{W}(\mathscr{R})$ is a join subsemilattice of $\mathscr{W}$, with $\mathscr{L}(R \times S)$ equal

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to the join of $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}$ for all R and S in $\mathscr{R}$. $\mathscr{W}$ and $\mathscr{W}(\mathscr{R})$ have continuously many elements. Observe that $\mathscr{W}(R)$ encodes the relations $R \precsim S$ and $R \sim S$ for rings R and S in $\mathscr{R}$ by the equivalents $\mathscr{L}(R) \subseteq \mathscr{L}(S)$ and $\mathscr{L}(R) = \mathscr{L}(S)$.

Let $\mathscr{R}_c$ denote the class of all commutative rings and $\mathscr{R}_{cm}$ the class of commutative rings with characteristic m ($m = 0$ or $m > 0$). Let $\mathscr{R}_m$ denote the class of all rings with characteristic m. Obviously $\mathscr{R}_c$, $\mathscr{R}_{cm}$, and $\mathscr{R}_m$ admit direct products, and so $\mathscr{W}(\mathscr{R}_c)$, $\mathscr{W}(\mathscr{R}_{cm})$, and $\mathscr{W}(\mathscr{R}_m)$ are join subsemilattices of $\mathscr{W}$.

After reviewing some known results in Section 2, we show in Section 3 that $\mathscr{W}(\mathscr{R}_c)$ is a complete lattice. Here, $\mathscr{L}(R \otimes S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R}_c)$, and the glb of an infinite family $\{\mathscr{L}(R_j)\}_{j \in J}$ in $\mathscr{W}(\mathscr{R}_c)$ can be formed using a suitable direct limit of a sequence of finite tensor products of rings in $\{R_j\}_{j \in J}$.

In Section 4, we consider $\mathscr{L}(R)$ for rings R with characteristic zero. For each prime p, we construct a ring R_p which either has characteristic p^k for some $k \geq 0$ or is equal to the localization of the integers $\mathbf{Z}$ at the prime ideal $p\mathbf{Z}$. If $\mathbf{Q}$ is the field of rationals, $\mathscr{L}(\mathbf{Q}) \subseteq \mathscr{L}(R)$ if and only if R is in $\mathscr{R}_0$. It is proved that $\mathscr{L}(R)$ is the join in $\mathscr{W}$ of $L(\mathbf{Q})$ and the $\mathscr{L}(R_p)$ for all primes p. In effect, $\mathscr{L}(R)$ is determined by aggregating its properties with respect to each prime p.

2. TERMINOLOGY AND KNOWN RESULTS

Based mainly on [14, 17, 22] and also on standard books [1, 2, 9, 10, 23, 24], now we review the notions and statements that will be used to achieve the main results.

Let $\mathrm{char}(R)$ denote the characteristic of a ring R.

2.1. (a) If R and S are rings with unit, then $\mathscr{L}(R) \subseteq \mathscr{L}(S)$ if and only if there exists an exact embedding functor F: R-Mod $\to$ S-Mod [14, 17].

(b) If there is a ring homomorphism f: $R \to S$, then $\mathscr{L}(S) \subseteq \mathscr{L}(R)$ [14, Proposition 2].

(c) If ${}_S M_R$ is an (S, R)-bimodule such that M_R is a faithfully flat right R-module, then $\mathscr{L}(R) \subseteq \mathscr{L}(S)$. (The tensor functor ${}_S M_R \otimes_R -$: R-Mod $\to$ S-Mod is then an exact embedding functor or cf. [14, Proposition 3].)

(d) If $\mathscr{L}_1$ is in $\mathscr{W}$ and $\mathscr{L}_0$ is a class of lattices such that, for each lattice Horn formula Λ, $\mathscr{L}_1 \vDash \Lambda$ implies $\mathscr{L}_0 \vDash \Lambda$, then $\mathscr{L}_0 \subseteq \mathscr{L}_1$.

(e) Suppose R is a ring and Λ is a universal Horn formula for lattices such that $\mathscr{L}(R) \vDash \Lambda$. Then there exists an (existentially quantified) system of equations Γ for rings with unit such that $R \models \Gamma$, and if $S \models \Gamma$, then $\mathscr{L}(S) \models \Lambda$ [22].

(f) Suppose rings R and S have characteristic d and e, respectively, and $\mathscr{L}(R) \subseteq \mathscr{L}(S)$. If $e \neq 0$, then d divides e. If $d \neq 0$, then $\mathscr{L}(R) \subseteq \mathscr{L}(S/dS)$ [14, Theorem 3].

(g) If R has prime characteristic p, the $\mathscr{L}(R) = \mathscr{L}(\mathbf{Z}/p\mathbf{Z})$. (Use 2.1(b) and (c) or cf. [14].)

(h) If $\{\mathscr{L}_j\}_{j \in J}$ is an infinite subfamily of $\mathscr{W}$ and $\mathscr{L} = \bigvee_{j \in J} \mathscr{L}_j$ in $\mathscr{W}$, then, for each lattice Horn formula Λ, $\mathscr{L} \models \Lambda$ if and only if $\mathscr{L}_j \models \Lambda$ for all j in J.

(i) Suppose R is a ring, $S = \prod_{j \in J} S_j$ is a product of a nonempty family $\{S_j\}_{j \in J}$ of rings, and there is a family of exact functors

$$\left\{F_j\colon R\text{-Mod} \to S_j\text{-Mod}\right\}_{j \in J}.$$

If $\{F_j(M)\}_{j \in J}$ contains some nonzero S_j-module whenever M is a nonzero R-module, then there exists an exact embedding functor $F\colon R\text{-Mod} \to S\text{-Mod}$. (As an additive group, take $F(M)$ isomorphic to $\bigoplus_{j \in J} F_j(M)$. Use projections $\pi_j\colon S \to S_j$ to make each $F_j(M)$ an S-module, hence $F(M)$ an S-module. Suppose $f\colon M \to N$ in R-Mod. Define $F(f)\colon F(M) \to F(N)$ from the S-homomorphisms $F_j(f)\colon F_j(M) \to F_j(N)$ as usual. Then F is an exact embedding functor.)

Hereafter, we will let $\operatorname{char}(R)$ denote the characteristic of R in $\mathscr{R}$.

2.2. DEFINITIONS. Tensor products $A \otimes B$ are taken over the integers $\mathbf{Z}$ unless otherwise indicated. Recall that $R \otimes S$ is a ring if R and S are rings.

(a) The tensor product $R \otimes S$ over $\mathbf{Z}$ is a coproduct for commutative rings R and S relative to $\mathscr{R}_c$. That is, there are homomorphisms $\alpha_R\colon R \to R \otimes S$ and $\alpha_S\colon S \to R \otimes S$ such that, given any homomorphisms $f\colon R \to T$ and $g\colon S \to T$ in $\mathscr{R}_c$, there exists a unique homomorphism $h\colon R \otimes S \to T$ such that $h\alpha_R = f$ and $h\alpha_S = g$. We have $\alpha_R(r) = r \otimes 1$ and $\alpha_S(s) = 1 \otimes s$. We use the matrix notation $h = [f\ g]$.

(b) If R is commutative, then $\mathscr{L}(R) = \mathscr{L}(R \otimes R)$ by 2.1(b) and the homomorphisms $\alpha_R\colon R \to R \otimes R$ and $[1_R\ \ 1_R]\colon R \otimes R \to R$ of 2.2(a).

(c) For R and S in $\mathscr{R}_c$, $R \otimes S$ and $S \otimes R$ are isomorphic, using isomorphisms obtained from the coproduct universal properties.

2.3. (a) If R and S are any rings, then $\mathscr{L}(R \times S)$ is the join of $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}$ [7, Proposition 4.2].

(b) If R and S have characteristics d and e, respectively, then $R \times S$ has characteristic equal to the lcm of d and e (defined as 0 if d or e is 0).

2.4. The direct limit of a sequence of rings

$$R_1 \to R_2 \to R_3 \to \cdots$$

is defined up to isomorphism by the colimit universal property for the above commutative diagram. Formally, a direct system $\{f_i^j: R_i \to R_j\}_{1 \le i \le j}$ of homomorphisms is defined such that $f_j^k f_i^j = f_i^k$ for all $1 \le i \le j \le k$ and $f_i^i = 1_{R_i}$ for $i \ge 1$. The direct limit R of this direct system has associated homomorphisms $f_i: R_i \to R$ for each $i \ge 1$, which satisfy $f_i = f_j f_i^j$ for $1 \le i \le j$. The colimit property is defined as follows: if $g_i: R_i \to S$ for $i \ge 1$ such that $g_i = g_j f_i^j$ for $1 \le i \le j$, then there exists a unique homomorphism $h: R \to S$ such that $hf_i = g_i$ for all $i \ge 1$. We can directly construct R by taking $X = \bigcup_{i \ge 1} R_i$ to be a pairwise disjoint union, forming the equivalence relation on θ on X generated by all pairs $\langle u, f_i^j(u)\rangle$ for $1 \le i \le j$ and u in R_i, and proving that there exists a unique ring structure for the quotient set $R = X/\theta$ such that each $f_i: R_i \to R$ given by $f_i(u) = \theta[u]$ for $i \ge 1$ and u in R_i is a homomorphism. We can verify:

(a) $f_i[R_i] \subseteq f_j[R_j]$ if $1 \le i \le j$, and $R = \bigcup_{i \ge 1} f_i[R_i]$.

(b) For u in R_i and v in R_j, $f_i(u) = f_j(v)$ in R if and only if there exists $n \ge \max\{i, j\}$ such that $f_i^k(u) = f_j^k(v)$ for all $k \ge n$.

2.5. Definition. Recall that $\mathbf{Z}$ is initial in $\mathscr{R}$, and let $\iota_R: \mathbf{Z} \to R$ denote the unique homomorphism $\mathbf{Z} \to R$. Elements of $\iota_R[\mathbf{Z}]$ are called $\mathbf{Z}$-images in R, and are central elements of R. Define $n \cdot r$ for integers $n > 0$ and r in R as the sum of n terms r. Also, let $0 \cdot r = 0$ and $n \cdot r = -(|n| \cdot r)$ if $n < 0$. So, $\iota_R(n) = n \cdot 1_R$ for all n in $\mathbf{Z}$.

Let P denote the set of prime numbers and P_n the set of the first n primes $\{p_1, p_2, \dots, p_n\}$ for $n \ge 0$. For p prime and R in $\mathscr{R}$, let $\mathrm{dgr}_R(p) = k$ if $k \ge 0$ is the smallest integer such that $R \models (\exists x)(p^{k+1} \cdot x = p^k \cdot 1)$. (The formula is equivalent to requiring that $\iota_R(p^{k+1})$ divides $\iota_R(p^k)$ in R.) If there is no such k, let $\mathrm{dgr}_R(p) = +\infty$.

(a) ι_R is one-to-one if and only if $\mathrm{char}(R) = 0$. If $m > 0$, then there is a homomorphism $\mathbf{Z}/m\mathbf{Z} \to R$ if and only if $\mathrm{char}(R)$ divides m. This homomorphism is one to one if and only if $m = \mathrm{char}(R)$.

(b) For any ring R, $\mathrm{dgr}_R(p) = 0$ if and only if $\iota_R(p)$ is a unit of R.

(c) If $\mathrm{char}(R) = m > 0$ and m has prime factorization $p_1^{k_1} p_2^{k_2} \cdots p_t^{k_t}$, then $\mathrm{dgr}_R(p_i) = k_i$ for $i \le t$ and $\mathrm{dgr}_R(p_i) = 0$ for $i > t$ [21, Proposition 1].

(d) If R and S are in $\mathscr{R}$ and $\mathscr{L}(R) \subseteq \mathscr{L}(S)$, then $\mathrm{dgr}_R(p) \leq \mathrm{dgr}_S(p)$ for all p in P. (This follows from [21, Proposition 6].)

2.6. DEFINITIONS. For $X \subseteq P$, let $\mathbf{Q}(X)$ denote the subring of the rationals generated by $\{1\} \cup \{1/p : p \in X\}$. Note that $X \mapsto \mathbf{Q}(X)$ defines a one-to-one correspondence between subsets of P and subrings of $\mathbf{Q}$ containing $\mathbf{Z}$. Let $\mathbf{Z}_{\langle p \rangle} = \mathbf{Q}(P - \{p\})$, which equals the localization of $\mathbf{Z}$ at $p\mathbf{Z}$ for p prime.

(a) If $X \subseteq Y \subseteq P$, then $\mathbf{Q}(X)$ is a subring of $\mathbf{Q}(Y)$, and $\mathscr{L}(\mathbf{Q}(Y)) \subseteq \mathscr{L}(\mathbf{Q}(X))$.

(b) For $X \subseteq P$, p in $P - X$, and $k \geq 0$, $\mathbf{Q}(X)/p^k\mathbf{Q}(X)$ is isomorphic to $\mathbf{Z}/p^k\mathbf{Z}$.

(c) If $X \subseteq P$, then any torsion-free $\mathbf{Q}(X)$-module is flat, since $\mathbf{Q}(X)$ is a Prüfer ring [24, p. 129].

(d) If A is an abelian group with an element of infinite order, then $\mathbf{Q} \otimes A \neq 0$. ($\mathbf{Q}$ is a flat $\mathbf{Z}$-module, so $\mathbf{Q} \otimes A$ has a submodule isomorphic to $\mathbf{Q} \otimes \mathbf{Z}$.)

(e) If A has an element of prime order p and $p \notin X \subset P$, then $\mathbf{Q}(X) \otimes A \neq 0$. (Since $\mathbf{Q}(X)$ is $\mathbf{Z}$-flat, $\mathbf{Q}(X) \otimes A$ has a subgroup isomorphic to $\mathbf{Q}(X) \otimes (\mathbf{Z}/p\mathbf{Z})$, hence to $\mathbf{Q}(X)/p\mathbf{Q}(X)$, hence to $\mathbf{Z}/p\mathbf{Z}$ by 2.6(b). So, $\mathbf{Q}(X) \otimes A \neq 0$.)

3. LATTICE STRUCTURE FOR SUBMODULE LATTICE QUASIVARIETIES

We first obtain meets in $\mathscr{W}(\mathscr{R}_c)$.

3.1. PROPOSITION. *If R, S, and T are commutative rings and $\mathscr{L}(R) \subseteq \mathscr{L}(S)$, then $\mathscr{L}(R \otimes T) \subseteq \mathscr{L}(S \otimes T)$.*

Proof. Assuming the hypotheses, there are exact embedding functors F: $R \otimes T$-Mod $\to$ R-Mod and G: R-Mod $\to$ S-Mod, by 2.1(a) and (b) and 2.2(a). Note that $F(M) = M$ and $F(f) = f$ as sets and functions. Let M be an $R \otimes T$-module. So, rv in $F(M)$ for r in R and v in M equals $(r \otimes 1_T)v$ in M. Now, $GF(M)$ is an S-module, and we define an $S \otimes T$-module $H(M)$ which equals $GF(M)$ as an additive group. For t in T, let t_M: $M \to M$ in $R \otimes T$-Mod be given by $t_M(v) = (1_R \otimes t)v$. Then $t \mapsto t_M$ is a ring homomorphism from T into the ring of endomorphisms $M \to M$ in $R \otimes T$-Mod. If f: $M \to N$ in $R \otimes T$-Mod, then $ft_M = t_N f$ for each t in T.

The formula $(s \otimes t)w = s(GF(t_M)(w))$ for s in S, t in T, and w in $H(M)$ uniquely determines a well-defined $S \otimes T$-module structure for $H(M)$. Some checking shows that H: $R \otimes T$-Mod $\to$ $S \otimes T$-Mod defined by $H(M)$ and $H(f) = GF(f)$ for $f\colon M \to N$ in $R \otimes T$-Mod is an exact embedding functor. But then $\mathscr{L}(R \otimes T) \subseteq \mathscr{L}(S \otimes T)$ by 2.1(a). Q.E.D

3.2. PROPOSITION. *If R and S are commutative rings, then $\mathscr{L}(R \otimes S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R}_c)$.*

Proof. By 2.1(b) and 2.2(a), $\mathscr{L}(R \otimes S)$ is contained in $\mathscr{L}(R)$ and $\mathscr{L}(S)$. Suppose $\mathscr{L}(T) \subseteq \mathscr{L}(R)$ and $\mathscr{L}(T) \subseteq \mathscr{L}(S)$ for rings R, S, and T in $\mathscr{R}_c$. Using 2.1(b), 2.2(b) and (c), and 3.1, we have

$$\mathscr{L}(T) = \mathscr{L}(T \otimes T) \subseteq \mathscr{L}(S \otimes T) = \mathscr{L}(T \otimes S) \subseteq \mathscr{L}(R \otimes S).$$

So, $\mathscr{L}(R \otimes S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R}_c)$. Q.E.D.

Suppose $\{R_j\}_{j \geq 1}$ is an ascending chain of subrings of R with union R. By 2.1(b), (d), and (e), we can see that $\{\mathscr{L}(R_j)\}_{j \geq 1}$ is a descending chain in $\mathscr{W}(\mathscr{R})$ such that $\mathscr{L}(R) = \bigcap_{j \geq 1} \mathscr{L}(R_j)$. We extend this to direct limits of sequences.

3.3. PROPOSITION. *Suppose $\{R_i\}_{i \geq 1}$ and $\{f_i^j\colon R_i \to R_j\}_{1 \leq i \leq j}$ are a direct system formed from a sequence of rings, and R is the direct limit of the sequence with associated homomorphisms $\{f_i\colon R_i \to R\}_{i \geq 1}$. Then $\{\mathscr{L}(R_i)\}_{i \geq 1}$ is a descending sequence of lattice quasivarieties, and $\mathscr{L}(R) = \bigcap_{i \geq 1} \mathscr{L}(R_i)$.*

Proof. Assuming the hypotheses, $\{\mathscr{L}(R_i)\}_{i \geq 1}$ is a descending chain, and each $\mathscr{L}(R_i) \supseteq \mathscr{L}(R)$ by 2.1(b). Let $\mathscr{L}_0 = \bigcap_{i \geq 1} \mathscr{L}(R_i)$, so $\mathscr{L}(R) \subseteq \mathscr{L}_0$. Suppose Λ is a Horn formula such that $\mathscr{L}(R) \vDash \Lambda$. By 2.1(e), there is an existential system of ring equations

$$(\exists x_1, x_2, \ldots, x_n)\Gamma(x_1, x_2, \ldots, x_n)$$

such that $R \vDash \Gamma$, and $\mathscr{L}(S) \vDash \Lambda$ whenever $S \vDash \Gamma$. Choose $r_1, r_2, \ldots, r_n$ in R so that $\Gamma(r_1, \ldots, r_n)$ is true. By 2.4(a), there exists $s \geq 1$ such that $r_i \in f_s[R_s]$ for all $i \leq n$. Choose w_i such that $f_s(w_i) = r_i$ for $i \leq n$. Suppose $g(x_1, x_2, \ldots, x_n) = h(x_1, x_2, \ldots, x_n)$ is an equation of Γ. Since $g(r_1, \ldots, r_n) = h(r_1, \ldots, r_n)$, there exists a $t \geq s$ such that

$$f_s^t(g(w_1, \ldots, w_n)) = f_s^t(h(w_1, \ldots, w_n))$$

by 2.4(b). Since Γ contains finitely many equations, we can choose u sufficiently large so that $R_u \vDash \Gamma$ using $x_i = f_s^u(w_i)$ for $i \leq n$. But then $\mathscr{L}(R_u) \vDash \Lambda$, so $\mathscr{L}_0 \vDash \Lambda$. This proves $\mathscr{L}_0 \subseteq \mathscr{L}(R)$ by 2.1(d), hence $\mathscr{L}(R) = \mathscr{L}_0$. Q.E.D.

3.4. THEOREM. *$\mathscr{W}(\mathscr{R}_c)$ is a complete lattice. For R and S in $\mathscr{R}_c$, $\mathscr{L}(R \times S)$ and $\mathscr{L}(R \otimes S)$ are the lub and glb, respectively, of $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R}_c)$. If $\{R_k\}_{k \in K}$ is an infinite family of rings in $\mathscr{R}_c$, then there* *exists a glb* $\mathscr{L}(S)$ *for* $\{\mathscr{L}(R_k)\}_{k\in K}$ *in* $\mathscr{W}(\mathscr{R}_c)$, *and* $\mathscr{L}(S) = \bigcap_{i\geq 1}\mathscr{L}(S_i)$ *for a descending sequence* $\{\mathscr{L}(S_i)\}_{i\geq 1}$ *such that each* S_i *is a tensor product of finitely many rings in* $\{R_k\}_{k\in K}$.

Proof. By 2.3(a), we know that $\mathscr{W}(\mathscr{R}_c)$ is a join subsemilattice of $\mathscr{W}$ such that $\mathscr{L}(R\times S) = \mathscr{L}(R)\vee\mathscr{L}(S)$. Also, $\mathscr{L}(R\otimes S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R}_c)$ by 3.2. Since $\mathscr{L}(\mathbf{Z})$ is the largest element of $\mathscr{W}(\mathscr{R}_c)$ by 2.1(b), $\mathscr{W}(\mathscr{R}_c)$ is complete if it admits infinite meets.

Suppose $\{R_k\}_{k\in K}$ is an infinite subfamily of $\mathscr{R}_c$. Let $H = \{\Lambda_i\}_{i\geq 1}$, where H consists exactly of Horn formulas Λ_i satisfied in some $\mathscr{L}(T_i)$, where T_i is a finite tensor product of elements of $\{R_k\}_{k\in K}$. Define $\{S_i\}_{i\geq 1}$ by $S_i = T_1\otimes T_2\otimes\cdots\otimes T_i$. For $i<j$, $S_j = S_i\otimes T_{ij}$ for $T_{ij} = T_{i+1}\otimes T_{i+2}\otimes\cdots\otimes T_j$, and so there is a ring homomorphism $\varphi_i^j\colon S_i\to S_j$ by 2.2(a). Clearly the S_i and φ_i^j form a direct system (with $\varphi_i^i = 1_{S_i}$), and so there is a direct limit S and homomorphisms $\varphi_i\colon S_i\to S$ for each $i\geq 1$. All S_i and S are in $\mathscr{R}_c$ by 2.2(a) and 2.4(a). By 3.3, $\mathscr{L}(S) = \bigcap_{i\geq 1}\mathscr{L}(S_i)$. By 2.1(d) and 3.2, $\mathscr{L}(T)\subseteq\mathscr{L}(R_k)$ for all k in K if and only if $\mathscr{L}(T)\subseteq\mathscr{L}(S_i)$ for all $i\geq 1$. So, $\mathscr{L}(S)$ is a glb for $\{\mathscr{L}(R_k)\}_{k\in K}$. Q.E.D.

3.5. COROLLARY. *If* $\mathscr{R}'$ *is a class of commutative rings such that* $\mathscr{W}(\mathscr{R}')$ *has a largest element and* $\mathscr{R}'$ *admits direct products, tensor products, and direct limits of sequences, then* $\mathscr{W}(\mathscr{R}')$ *is a sublattice of* $\mathscr{W}(\mathscr{R}_c)$, $\mathscr{W}(\mathscr{R}')$ *is complete, and the inclusion* $\mathscr{W}(\mathscr{R}')\to\mathscr{W}(\mathscr{R}_c)$ *preserves infinite meets.*

3.6. PROPOSITION. *For* R *in* $\mathscr{R}$, $\mathscr{L}(\mathbf{Q})\subseteq\mathscr{L}(R)$ *if and only if* $\operatorname{char}(R) = 0$.

Proof. The forward implication is by 2.1(f). Assume $\operatorname{char}(R) = 0$, so ${}_RV_{\mathbf{Q}} = {}_RR\otimes\mathbf{Q}_{\mathbf{Q}}\neq 0$ by 2.6(d). Since $V_{\mathbf{Q}}$ is free, $\mathscr{L}(\mathbf{Q})\subseteq\mathscr{L}(R)$ by 2.1(c). Q.E.D.

3.7. COROLLARY. *For all* $m\geq 0$, $\mathscr{W}(\mathscr{R}_{cm})$ *is a complete sublattice of* $\mathscr{W}(\mathscr{R}_c)$, *and the inclusion* $\mathscr{W}(\mathscr{R}_{cm})\to\mathscr{W}(\mathscr{R}_c)$ *preserves both infinite joins and meets.*

Proof. By 2.3(b), $\mathscr{W}(\mathscr{R}_{cm})$ is a join subsemilattice of $\mathscr{W}(\mathscr{R}_c)$. Now $\mathscr{R}_{cm}$ admits tensor products. This is by 3.6 and 3.2 if $m = 0$. If $m > 0$, then for $\mathbf{Z}(m) = \mathbf{Z}/m\mathbf{Z}$ we have additive group decompositions $R\approx\mathbf{Z}(m)\oplus M$ and $S\approx\mathbf{Z}(m)\otimes M'$ if R and S are in $\mathscr{R}_m$. So, $\mathscr{W}(\mathscr{R}_{cm})$ is a sublattice of $\mathscr{W}(\mathscr{R}_c)$, and $\mathscr{L}(\mathbf{Z}/m\mathbf{Z})$ is a largest element for $\mathscr{W}(\mathscr{R}_{cm})$. By 2.4(b), a direct limit of a sequence of rings in $\mathscr{R}_{cm}$ has characteristic m, and so $\mathscr{W}(\mathscr{R}_{cm})$ is complete and the inclusion $\mathscr{W}(\mathscr{R}_{cm})\to\mathscr{W}(\mathscr{R}_c)$ preserves infinite meets.

Suppose $\{R_k\}_{k\in K}$ is an infinite subfamily of $\mathscr{R}_{cm}$. Let $\{\mathscr{L}(S_j)\}_{j\in J}$ be the subset of $\mathscr{W}(\mathscr{R}_c)$ such that, for each j, $\mathscr{L}(S_j)\supseteq\mathscr{L}(R_k)$ for all k in K. Let $J_0 = \{j\in J : S_j\in\mathscr{R}_{cm}\}$. Let $\mathscr{L}(S)$ be the glb in $\mathscr{W}(\mathscr{R}_{cm})$ of $\{\mathscr{L}(S_j) : j\in J_0\}$. If $m = 0$, then $J = J_0$ by 2.1(f), and $\mathscr{L}(S)$ is the lub of $\{\mathscr{L}(R_k)\}_{k\in K}$ in $\mathscr{W}(\mathscr{R}_c)$. If $m\neq 0$, then $\mathscr{L}(R_k)\subseteq\mathscr{L}(S_j/mS_j)\subseteq\mathscr{L}(S_j)$ for $k\in K$ and $j\in J$, and $\mathscr{L}(S_j/mS_j) = \mathscr{L}(S_q)$ for some q in J_0. So, $\mathscr{L}(S)$ is also the glb for $\{\mathscr{L}(S_j)\}_{j \in J}$ in $\mathscr{W}(\mathscr{R}_c)$, and it equals the lub for $\{\mathscr{L}(R_k)\}_{k \in K}$ in $\mathscr{W}(\mathscr{R}_c)$. Therefore, the inclusion $\mathscr{W}(\mathscr{R}_{cm}) \to \mathscr{W}(\mathscr{R}_c)$ preserves infinite joins also. Q.E.D.

It is not known whether $\mathscr{L}(R \otimes S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R})$. In particular, $\mathscr{L}(R) \subseteq \mathscr{L}(R \otimes R)$ might not hold for noncommutative R.

Let $R \cdot S$ denote the (noncommutative) coproduct of rings R and S in $\mathscr{R}$. Essentially, $R \cdot S$ can be formed from a ring with unit, freely generated by a disjoint union $R \cup S$ and then divided by the two-sided ideal generated by relations true in R, relations true in S, and $1 = 1_R = 1_S$. As in 2.2(a)–(c), $\mathscr{L}(R \cdot S)$ is a lower bound for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(\mathscr{R})$, $\mathscr{L}(R \cdot R) = \mathscr{L}(R)$, and $\mathscr{L}(R \cdot S) = \mathscr{L}(S \cdot R)$.

It is not clear whether $\mathscr{L}(R) \subseteq \mathscr{L}(S)$ implies $\mathscr{L}(R \cdot T) \subseteq \mathscr{L}(S \cdot T)$ in general. If this is true, then we can prove $\mathscr{L}(R \cdot S)$ is a glb for $\mathscr{L}(R)$ and $\mathscr{L}(S)$ in $\mathscr{W}(R)$ as in 3.2. In that case, adapting the proof of 3.4 shows that $\mathscr{W}(\mathscr{R})$ is a complete lattice, and similarly for 3.7 and $\mathscr{W}(\mathscr{R}_m)$.

4. SUBMODULE LATTICE QUASIVARIETIES FOR RINGS WITH CHARACTERISTIC ZERO

In the following, we show that $\mathscr{L}(R)$ for rings with characteristic zero can be determined from char(R), $\mathrm{dgr}_R(p)$ for primes p, and $\mathscr{L}(R_p)$ for certain associated rings R_p which have prime power characteristic.

4.1. Proposition. *Suppose* $\mathrm{dgr}_R(p) = k < +\infty$ *for* R *in* $\mathscr{R}$, *so* $p^{k+1} \cdot r = p^k \cdot 1_R$ *for some* r *in* R. *Choosing such an* r, *we have*:

(a) *If* $n \geq k$ *and* $m \geq j > 0$, *then* $(p^j \cdot 1_R)(p^n \cdot r^m) = p^n \cdot r^{m-j}$. *Therefore*, $p^n \cdot r^n = p^k \cdot r^k$ *if* $n > k$, *and so* $p^k \cdot r^k$ *is an idempotent* $(p^{2k} \cdot r^{2k} = p^k \cdot r^k)$. *Also*, $e = 1 - p^k \cdot r^k$ *is an idempotent, and* $p^k \cdot e = 0$. *Note that* R/p^kR *has characteristic* p^k *in this case.*

(b) *If* $e = 1 - p^k \cdot r^k$ *as above, then* $h(x + p^kR) = exe$ *determines a ring isomorphism* $h\colon R/p^kR \to eRe$.

4.2. Definitions. Suppose R is in $\mathscr{R}$ and p is a prime. Let R_p denote $\mathbf{Q}(P - \{p\})$ if $\mathrm{dgr}_R(p) = +\infty$, and R/p^kR if $\mathrm{dgr}_R(p) = k$, $0 \leq k < +\infty$.

Define Reduct(R) for R in $\mathscr{R}$ to equal $\mathbf{Q} \times \prod_{p \in P} R_p$ if char$(R) = 0$, and to equal $\prod_{p \in P} R_p$ if char$(R) > 0$.

(a) If R is a ring and char$(R) = m > 0$, then $R \approx$ Reduct(R). (If m has prime factorization $q_1^{k_1} q_2^{k_2} \cdots q_n^{k_n}$ for some primes q_i and $k_i \geq 1$, $i = 1, 2, \ldots, n$, then $R_{q_i} = R/q_i^{k_i}R$ for $i \leq n$ and R_p is trivial for other primes p by 2.5(b) and (c). Since $\mathrm{Reduct}(R) \approx \prod_{i \le n} R/q_i^{k_i}R$, we can apply the Chinese remainder theorem [1, Exercise 7.13, p. 103]. If $m = 1$, then R and all R_p are trivial.)

(b) For each R in $\mathscr{R}$, $\mathrm{char}(R) = \mathrm{char}(\mathrm{Reduct}(R))$ and R_p is isomorphic to $\mathrm{Reduct}(R)_p$ for each prime p. So, $\mathrm{Reduct}(\mathrm{Reduct}(R)) \approx \mathrm{Reduct}(R)$.

(c) If $\mathrm{dgr}_R(p) = k < +\infty$, then $\mathrm{char}(R/p^iR) = p^j$ for $i \ge 0$ and $j = \min\{i, k\}$. If $\mathrm{dgr}_R(p) = +\infty$, then $\mathrm{char}(R/p^iR) = p^i$ for $i \ge 0$.

4.3. PROPOSITION. *If R is in $\mathscr{R}$ and p is prime, then $\mathscr{L}(R_p) \subseteq \mathscr{L}(R)$.*

Proof. Suppose $\mathrm{dgr}_R(p) = +\infty$. So, $R_p = \mathbf{Z}_{\langle p \rangle} = \mathbf{Q}(P - \{p\})$ and $\mathrm{char}(R) = 0$ by 2.5(c). Let $S = R_p$ and $T_0 = {}_RR \otimes S_S$, an (R, S)-bimodule. By localization properties, T_0 can be regarded as consisting of fractions r/u with r in R and u in $\mathbf{Z} - p\mathbf{Z}$, where $r/u = r'/u'$ if and only if $v(ru' - r'u) = 0$ for some v in $\mathbf{Z} - p\mathbf{Z}$. Let $\mathrm{Ker}\, m \cdot 1_{T_0} = \{v \in T_0 : m \cdot v = 0\}$, and let ${}_RT_S = T_0/T_p$ for $T_p = \bigcup_{i \ge 0} \mathrm{Ker}\, p^i \cdot 1_{T_0}$. If T_S is not torsion-free, then $vq = 0$ for some prime q and $v \ne 0$ in T_S. If $q \ne p$, then $1/q$ is in S, and we get the contradiction $v = vq(1/q) = 0$. Suppose $q = p$, and $v = w + T_p$ in $T_S = T_0/T_p$. Since $wp \in T_p$, $(wp)p^k = 0$ for some $k \ge 0$. But then $w \in T_p$ and again $v = 0$. This proves that T_S is torsion-free, and so is flat by 2.6(c). Since S is local with maximal ideal pS, T_S is faithfully flat if $T \otimes_S (S/pS) \ne 0$. By 2.6(b), we can prove this is equivalent to $T/pT \ne 0$. For more details, cf. [14, Proposition 11]. Assume the contrary, so that $pT = T$. Then $(r/u + T_p)p = 1_R/1 + T_p$ for some r in R and u in $\mathbf{Z} - p\mathbf{Z}$. Let $\bar{1}$ denote $1_R/1$, so $(r/u)p - \bar{1} = y$ in T_0, for y satisfying $yp^k = 0$ for some $k \ge 0$. Choose integers a and b such that $au + bp = 1$, so that

$$\begin{aligned}(ra/1 + \bar{1}b)p^{k+1} &= (r/u)aup^{k+1} + \bar{1}(1 - au)p^k \\ &= (rp/u - \bar{1})aup^k + \bar{1}p^k \\ &= yaup^k + \bar{1}p^k = \bar{1}p^k.\end{aligned}$$

So, $xz = 0$ for $x = rap^{k+1} + 1_Rbp^{k+1} - 1_Rp^k$ in R and some z in $\mathbf{Z} - p\mathbf{Z}$. Choosing integers c and d with $cz + dp = 1$, we see that $x(1 - dp) = 0$ in R, which leads to the contradiction $\mathrm{dgr}_R(p) \le k < +\infty$. So $pT \ne T$, and $\mathscr{L}(R_p) \subseteq \mathscr{L}(R)$ follows by using 2.1(c) with ${}_RT_S$.

If $\mathrm{dgr}_R(p) = k < +\infty$, then $R_p = R/p^kR$ and $\mathscr{L}(R_p) \subseteq \mathscr{L}(R)$ by 2.1(b). Q.E.D.

In general, we do not assert that 2.3(a) can be extended to infinite products of rings. However, this extension is possible for products of the form $\mathrm{Reduct}(R)$.

4.4. PROPOSITION. *For all R in $\mathscr{R}_0$, $\mathscr{L}(\text{Reduct}(R))$ is the join in $\mathscr{W}$ of $\mathscr{L}(\mathbf{Q})$ and $\mathscr{L}(R_p)$ for all p in P.*

Proof. Suppose R is in $\mathscr{R}$ and $\text{char}(R) = 0$. Let $\mathscr{L}_1 = \mathscr{L}(\mathbf{Q}) \vee \bigvee_{p \in P} \mathscr{L}(R_p)$ in $\mathscr{W}$. Then $\mathscr{L}_1 \subseteq \mathscr{L}(\text{Reduct}(R))$ by 3.6, 4.2(b), and 4.3. Suppose $\mathscr{L}_1 \models \Lambda$ for some lattice Horn formula Λ. Since $\mathscr{L}(\mathbf{Q}) \models \Lambda$, there is a system of ring equations $(\exists \mathbf{x})\Gamma_0(\mathbf{x})$ for $\mathbf{x} = \langle x_1, x_2, \ldots, x_s \rangle$ such that $\mathbf{Q} \models (\exists \mathbf{x})\Gamma_0(\mathbf{x})$, and $\mathscr{L}(S) \models \Lambda$ if $S \models (\exists \mathbf{x})\Gamma_0(\mathbf{x})$ by 2.1(e). Suppose $\mathbf{Q} \models \Gamma_0(m_1/n_1, m_2/n_2, \ldots, m_s/n_s)$. Choose n large enough so that m_i/n_i is in $\mathbf{Q}(P_n)$ for $i = 1, 2, \ldots, s$. By construction, $\mathscr{L}(\mathbf{Q}(P_n)) \models \Lambda$. Since $\mathscr{L}_1 \models \Lambda$, $\mathscr{L}(R_p) \models \Lambda$ for p prime. In particular, $\mathscr{L}(\mathbf{Q}(P_n) \times R_{p_1} \times R_{p_2} \times \cdots \times R_{p_n}) \models \Lambda$ by 2.1(h) and 2.3(a). Again using 2.1(e) and adjusting s, there is a system of ring equations $(\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$ such that

$$\mathbf{Q}(P_n) \times R_{p_1} \times R_{p_2} \times \cdots \times R_{p_n} \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x})),$$

and $\mathscr{L}(S) \models \Lambda$ if $S \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$. We assert that $\text{Reduct}(R) \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$. Since $\mathbf{Q} \supseteq \mathbf{Q}(P_n) \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$, it suffices to prove that $R_p \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$ for all primes p. If $p = p_i$ for $i \leq n$, this follows from the definition of Γ_1. If $p = p_i$ for $i > n$ and $\text{dgr}_R(p) = +\infty$, then it follows because $\mathbf{Q}(P_n) \subseteq \mathbf{Q}(P - \{p\}) = R_p$ by 2.6(a). If $p = p_i$ for $i > n$ and $\text{dgr}_R(p) = k < +\infty$, then $R_p = R/p^k R$ and there are ring homomorphisms

$$\mathbf{Q}(P_n) \to \mathbf{Q}(P_n)/p^k\mathbf{Q}(P_n) \to \mathbf{Z}/p^k\mathbf{Z} \to R/p^k R$$

by 2.6(b) and 4.1(a). But then $R_p \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$ for all primes p, and so $\text{Reduct}(R) \models (\exists \mathbf{x})(\Gamma_1(\mathbf{x}))$, and $\mathscr{L}(\text{Reduct}(R)) \models \Lambda$. It follows by 2.1(d) that $\mathscr{L}(\text{Reduct}(R)) \subseteq \mathscr{L}_1$, and so $\mathscr{L}(\text{Reduct}(R)) = \mathscr{L}_1$. Q.E.D.

4.5. THEOREM. *If R is a ring, then $\mathscr{L}(R) = \mathscr{L}(\text{Reduct}(R))$.*

Proof. If $\text{char}(R) = m \geq 1$, then $\mathscr{L}(R) = \mathscr{L}(\text{Reduct}(R))$ by 4.2(a). So, assume $\text{char}(R) = 0$. By 3.6, 4.3, and 4.4, $\mathscr{L}(\text{Reduct}(R)) \subseteq \mathscr{L}(R)$. To prove $\mathscr{L}(R) \subseteq \mathscr{L}(\text{Reduct}(R))$, we construct exact functors F_p: R-Mod $\to R_p$-Mod for each prime p and F_0: R-Mod $\to \mathbf{Q}$-Mod such that 2.1(i) applies. For F_0, we compose the functor H: R-Mod $\to \mathbf{Z}$-Mod from 2.1(b) with the functor ${}_{\mathbf{Q}}\mathbf{Q}_{\mathbf{Z}} \otimes -$ from $\mathbf{Z}$-Mod into $\mathbf{Q}$-Mod. This is an exact functor by 2.6(c).

Suppose p is prime and $\text{dgr}_R(p) = +\infty$. Let $T = R_p = \mathbf{Q}(P - \{p\})$. Here, T is torsion-free as a $\mathbf{Z}$-module, and so we can compose H with ${}_T T_{\mathbf{Z}} \otimes -$ to obtain an exact functor F_p: R-Mod $\to R_p$-Mod.

Suppose p is prime and $\text{dgr}_R(p) = k < +\infty$. There is an idempotent $e = 1 - p^k \cdot r^k$ in R such that $p^k e = 0$ by 4.1(a), and $G(M) = eM$ determines an exact functor G: R-Mod $\to eRe$-Mod, using $G(f)$: $eM \to eN$

induced by $f: M \to N$ in R-Mod. Since $R_p = R/p^kR$ is isomorphic to eRe by 4.1(b), we see that G can be regarded as an exact functor F_p: R-Mod $\to$ R_p-Mod.

Suppose $M \neq 0$ in R-Mod. If M contains an element of infinite order, then $\mathbf{Q} \otimes H(M) \neq 0$ by 2.6(d), and so $F_0(M) \neq 0$. If there is no element of infinite order, then we can find $v \neq 0$ in M and a prime p such that $p \cdot v = 0$. If $\text{dgr}_R(p) = +\infty$, then there is a $\mathbf{Z}$-submodule of $H(M)$ isomorphic to $\mathbf{Z}/p\mathbf{Z}$, hence $F_p(M) \neq 0$ by 2.6(e). Suppose $\text{dgr}_R(p) = k < +\infty$. Then $k > 0$, since otherwise $p \cdot 1_R$ is invertible by 2.5(b). So $v = ev$ for $e = 1 - p^k \cdot r^k$, where $p^{k+1} \cdot r = p^k \cdot 1_R$. Again, $F_p(M) \neq 0$.

Therefore, $\{F_0(M)\} \cup \{F_p(M)\}_{p \in P}$ always contains a nonzero module if M is a nonzero R-module. By 2.1(i), there exists an exact embedding functor R-Mod $\to$ Reduct(R)-Mod. So, $\mathscr{L}(R) = \mathscr{L}(\text{Reduct}(R))$ by 2.1(a). Q.E.D.

By 4.5, we can conclude that $\mathscr{L}(R)$ is determined by $\text{char}(R)$ and $\mathscr{L}(R_p)$ for p in P. Alternatively, $\mathscr{L}(R)$ is determined by $\text{char}(R)$, $\text{dgr}_R(p)$ for p in P, and $\mathscr{L}(R_p)$ for p such that $2 \leq \text{dgr}_R(p) < +\infty$.

4.6. THEOREM. *For rings R and S, the following are equivalent*:

(a) $\mathscr{L}(R) \subseteq \mathscr{L}(S)$.

(b) $\mathscr{L}(\text{Reduct}(R)) \subseteq \mathscr{L}(\text{Reduct}(S))$.

(c) $\mathscr{L}(R_p) \subseteq \mathscr{L}(S_p)$ *for all p in P, and* $\text{char}(R)$ *divides* $\text{char}(S)$ *or* $\text{char}(S) = 0$.

(d) $\text{char}(S) = 0$ *if* $\text{char}(R) = 0$, *and* $\text{dgr}_R(p) \leq \text{dgr}_S(p)$ *for all p in P, and* $\mathscr{L}(R/p^kR) \subseteq \mathscr{L}(S/p^kS)$ *whenever* $2 \leq k = \text{dgr}_R(p) \leq \text{dgr}_S(p) < +\infty$.

Proof. Suppose $2 \leq k = \text{dgr}_R(p) \leq \text{dgr}_S(p) < +\infty$ and $\mathscr{L}(R) \subseteq \mathscr{L}(S)$. Then there is an exact embedding functor R/p^kR-Mod $\to$ S-Mod by 2.1(b), which clearly induces an exact embedding functor R/p^kR-Mod $\to$ S/p^kS-Mod. So, $\mathscr{L}(R/p^kR) \subseteq \mathscr{L}(S/p^kS)$. But then 4.6(a) implies 4.6(d), using 2.1(f) and 2.5(d).

Assume 4.6(d), so $\text{char}(R)$ divides $\text{char}(S)$ or $\text{char}(S) = 0$ by 2.5(c). Suppose p is a prime, $a = \text{dgr}_R(p)$, and $b = \text{dgr}_S(p)$, so $a \leq b$. If $a = b = +\infty$, then $R_p = S_p = \mathbf{Z}_{\langle p \rangle}$. If $a < b = +\infty$, then there is a homomorphism $S_p \to R_p$ by 2.6(b) and 2.5(a), so $\mathscr{L}(R_p) \subseteq \mathscr{L}(S_p)$ by 2.1(b). Suppose $a \leq b < +\infty$. If $a \geq 2$, then $\mathscr{L}(R_p) \subseteq \mathscr{L}(S_p)$ by the assumption 4.6(b) and 2.1(b). If $a \leq 1$, then $\mathscr{L}(R_p) = \mathscr{L}(S_p/p^aS_p) \subseteq \mathscr{L}(S_p)$, because R_p and S_p/p^aS_p are trivial if $a = 0$, and $\mathscr{L}(R_p) = \mathscr{L}(S_p/p^aS_p) = \mathscr{L}(\mathbf{Z}/p\mathbf{Z})$ if $a = 1$ by 2.1(g). So, $\mathscr{L}(R_p) \subseteq \mathscr{L}(S_p)$ in all cases, proving that 4.6(d) implies 4.6(c).

Using 2.3(a), 4.2(a), and 4.4, we see that 4.6(c) implies 4.6(b). Finally, 4.6(b) implies 4.6(a), by 4.5. Q.E.D.

By 4.2(c), each relation $\mathscr{L}(R/p^kR) \subseteq \mathscr{L}(S/p^kS)$ of 4.6(d) compares rings with the same power characteristic p^k.

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