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The competition number of a graph with exactly h holes, all of which are independent[☆]

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ABSTRACT

Given an acyclic digraph D, the competition graph $C(D)$ of D is the graph with the same vertex set as D where two distinct vertices x and y are adjacent in $C(D)$ if and only if there is a vertex v in D such that (x, v) and (y, v) are arcs of D. The competition number $\kappa(G)$ of a graph G is the least number of isolated vertices that must be added to G to form a competition graph. The purpose of this paper is to prove that the competition number of a graph with exactly h holes, all of which are independent, is at most $h + 1$. This generalizes the result for $h = 0$ given by Roberts, and the result for $h = 1$ given by Cho and Kim.

1. Introduction

Given an acyclic digraph D, the *competition graph* $C(D)$ of D is the graph with the same vertex set as D where two distinct vertices x and y are adjacent in $C(D)$ if and only if there is a vertex v in D such that (x, v) and (y, v) are arcs of D. The notion of a competition graph was introduced by Cohen [1] for studying ecological systems. Since then, several variations have been defined and studied by many authors (see, for examples, [2–7]). Besides the application to ecology, the concept of competition graph can be applied in the study of communication over noisy channels (see [8,9]) and to the problem of assigning channels to radio or television transmitters (see [10–12]).

While not all graphs are competition graphs, Roberts [8] observed that any graph G together with sufficiently many isolated vertices is the competition graph of some acyclic digraph. In fact, $|E(G)|$ isolated vertices are enough, as $G \cup I_{|E(G)|}$ is the competition graph of D with $V(D) = V(G) \cup E(G)$ and $E(D) = \{(x, e) : x \text{ is incident to } e\}$, where I_r is the graph of r vertices and no edges and $G \cup I_r$ is the disjoint union of G and I_r. Roberts then defined the *competition number* $\kappa(G)$ of a graph G to be the smallest number r such that $G \cup I_r$ is the competition graph of an acyclic digraph. It is clear that G is a competition graph if and only if $\kappa(G) = 0$. For graphs whose competition numbers are known, see [4,5]. From an algorithmic point of view, Opsut [12] proved that determining the competition number of a graph is NP-hard.

In a graph G, a *chord* of a path $(v_1, v_2, \ldots, v_r)$ is an edge v_iv_j with $|i-j| \geq 2$. Similarly, a *chord* of a cycle $(v_1, v_2, \ldots, v_r, v_1)$ is an edge v_iv_j with $|i - j|_r \geq 2$, where $|i - j|_r = \min\{|i - j|, r - |i - j|\}$. A *chordless path* (respectively, *cycle*) is a path (respectively, cycle) with no chord. We remark that a “chordless path/cycle” is also called an “induced path/cycle” by other

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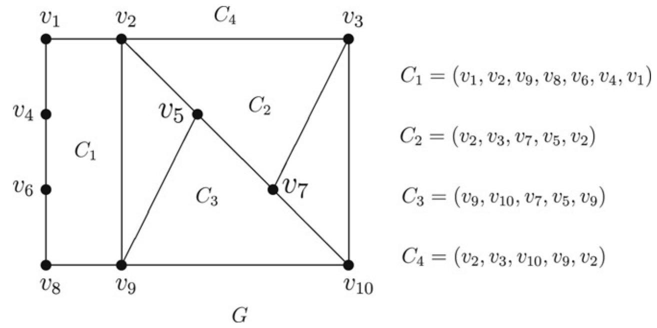

Fig. 1. A graph G with exactly four holes, where C_1 is the only independent hole.

authors. A *hole* is a chordless cycle of length at least 4. A *chordal graph* is a graph with no hole. For any integer $n \geq 4$, Harray, Kim and Roberts [13] showed that the maximum competition number of a graph on n vertices is achieved uniquely by the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$ which has a lot of holes. On the other hand, Roberts [8] proved that the competition number of a chordal graph is at most 1. Cho and Kim [14] established that the competition number of a graph with exactly one hole is at most 2. They also gave a sufficient condition for a graph with exactly one hole to have competition number at most 1. They raised the problem of determining graphs with exactly one hole and with competition number at most 1. Kim [15] gave another sufficient condition for a graph with exactly one hole to have competition number 1. He then asked an interesting question: that of whether $h + 1$ is the maximum competition number of a graph with exactly h holes.

The purpose of this paper is to partially answer Kim's question. Roughly speaking, we confirm that the answer is yes when the holes do not 'overlap' much. More precisely, in a graph G, a hole C is *independent* if the following two conditions hold for any other hole C' of G.

1. C and C' have at most two common vertices.
2. If C and C' have two common vertices, then they have one common edge and C is of length at least 5.

Fig. 1 shows a graph G with exactly four holes $C_1 = (v_1, v_2, v_9, v_8, v_6, v_4, v_1)$, $C_2 = (v_2, v_3, v_7, v_5, v_2)$, $C_3 = (v_9, v_{10}, v_7, v_5, v_9)$ and $C_4 = (v_2, v_3, v_{10}, v_9, v_2)$. The hole C_1 is the only independent hole. Notice that C_2, C_3 and C_4 are pairwise intersecting an edge, but they are of length 4, and so are not independent by point 2 in the definition. The reason that we need the condition "C is of length at least 5" in point 2 will become clear after Lemma 3.

Notice that if a graph has exactly one hole then the hole is independent. In this paper, we prove that if G is a graph with exactly h holes, all of which are independent, then its competition number is at most $h + 1$.

2. Preliminaries

In this section, we establish some properties that are useful in this paper. First, we fix some notation. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph *induced* by a subset $S \subseteq V(G)$ is the graph $G[S]$ with vertex set S and edge set $\{xy \in E(G) : x, y \in S\}$. The *deletion* of a subset $S \subseteq V(G)$ from G results in the graph $G - S$ which is $G[V(G) - S]$. We denote $G - \{v\}$ by $G - v$. We use $G - uv$ to denote the graph obtained from G by deleting edge uv. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices adjacent to v; and the *closed neighborhood* of v is $N[v] = \{v\} \cup N(v)$.

Next we state two easy lemmas whose proofs we have omitted.

Lemma 1. *In any graph, if a vertex is not in an independent hole but is adjacent to two non-adjacent vertices of this hole, then it is adjacent to all vertices of this hole.*

Lemma 2. *In any graph, the set of vertices adjacent to all vertices of an independent hole is a clique.*

For a hole C, a C*-avoiding walk* is a walk whose internal vertices are not in $V(C) \cup X$, where X is the set of vertices adjacent to all vertices of C. Notice that repeated vertices are allowed in a C-avoiding walk. Notice that we need the concept of a C-avoiding walk rather than only a C-avoiding path, as you will see that the former is essential in the third paragraph of the proof of Theorem 6.

Lemma 3. *For any two distinct non-adjacent vertices v_i and v_j in an independent hole $C = (v_1, v_2, \ldots, v_r, v_1)$ of a graph G, there is no C-avoiding walk from v_i to v_j.*

Proof. Suppose to the contrary that there is a C-avoiding walk $P = (v_i, u_1, u_2, \ldots, u_s, v_j)$ from v_i to v_j. First, $s \geq 2$ as C has no chord and u_1 is not in X by Lemma 1, where X is the set of vertices adjacent to all vertices of C. Without loss of generality, we may assume that $i = 1$ and $3 \leq j \leq r - 1$. We may also assume that v_i, v_j and P are chosen so that $|P| = s + 1$ is minimal, where $|P|$ denotes the number of edges of P. In this case, P is a chordless path.

We now consider the two cycles $C_1 = (v_1, u_1, u_2, \ldots, u_s, v_j, v_{j-1}, \ldots, v_2, v_1)$ and $C_2 = (v_1, u_1, u_2, \ldots, u_s, v_j, v_{j+1}, \ldots, v_r, v_1)$. Since C_1 intersects C at v_1 and v_j, by the independence of C, C_1 is not a hole and so there is a chord $u_k v_{k'}$ where $1 \leq k \leq s$ and $2 \leq k' \leq j - 1$. Similarly, C_2 has a chord $u_\ell v_{\ell'}$ where $1 \leq \ell \leq s$ and $j + 1 \leq \ell' \leq r$. In fact, for all such u_k and u_ℓ, we always have $u_k \neq u_\ell$ for otherwise $u_k = u_\ell \in X$ by Lemma 1, violating that P is a C-avoiding walk. For simplicity, assume that $k < \ell$. Since $P' = (v_{k'}, u_k, u_{k+1}, \ldots, u_\ell, v_{\ell'})$ is a C-avoiding walk between two non-adjacent vertices $v_{k'}$ and $v_{\ell'}$ in C, by the minimality of $|P|$, we have $s + 1 \leq 2 + \ell - k \leq 2 + s - 1$ and so $k = 1$ and $\ell = s$. Again, by Lemma 1, we have $k' = 2$ and $\ell' = j + 1$. Thus the only edges between P and C are u_1v_1, u_1v_2, u_sv_j and u_sv_{j+1}. Then, $C_1' = (v_2, u_1, u_2, \ldots, u_s, v_j, v_{j-1}, \ldots, v_2)$ is a hole intersecting C at $j - 1$ vertices, and $C_2' = (v_1, u_1, u_2, \ldots, u_s, v_{j+1}, v_{j+2}, \ldots, v_r, v_1)$ is a hole intersecting C at $r - j + 1$ vertices. By the independence of C, $j - 1 \leq 2$ and $r - j + 1 \leq 2$, so $r = 4$. But this is still a contradiction as C and C_1' intersect at two vertices while C is of size 4 only. □

We notice that it is possible to have a C-avoiding walk between two adjacent vertices v_i and v_{i+1} of an independent hole C. In graph G of Fig. 1, (v_2, v_5, v_9) is a C_1-avoiding walk between two adjacent vertices v_2 and v_9 in C_1. On the other hand, (v_2, v_9, v_{10}, v_7) is a C_2-avoiding walk between two non-adjacent vertices v_2 and v_7 in C_2. This justifies the inclusion of the second point in the definition of an independent hole, since without it, Lemma 3 would fail.

We now consider the case when G has exactly h holes $C_1, C_2, \ldots, C_h$, all of which are independent. Let X_i be the set of vertices adjacent to all the vertices of the hole C_i for $i = 1, 2, \ldots, h$. For any edge uv of hole C_i, define the set $S_{i,uv} = \{w :$ w is an internal vertex of a C_i-avoiding walk from u to $v\}$. Notice that the set $S_{i,uv}$ may possibly be empty.

Lemma 4. *Suppose a graph G has exactly h holes $C_1, C_2, \ldots, C_h$, all of which are independent. For any edge uv in C_h, if $S_{h,uv}$ is empty then $G - uv$ has exactly $h - 1$ holes, all of which are independent.*

Proof. Suppose that $uv \in E(C_i)$ for some $i \neq h$. Since C_i is a hole, any vertex in $V(C_i) - \{u, v\}$ is not adjacent to both u and v, and hence is not in X_h. Then, $C_i - uv$ is a C_h-avoiding walk from u to v, a contradiction to the fact that $S_{h.uv}$ is empty. This proves that $uv \notin E(C_i)$ for all $i \neq h$ and so $C_1, C_2, \ldots, C_{h-1}$ are holes in $G - uv$.

Next, we show that $G - uv$ has only these $h - 1$ holes and so they are also independent in $G - uv$. Suppose to the contrary that $G - uv$ has another hole C' which is a cycle other than C_h in G. In G, the edge uv is the only chord of C' and so it divides C' into two chordless cycles. As these two cycles contain u and v, either one is C_h and the other is a triangle uvw or else they are two triangles uvw and uvw'. For the former case, $w \notin X_h$ and so (u, w, v) is a C_h-avoiding walk, a contradiction to the fact that $S_{h,uv}$ is empty. For the latter case, (u, w, v, w', u) is a hole in $G - uv$ and so $ww' \notin E(G)$. By Lemma 2, one of w and w' is not in X_h; without loss of generality assume that $w \notin X_h$. Again, (u, w, v) is a C_h-avoiding walk, a contradiction to the fact that $S_{h,uv}$ is empty. □

3. Main result

This section gives the main result that the competition number of a graph with exactly h holes, all of which are independent, is at most $h + 1$. First, we state a useful result for the case of $h = 0$.

Theorem 5 ([8]). *For any clique Q of a chordal graph G, there exists an acyclic digraph D such that $C(D) = G \bigcup I_1$ and the vertices of Q have only outgoing arcs in D.*

We now have our main result as follows.

Theorem 6. *Suppose G is a graph with exactly h holes $C_1, C_2, \ldots, C_h$, all of which are independent. If Q is a clique of G, then there exists an acyclic digraph D such that $C(D) = G \bigcup I_{h+1}$ and the vertices of Q have only outgoing arcs in D. Consequently, $\kappa(G) \leq h + 1$.*

Proof. We shall prove the theorem by induction on h. The theorem is true for $h = 0$ by Theorem 5. Suppose $h \geq 1$ and the theorem is true for $h' < h$.

Suppose $(e \cup S_{h,e}) \cap Q$ contains some vertex x for some edge e in C_h. Since Q is a clique and C_h is a hole, we may assume $e' \cap Q = \emptyset$ for any edge e' of C_h disjoint from e. For such e' we have $(e' \cup S_{h,e'}) \cap Q = \emptyset$, for otherwise if $x' \in S_{h,e'} \cap Q$ then $x' \in N[x]$. By the definitions of $S_{h,e}$ and $S_{h,e'}$, there is a C_h-avoiding walk from an end vertex y of e to any end vertex y' of e', which can be chosen so that y and y' are not adjacent, a contradiction to Lemma 3. Since C_h has at most three edges e' that are not disjoint from e, the set $(e \cup S_{h,e}) \cap Q$ is nonempty for at most three edges e in $E(C_h)$. Now, since C_h has at least four edges, we may choose an edge uv in $E(C_h)$ such that $(\{u, v\} \cup S_{h,uv}) \cap Q$ is empty. Consider the two induced subgraphs $G_1 = G - S_{h,uv}$ and $G_2 = G[X_h \cup \{u, v\} \cup S_{h,uv}]$ of G; see Fig. 2.

We claim that no vertex of $S_{h,uv}$ is adjacent to a vertex of $V(G) - (X_h \cup V(C_h) \cup S_{h,uv})$. For otherwise there is a C_h-avoiding walk W from u to v that contains a vertex x adjacent to a vertex $y \notin X_h \cup V(C_h) \cup S_{h,uv}$. The walk W' obtained from W by replacing x with xyx is then C_h-avoiding, contradicting the fact that $y \notin S_{h,uv}$. By Lemma 3, no vertex of $S_{h,uv}$ is adjacent to a vertex of $V(C_h) - \{u, v\}$. Hence, $X_h \cup \{u, v\}$ is a vertex cut of G and no vertex in $S_{h,uv}$ belongs to the component that includes $V(C_h) - \{u, v\}$. Since $V(G_1) \cap V(G_2) = X_h \cup \{u, v\}$ is a clique vertex cut of G, we have that G_1 has exactly h_1 holes, all of which are independent; and G_2 has exactly $h_2 = h - h_1$ holes, all of which are independent.

Since C_h is not in G_2, we have $h_2 < h$. By the induction hypothesis, there exists an acyclic digraph D_2 such that $C(D_2) = G_2 \cup I_{h_2+1}$ and the vertices of $X_h \bigcup \{u, v\}$ have only outgoing arcs in D_2. Notice that C_h is a hole in G_1 which has no C_h-avoiding walk from u to v. By Lemma 4, $G_1 - uv$ has exactly $h_1 - 1$ holes, all of which are independent. As Q is a clique in

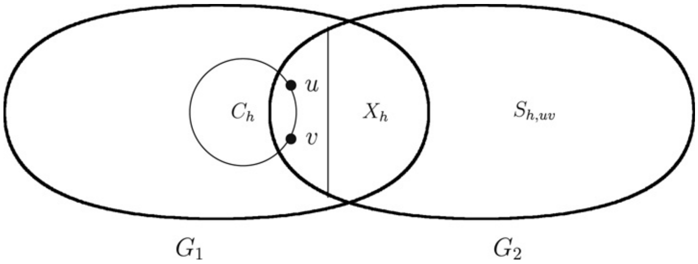

Fig. 2. A graph with exactly h holes, all of which are independent.

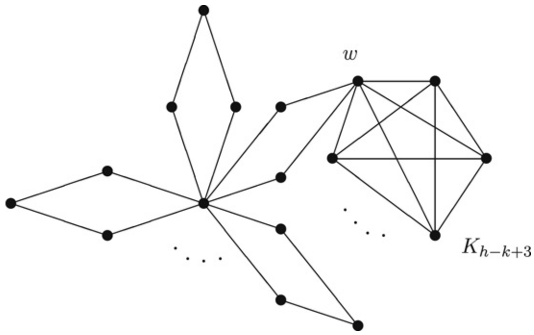

Fig. 3. A graph G has exactly h holes with $\kappa(G) = k$, where $1 \le k \le h$.

$G_1 - uv$, by the induction hypothesis, there exists an acyclic digraph D_1 such that $C(D_1) = (G_1 - uv) \cup I_{h_1}$ and the vertices of Q have only outgoing arcs in D_1. Having D_1 and D_2 at hand, we now construct the digraph D with $V(D) = V(D_1) \cup V(D_2)$ and $E(D) = E(D_1) \cup E(D_2)$. It is then easy to check that D is an acyclic digraph with the vertices of Q having only outgoing arcs in D and $C(D) = G \cup I_{h+1}$. Consequently, $\kappa(G) \le h + 1$. □

We remark that the upper bound in Theorem 6 is sharp as the following examples show. Kim [4] observed that for $1 \le k \le h + 1$ there is a graph G with exactly h holes and $\kappa(G) = k$. In fact, G is the graph obtained from h copies of 4-cycles and a copy of a complete graph K_{h-k+3} by first identifying a vertex at each 4-cycle and then another vertex of a 4-cycle with a vertex of the complete graph K_{h-k+3}; see Fig. 3.

Notice that if G has exactly one hole then the hole is independent. Consequently, we have the following corollary.

Corollary 7 (*[14]*). *If G has exactly one hole, then $\kappa(G) \le 2$.*

Another interesting consequence is as follows.

Corollary 8. *Suppose G has exactly r components $G_1, G_2, \ldots, G_r$, where each component G_i has a clique of size ω_i and exactly h_i holes, all of which are independent. If $h'_0 = \omega'_0 = 0$ and $h'_i = h'_{i-1} + \max\{0, h_i + 1 - \omega'_{i-1}\}$ and $\omega'_i = \omega_i + \max\{0, \omega'_{i-1} - h_i - 1\}$ for $1 \le i \le h$, then $\kappa(G) \le h'_r$.*

Proof. For each component G_i of G, choose a clique Q_i of size ω_i in G_i. By Theorem 6, there exists an acyclic digraph D_i such that $C(D_i) = G_i \cup I_{h_i+1}$ and the vertices of Q_i have only outgoing arcs in D_i. Since the vertices of Q_j have only outgoing arcs in D_j for all j, $\min\{h_i + 1, \omega'_{i-1}\}$ new vertices of D_i can be replaced by vertices in the Q_j with $j < i$, while $\max\{0, h_i + 1 - \omega'_{i-1}\}$ new vertices of D_i remain unreplaced, which gives the formula for h'_i. On the other hand, $\max\{0, \omega'_{i-1} - h_i - 1\}$ vertices in the cliques Q_j with $j < i$ remain. This together with the ω_i vertices in Q_i gives the formula for ω'_i. Thus, we can construct an acyclic digraph D from the digraphs $D_1, D_2, \ldots, D_r$ such that $C(D) = G \cup I_{h'_r}$. This gives that $\kappa(G) \le h'_r$. □

In particular, we have:

Corollary 9. *If G is a graph in which each component has at most one hole, then $\kappa(G) \le 2$. If, in addition, G has a component containing no hole, then $\kappa(G) \le 1$.*

An interesting question is how to determine graphs G with exactly one hole such that $\kappa(G) \le 1$.

4. The sufficient condition for $\kappa(G) \le h$

We close this paper by giving a sufficient condition for a graph having exactly h holes, all of which are independent, to have the competition number at most h. First, we need a well known lemma. A vertex is *simplicial* if its neighbors form a clique.

Lemma 10 ([16]). *Every chordal graph has a simplicial vertex. Moreover, every chordal graph that is not a complete graph has two non-adjacent simplicial vertices.*

Theorem 11. *Suppose G is a graph with exactly h holes, all of which are independent. If $S_{i,uv}$ is not empty and $G[X_i \cup \{u, v\} \cup S_{i,uv}]$ has no hole for some i and $uv \in E(C_i)$, then $\kappa(G) \le h$.*

Proof. We may assume that $S_{1,uv}$ is not empty and $G_1 = G[X_1 \cup \{u, v\} \cup S_{1,uv}]$ has no hole. Then there is a shortest C_1-avoiding walk P, which is a chordless path, from u to v in G. If P is of length at least 3, then P together with uv is a hole in G_1, a contradiction to the fact that G_1 has no hole. Therefore, there is a vertex w in $S_{1,uv}$ such that w is adjacent to u and v. By Lemma 10, there exists a vertex ordering $v_1, v_2, \ldots, v_n$ of G_1 with $X_1 \cup \{u, v\} = \{v_1, v_2, \ldots, v_t\}$ for some $t < n$ such that $Q_i = \{v_j : 1 \le j < i, v_j v_i \in E(G_1)\} \cup \{v_i\}$ is a clique for $1 \le i \le n$. We construct a digraph D_1 with $V(D_1) = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$ and $E(D_1) = \bigcup_{t+1 \le i \le n} \{(y, v_{i+1}) : y \in Q_i\}$. Then D_1 is acyclic and the vertices of $X_1 \cup \{u, v, v_{t+1}\}$ have only outgoing arcs in D_1. Notice that $C(D_1)$ is a subgraph of G_1 such that $E(C(D_1))$ contains the set $E(G_1) - E(X_1 \cup \{u, v\})$. Since $w \notin X_1 \cup \{u, v\}$, we have $w = v_j$ for some $j > t$ with $\{u, v\} \subseteq Q_j$, and so $uv \in E(C(D_1))$. Let $G_2 = G - S_{1,uv}$. Notice that G_2 is a graph with exactly h holes, all of which are independent, and G_2 has no C_1-avoiding walk from u to v. By Lemma 4, we have that $G_2 - uv$ is a graph with exactly $h - 1$ holes, all of which are independent. By Theorem 6, there exists an acyclic digraph D_2 such that $C(D_2) = (G_2 - uv) \cup I_h$, where $v_{t+1} \in V(I_h)$. Now we construct a digraph D with $V(D) = V(D_1) \cup V(D_2)$ and $E(D) = E(D_1) \cup E(D_2)$. It can be easily checked that D is acyclic and $C(D) = G \cup I_h$. This gives that $\kappa(G) \le h$. □

Corollary 12. *If G is a graph with exactly one hole C_1 and $S_{1,uv}$ is not empty for some edge uv in $E(C_1)$, then $\kappa(G) \le 1$.*

Corollary 13 ([15]). *If G is a graph with exactly one hole C_1 and there is a vertex w adjacent to u and v for some edge uv in $E(C_1)$, then $\kappa(G) \le 1$.*

Proof. The corollary follows from Corollary 12 and the fact that "there is a vertex w adjacent to u and v" implies "$S_{1,uv}$ is not empty". □

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References

[1] J.E. Cohen, Interval graphs and food webs: A finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, CA, 1968.
[2] C. Cable, K.F. Jones, J.R. Lundgren, S. Seager, Niche graphs, Discrete Appl. Math. 23 (1989) 231–241.
[3] P.C. Fishburn, W.V. Gehrlein, Niche numbers, J. Graph Theory 16 (1992) 131–139.
[4] S.-R. Kim, The competition number and its variants, in: J. Gimbel, J.W. Kennedy, L.V. Quintas (Eds.), Quo Vadis, Graph Theory? in: Annals of Discrete Mathematics, vol. 55, North-Holland B. V, Amsterdam, The Netherlands, 1993, pp. 313–326.
[5] S.-R. Kim, F.S. Roberts, Competition numbers of graphs with a small number of triangles, Discrete Appl. Math. 78 (1997) 153–162.
[6] J.R. Lundgren, Food webs, competition graphs, competition–common enemy graphs, and Niche graphs, in: F.S. Roberts (Ed.), Applications of Combinatorics and Graph Theory to the Biological and Social Sciences, in: IMH Volumes in Mathematics and its Application, vol. 17, Springer, New York, 1989, pp. 221–243.
[7] D. Scott, The competition–common enemy graph of a digraph, Discrete Appl. Math. 17 (1987) 269–280.
[8] F.S. Roberts, Food webs, competition graphs, and the boxicity of ecological phase space, in: Y. Alavi, D. Lick (Eds.), Theory and Applications of Graphs, Springer, New York, 1978, pp. 477–490.
[9] C.E. Shannon, The zero capacity of a noisy channel, IRE Trans. Inform Theory IT-2 (1956) 8–19.
[10] M.B. Cozzens, F.S. Roberts, T-colorings of graphs and the channel assignment problem, Congr. Numer. 25 (1982) 191–208.
[11] W.K. Hale, Frequency assignment: Theory and application, Proc. IEEE 68 (1980) 1497–1514.
[12] R.J. Opsut, On the computation of the competition number of a graph, SIAM J. Alg. Discrete Math. 3 (1982) 420–428.
[13] F. Harary, S.-R. Kim, F.S. Roberts, Extremal competition numbers as a generalization of Turan's theorem, J. Ramanujan Math. Soc. 5 (1990) 33–43.
[14] H.H. Cho, S.-R. Kim, The competition number of a graph having exactly one hole, Discrete Math. 303 (2005) 32–41.
[15] S.-R. Kim, Graphs with one hole and competition number one, J. Korean Math. Soc. 42 (2005) 1251–1264.
[16] G.A. Dirac, On rigid circuit graphs, Abh. Math. Sem. Univ. Hamburg 25 (1961) 71–76.