A characterization of even doubly-stochastic matrices

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Abstract

Even doubly-stochastic matrices are characterized with the aid of the minima of functionals defined by the even diagonals contained in the matrix.

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1. Introduction and preliminaries

In 1961, L. Mirsky gave the following necessary condition for a doubly-stochastic $n \times n$-matrix with $n \geq 3$ to be even, i.e. a convex combination of even permutation matrices,

$$\forall \pi \in A_n, \forall k \in \{1, \ldots, n\}: \sum_{i=1}^{n} a_{i\pi(i)} - a_{k\pi(k)} \leq n - 3,$$

and he conjectured that this condition would also be sufficient [10]. This is true for $n = 3$, but for $n \geq 4$, this is false, see [1], and a characterization of the polyhedron defined by these doubly-stochastic matrices remained desirable. In 1991 R.A. Brualdi and B.L. Liu [4] conjectured that the number of linear inequalities necessary to characterize the even permutation matrices is not bounded by a polynomial in $n$. This conjecture was confirmed by W. H. Cunningham and Y. Wang [5] and J. Hood and D. Perkinson [6] in 2004, see also [3,4] for related topics. In the present paper we present a characterization of even doubly-stochastic matrices in terms of the even diagonals contained in the matrix, of the number of fixed points of their quotients and of a natural quadratic functional associated to them. It yields criteria to decide whether a given doubly-stochastic matrix is even or not. Moreover, it can be detected whether the convex combination in terms of permutation matrices is unique or not. The main result of Section 2 can be summarized as follows: For a given doubly-stochastic matrix $A = (a_{ij})_{n \times n}$, denote its distinct even diagonals by $\pi_1, \ldots, \pi_p$ and set $b = (b_m)_{p \times 1}$ with

$$b_m = \sum_{i=1}^{n} a_{i\pi_m(i)},$$

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and introduce the matrix $H = 2 \left( \text{tr}(\pi_m \pi_q^{-1}) \right)_{p \times p}$. Then $H$ is semi-positive definite, see Lemma 2.2, and $A$ is even iff the equation $Hy = 2b$ has a non-negative solution $y$ with coordinate sum equal to 1 such that $f(y) = 0$ where

$$f(x) = \sum_{i,j=1}^{n} \left( a_{ij} - \frac{1}{p} \sum_{k=1}^{p} x_k \delta_{\pi(i),j} \right)^2.$$ 

In that case, the convex representation is unique if $\det H > 0$, while there can be several distinct ones if $\det H = 0$. In Section 3 the characterization is refined in the presence of an even diagonal, while Section 4 presents some examples.

By definition, a doubly-stochastic matrix is a real non-negative square matrix whose sum of each row and of each column amounts to 1. The polyhedron of all doubly-stochastic $n \times n$ matrices will be denoted by $\Omega_n$. A famous result by G. Birkhoff states that the permutation matrices are precisely the vertices of $\Omega_n$, see e.g. [11]. The symmetric group on $n$ elements will be denoted by $S_n$ and the alternating group by $A_n$. We say that a permutation $\sigma \in S_n$ is contained in a real matrix $A = (a_{ij})_{n \times n}$ or a diagonal of $A$ if

$$\prod_{i=1}^{n} a_{i\sigma(i)} \neq 0,$$

which will be denoted by $\sigma \prec A$. Unless precision is required, we shall identify a permutation $\pi \in S_n$ with the corresponding permutation matrix $\pi = P_{\pi} = (\delta_{\pi(i),j})_{n \times n}$.

The polyhedron of even doubly-stochastic matrices, i.e. the convex hull of the alternating group $A_n$ in $\Omega_n$ will be denoted by $\Delta_n$. Accordingly, a convex combination of odd permutation matrices will be called an odd doubly-stochastic matrix. This apparent parity is neither exclusive nor complete: There are doubly-stochastic matrices that are even and odd, while there are others that are neither even nor odd. Conceivably, we call a matrix $A$ totally even (respectively totally odd) if no odd permutation (resp. no even permutation) is contained in $A$. In 1916, Denes Kőnig [7], see also [8], showed that any doubly-stochastic matrix possesses a diagonal and, thereby, is a convex combination of permutation matrices using Birkhoff’s algorithm, see e.g. [8]. In [2] the existence of even diagonals and their minimal number in a given doubly-stochastic matrix as a function of the number of positive elements has been established.

Throughout we shall use the following notations.

**Definition 1.1.**

- $\mathcal{M}_n(\mathbb{R}) = \text{algebra of real } n \times n \text{ matrices}$
- $I = I_n = n \times n$-identity matrix
- $e = e_n = n \times 1$-column vector with constant entries equal to 1
- $\text{tr}(A) = \text{trace of the matrix } A$
- $\mathbb{E}^p = \{ x \in (\mathbb{R}^+)^p \mid x^t e_n = 1 \}$
- $\mathbb{F}^p = \{ x \in (\mathbb{R}^+)^p \mid x^t e_n \leq 1 \}$.

For further matrix theoretical terminology we refer to [9].

The algebra $\mathcal{M}_n(\mathbb{R})$ is endowed with the Euclidean or Frobenius-norm

$$\|A\| := \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}^2} = \sqrt{\text{tr}(AA^t)}.$$ 

This norm is not an operator norm, but it is consistent with the Euclidean norm in $\mathbb{R}^n$ and fulfills $\|AB\| \leq \|A\| \|B\|$.
2. The variational approach

For a given matrix \( A \in \Omega_n \), let \( \pi_1, \ldots, \pi_p \) denote the distinct even diagonals of \( A \), \( \{ \pi \in A_n \mid \pi \prec A \} \) and define

\[
 f = f_A : \mathbb{R}^p \rightarrow \mathbb{R}^+ \text{ by }
\]

\[
f(x_1, \ldots, x_p) = \left\| A - \sum_{k=1}^{p} x_k \pi_k \right\|^2 = \sum_{i,j=1}^{n} \left( a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)\pi_k(j)} \right)^2.
\]

(3)

We note in passing that \( p = \frac{1}{2} \text{ (per}(P(A)) + \text{det}(P(A))) \), where \( P(A) \) is the adjacency matrix defined by \( A \). For more details and further estimates related to \( p \) we refer to [2]. Since \( f(x) \rightarrow +\infty \text{ as } \|x\| \rightarrow +\infty \), \( f \) attains its infimum in \( \mathbb{R}^p \) and, thereby, always has critical points. Clearly, if \( A \in \Omega_n \) is even, then there is some non-negative vector \( y \in \mathbb{R}^p \) with \( y^t e = 1 \) such that \( f(y) = 0 \) and

\[
 A \in \Delta_n \Rightarrow \min_{x \in \mathbb{R}^p} f(x) = 0.
\]

(4)

But this condition is by no means sufficient, see Example 4.3 and it has to be complemented by an extra condition in order to characterize \( \Delta_n \). Therefore, we have to analyse in detail the critical points of the functional \( f \). Denote the Hessian matrix of \( f \) by

\[
 H(f)(x_1, \ldots, x_p) = \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)_{p \times p} (x_1, \ldots, x_p).
\]

We readily calculate

\[
 \frac{\partial f}{\partial x_i} (x_1, \ldots, x_p) = \sum_{i=1}^{n} \sum_{j=1}^{n} 2(-\delta_{\pi_j(i)}) \left( a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)} \right)
\]

\[
 = -2 \sum_{i=1}^{n} \left( a_{i\pi_j(i)} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)\pi_q(i)} \right).
\]

Thus

\[
 \frac{1}{2} \frac{\partial^2 f}{\partial x_q \partial x_l} (x_1, \ldots, x_p) = \sum_{i=1}^{n} \delta_{\pi_j(i)\pi_q(i)}.
\]

Since the r.h.s. is just the number of common images of \( \pi_i \) and \( \pi_q \), i.e. the number of fixed points of \( \pi_i \pi_q^{-1} \), \( H(f) \) is seen to be a constant symmetric non-negative matrix given by

\[
 \text{Lemma 2.1.} \quad H(f) = 2 \left( \text{tr}(\pi_m \pi_q^{-1}) \right)_{p \times p}.
\]

Especially, \( H(f) \) does not depend on the critical point \( \nabla f(x) = 0 \). This fact enables us to show the definite character of \( H(f) \).

\[
 \text{Lemma 2.2.} \quad \text{The matrix } H(f) \text{ is semi-positive definite.}
\]

\[
 \text{Proof.} \quad \text{Introduce the following even doubly-stochastic matrix}
\]

\[
 B = \sum_{k=1}^{p} \frac{1}{p} \pi_k.
\]

Then \( B \) contains exactly the same even diagonals as \( A \) and

\[
 H(f_B) = 2 \left( \text{tr}(\pi_i \pi_q^{-1}) \right)_{q,d} = H(f_B).
\]

But \( f_B \) attains its minimum 0 by Eq. (4), which shows that \( H(f_B) \) is semi-positive definite, as well as \( H(f_A) \). □
Thus \( H(f) \) cannot have negative eigenvalues. If all the eigenvalues are positive, then \( A \) is not necessarily even, as the following example shows.

**Example 2.3.** For \( n = 3 \) choose \( A = \frac{1}{4}(123) + \frac{1}{4}(132) + \frac{1}{4}1_{S_n} + \frac{1}{4}(12) = \frac{1}{4} \begin{pmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \not\in \Delta_3 \). The even diagonals are \( \pi_1 = (123), \pi_2 = (132), \pi_3 = 1_{S_n} \) and \( H(f) = 2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \) having the determinant 216.

An element \( y = (y_1, \ldots, y_p) \in \mathbb{R}^p \) is a critical point of \( f \) iff for each \( l = 1, \ldots, p \)

\[
\frac{\partial f}{\partial x_l}(y_1, \ldots, y_p) = 0
\iff \sum_{i=1}^n \left( a_i\pi_l(i) - \sum_{k=1}^p y_k \delta_{\pi_l(i)\pi_k(i)} \right) = 0
\iff \sum_{i=1}^n a_i\pi_l(i) = \sum_{k=1}^p y_k \sum_{i=1}^n \delta_{\pi_l(i)\pi_k(i)}.
\]

This corresponds to the \( p \) equations

\[
\begin{cases}
\sum_{k=1}^p \text{tr}(\pi_1\pi_k^{-1}) y_k = \sum_{i=1}^n a_i\pi_1(i) \\
\vdots \\
\sum_{k=1}^p \text{tr}(\pi_p\pi_k^{-1}) y_k = \sum_{i=1}^n a_i\pi_p(i)
\end{cases}
\]

or

\[
M \begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} = b
\]

with

\[
M = \begin{pmatrix} \text{tr}(\pi_1\pi_1^{-1}) & \cdots & \text{tr}(\pi_1\pi_p^{-1}) \\ \vdots & \ddots & \vdots \\ \text{tr}(\pi_p\pi_1^{-1}) & \cdots & \text{tr}(\pi_p\pi_p^{-1}) \end{pmatrix} = \frac{1}{2} H(f) \quad \text{and} \quad b = \begin{pmatrix} \sum_{i=1}^n a_i\pi_l(i) \end{pmatrix}_{p \times 1}.
\]

Thus we can state

**Lemma 2.4.** A vector \( y \in \mathbb{R}^p \) is a critical point of the functional \( f \) iff

\[
\frac{1}{2} H(f) y = \left( \text{tr}(\pi_l\pi_q^{-1}) \right)_{p \times p} y = \begin{pmatrix} \sum_{i=1}^n a_i\pi_l(i) \end{pmatrix}_{p \times 1}.
\]  \hspace{1cm} (5)

If \( \det H(f) > 0 \) then (5) has a unique solution, namely

\[
y = (y_1, \ldots, y_p) = 2 H(f)^{-1} \begin{pmatrix} \sum_{i=1}^n a_i\pi_1(i) \\ \vdots \\ \sum_{i=1}^n a_i\pi_p(i) \end{pmatrix}.
\]  \hspace{1cm} (6)

and there is exactly one critical point. Since \( H(f) \) is positive definite, we are led to
Corollary 2.5. If $\det H(f) > 0$, then the doubly-stochastic matrix $A$ is even if the vector $y$ defined in Eq. (6) is non-negative and satisfies $y^t e = 1$ and $f(y) = 0$.

Without the constraint $f(y) = 0$ the result is false. Take the Example 2.3, where the unique solution of $H(y)y = 2b$ is given by $y^t = \frac{1}{3}(1, 1, 1)$. Since $\det H(f) > 0$, $y$ is the absolute minimum, but $f(y) > 0$ and $A \notin \Delta_3$.

The condition $\det H(f) = 0$ can be fulfilled by even and non-even matrices. Take e.g. $A = \frac{1}{3}(123) + \frac{1}{2}(456) \in \Delta_6$ having the even diagonals. $\pi_1 = (123), \pi_2 = (456), \pi_3 = (123)(456)$ and $\pi_4 = 1s_n$ and

$$\det H(f_A) = \begin{vmatrix} 12 & 0 & 6 & 6 \\ 0 & 12 & 6 & 6 \\ 6 & 6 & 12 & 0 \\ 6 & 6 & 0 & 12 \end{vmatrix} = 0.$$ 

A non-even example is given by Example 4.3. For the general case, let us derive the following lemmata.

Lemma 2.6. Suppose $\det H(f) = 0$. If $f$ admits a minimum 0, then this minimum is not isolated and $\dim_{\mathbb{R}}(\pi_k | 1 \leq k \leq p) < p$.

Proof. Suppose $f(y) = 0$ with $y = (y_1, \ldots, y_p) \in \mathbb{R}^p$. Then $\nabla f(y) = 0$ and

$$\frac{1}{2} H(f)y = b,$$ 

where we have used Lemma 2.4. Since $q := \text{rank } H(f) < p$, $y$ is seen to belong to an affine subspace of dimension $p - q$ of critical points at which $f$ vanishes.

Finally, the existence of distinct vectors $y = (y_1, \ldots, y_p)$ and $z = (z_1, \ldots, z_p)$ with $f(y) = f(z) = 0$ implies

$$A = \sum_{k=1}^p y_k \pi_k = \sum_{k=1}^p z_k \pi_k \quad \text{and} \quad \sum_{k=1}^p (y_k - z_k) \pi_k = 0.$$ 

This shows that the diagonals of $A, \pi_1, \ldots, \pi_p$ are linearly dependent. \qed

Lemma 2.7. The functional $f$ is constant on the set of critical points of $f$ being precisely

$$S := \{y \in \mathbb{R}^p | H(f)y = 2b \}.$$ 

Proof. Indeed, since $S$ is either a singleton or an affine subspace of $\mathbb{R}^p$, $S$ is connected, and $\nabla f$ vanishes in $S$. This permits us to conclude. \qed

Lemma 2.8. If $\dim_{\mathbb{R}}(\pi_k | 1 \leq k \leq p) < p$ then $\det H(f) = 0$.

Proof. Suppose that there are reals $\lambda_1, \ldots, \lambda_p$, not all vanishing, such that

$$\sum_{k=1}^p \lambda_k \pi_k = 0.$$ 

Then for each $i \in \{1, \ldots, p\}$, the matrix $\sum_{k=1}^p \lambda_k \pi_k \pi_i^{-1}$ is the zero matrix and, thereby,

$$\sum_{k=1}^p \lambda_k \text{tr}(\pi_k \pi_i^{-1}) = 0,$$ 

showing that the rows of $H(f)$ are linearly dependent. \qed

Corollary 2.9. Suppose $A \in \Delta_n$. Then the following conditions are equivalent:

(a) $\det H(f) = 0$.
(b) $\dim_{\mathbb{R}}(\pi_k | 1 \leq k \leq p) < p$. 
(c) The matrix $A$ can be written in more than one way as a linear combination of even permutation matrices contained in $A$.

Note that $\det H(f) = 0$ can hold, while there is only one way to express $A$ as a convex combination of even permutation matrices, see Example 4.2. If $f(y) = 0$ for some $y \in S \cap E^p$, then the matrix $A$ must be even. Thus we can state the following criteria.

**Theorem 2.10.** A doubly-stochastic matrix $A$ is even iff $S \cap E^p \cap f^{-1}(0) \neq \emptyset$.

**Corollary 2.11.** If $f(z) > 0$ for some $z \in S$, then $A$ is not even.

Clearly the same holds if $f > 0$ in $E^p$.

### 3. Taking into account one even diagonal

The above characterization can be improved by bearing in mind that an even doubly-stochastic matrix necessarily possesses even diagonals. It is not necessary to include the case of totally odd doubly-stochastic matrices. Thus, assuming that $A \in \Omega_h$ has the even diagonal $\sigma \in A_h$, all entries of the principal diagonal in the matrix $\sigma^{-1}A$ are positive. Working with the latter matrix instead of $A$, we can assume w.l.o.g. that

$$1_{S_n} < A.$$  \hspace{1cm} (7)

Let $\pi_1, \ldots, \pi_r$ denote the distinct even diagonals of $A$ different from $1_{S_n}$, i.e. $r = p - 1$ and $\pi_{r+1} = 1_{S_n}$. The aim is to control whether the matrix $A$ can be written in the form

$$\sum_{k=1}^{r} x_k \pi_k + \left( 1 - \sum_{k=1}^{r} x_k \right) 1_{S_n}$$

with $(x_1, \ldots, x_r)$ varying in the polyhedron $\mathbb{R}^r = \{ x \in (\mathbb{R}^+)^r \mid x_1 + \ldots + x_r \leq 1 \}$ or not. Accordingly, we modify the functional $f$ into $g = g_A : (\mathbb{R}^+,r) \longrightarrow \mathbb{R}^+$ by setting

$$g(x_1, \ldots, x_r) = \left\| A - \sum_{k=1}^{r} x_k \pi_k - \left( 1 - \sum_{k=1}^{r} x_k \right) 1_{S_n} \right\|^2 = \sum_{i,j=1}^{n} \left( a_{ij} - \sum_{k=1}^{p} x_k \delta_{\pi_k(i)j} - \left( 1 - \sum_{k=1}^{p} x_k \right) \delta_{ij} \right)^2.$$  \hspace{1cm} (8)

The minimum of $g$ restricted to $\mathbb{R}^r$ is always attained therein, but in general, these minima are not critical points of $g$. But, if the minimal value of $g$ amounts to 0, then the minimum is a critical point and shows that $A$ is even:

$$A \in \Delta_n \iff \min_{\mathbb{R}^p} g = 0.$$  \hspace{1cm} (8)

**Lemma 3.1.** The Hessian matrix of the functional $g$ is given by

$$H(g) = 2 \left( n + \operatorname{tr} \left( \pi_m \pi_q^{-1} - \pi_m - \pi_q \right) \right)_{r \times r}.$$  \hspace{1cm} (8)

It is a constant, symmetric, non-negative and semi-positive definite matrix.

**Proof.** For $1 \leq m, q \leq r$ we have

$$\frac{\partial g}{\partial x_m} (x_1, \ldots, x_r) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \left( -\delta_{j \pi_m(i)} + \delta_{ij} \right) \left( a_{ij} - \sum_{k=1}^{r} x_k \delta_{\pi_k(i)j} - \delta_{ij} + \sum_{k=1}^{r} x_k \delta_{ij} \right)$$

$$= 2 \sum_{i=1}^{n} \left( a_{ij} - \sum_{k=1}^{r} x_k \delta_{\pi_k(i)} - 1 + \sum_{k=1}^{r} x_k - a_{i \pi_m(i)} + \sum_{k=1}^{r} x_k \delta_{\pi_m(i)j} + \delta_{i \pi_m(i)} - \sum_{k=1}^{r} x_k \delta_{\pi_m(i)j} \right).$$
Thus
\[
\frac{\partial^2 f}{\partial x_q \partial x_m}(x_1, \ldots, x_r) = 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (-\delta j \pi_m(i) + \delta_{ij}) (-\delta j \pi_q(i) + \delta_{ij}) \\
= 2 \sum_{i=1}^{n} \sum_{j=1}^{n} (\delta_j \pi_m(i) \delta_j \pi_q(i) + \delta_{ij}^2 - \delta_j \pi_m(i) \delta_{ij} - \delta_j \pi_q(i) \delta_{ij}) \\
= 2 \sum_{i=1}^{n} (\delta \pi_m(i) \pi_q(i) + 1 - \delta \pi_m(i) i - \delta \pi_q(i) i) \\
= 2 \sum_{i=1}^{n} (\delta \pi_m(i) \pi_q(i) + 2n - 2 \sum_{i=1}^{n} (\delta \pi_m(i) i) - 2 \sum_{i=1}^{n} (\delta \pi_q(i) i)) \\
= 2n + 2 \text{tr} \left( \pi_l \pi_q^{-1} - \pi_l - \pi_q \right).
\]

But \(\text{tr} \left( \pi_m \pi_q^{-1} - \pi_m - \pi_q \right)\) counts just the elements of the common support of \(\pi_m\) and \(\pi_q\), reduced by the number of fixed points of \(\pi_m\) and reduced by the number of fixed points of \(\pi_q\) that are not fixed points of \(\pi_m\). Since the three involved sets are mutually disjoint, \(\text{tr} \left( \pi_m \pi_q^{-1} - \pi_m - \pi_q \right)\) is bounded from below by \(-n\). Thus \(H(g)\) is non-negative.

Finally, for the semi-positive definite character, we observe as in the proof of Lemma 2.2 that for the even doubly-stochastic matrix
\[
B = \sum_{k=1}^{r} \frac{1}{r+1} \pi_k + \frac{1}{r+1} 1_{sa}
\]
the Hessian matrices of \(g\) coincide:
\[
H(g_B) = 2 \left( n + \text{tr} \left( \pi_l \pi_q^{-1} - \pi_l - \pi_q \right) \right)_{q,l} = H(g_A).
\]

By Eq. (8), \(g_B\) attains its absolute minimum 0. Thus \(H(g_B)\) is semi-positive definite as well as \(H(g_A)\). □

Introduce the vector \(c = (c_m)_{r \times 1}\) with
\[
c_m = n + \sum_{i=1}^{n} (a_i \pi_m(i) - a_{ii} - \delta_{i \pi_m(i)}).
\]

**Lemma 3.2.** A vector \(y \in \mathbb{R}^r\) is a critical point of the functional \(g\) iff
\[
\frac{1}{2} H(g_A) y = \left( \sum_{k=1}^{r} y_k \left( n + \text{tr} \left( \pi_m \pi_k^{-1} - \pi_k - \pi_m \right) \right) \right)_{r \times 1} = c.
\]

**Proof.** For \(1 \leq m \leq r\), \(\frac{\partial f}{\partial x_m}(x_1, \ldots, x_r) = 0\) iff
\[
\sum_{i=1}^{n} (1 - a_{ii} + a_{i \pi_m(i)} - \delta_{i \pi_m(i)}) = \sum_{i=1}^{n} \left( \sum_{k=1}^{r} x_k - \sum_{k=1}^{r} x_k \delta_{i \pi_k(i)} + \sum_{k=1}^{r} x_k \delta_{\pi_m(i) \pi_k(i)} - \sum_{k=1}^{r} x_k \delta_{\pi_m(i) i} \right) \\
= \sum_{k=1}^{r} x_k \left( \sum_{i=1}^{n} (1 - \delta_{i \pi_k(i)} + \delta_{\pi_m(i) \pi_k(i)} - \delta_{\pi_m(i) i}) \right).
\]

Thus, \(y = (y_1, \ldots, y_r)^t\) is a critical point of \(g\) iff for all \(m\)
\[
c_m = \sum_{k=1}^{r} y_k \left( n + \text{tr} \left( \pi_m \pi_k^{-1} - \pi_k - \pi_m \right) \right) = \left( \frac{1}{2} H(g_A) y \right)_m.
\]

□
We note in passing that a critical point of $g$ can yield the absolute minimum in $F^p$ while being of positive value. Take e.g. the matrix

$$A = \frac{1}{4} \begin{pmatrix}
1 & 2 & 1 \\
2 & 1 & 1 \\
1 & 1 & 2
\end{pmatrix} \quad \text{with} \quad H(g) = 2 \begin{pmatrix}
6 & 3 \\
3 & 6
\end{pmatrix} \quad \text{and} \quad c = \begin{pmatrix}
3 \\
3
\end{pmatrix}$$

having the non-trivial even diagonals $\pi_1 = (123)$ and $\pi_2 = (132)$. The unique solution of $H(g)y = 2c$ is given by $y^t = \left( \frac{3}{5}, \frac{4}{5} \right)$ and yields an absolute minimum $g(y) > 0$ in $F^2$. Thus, we conclude that $A \notin \Delta_3$.

As above, the affine subspace of $R^r$ 

$$\mathcal{R} := \{ y \in R^r \mid H(g)y = 2c \}$$

is connected, and the functional $g$ defined on $R^r$ is constant on the set of its critical points being precisely $\mathcal{R}$. Now we can state the following

**Theorem 3.3.** Suppose $A \in \Omega_n$ with $1_{S_n} < A$.

(a) If $\det H(g) > 0$, then $A \in \Delta_n$ iff the vector $y = 2H(g)^{-1}c$ belongs to $F^p \cap g^{-1}(0)$.

(b) If $\det H(g) = 0$, then $A \in \Delta_n$ iff $\mathcal{R} \cap F^p \cap g^{-1}(0) \neq \emptyset$.

**Corollary 3.4.** If $g(z) > 0$ for some $z \in \mathcal{R}$, then $A$ is not even.

As pointed out above, the condition $1_{S_n} < A$ causes no restriction. If $A$ has no even diagonal, then evidently $A \notin \Delta_n$. If $A$ has the diagonal $\sigma \in A_n$, then $A \in \Delta_n$ iff $\sigma^{-1}A \in \Delta_n$.

4. Examples

**Example 4.1.** The (even) doubly-stochastic matrix

$$A = \frac{1}{10} (123) + \frac{3}{10} (134) + \frac{4}{10} (145) + \frac{2}{10} (12345) = \frac{1}{10} \begin{pmatrix}
0 & 3 & 3 & 4 & 0 \\
0 & 7 & 3 & 0 & 0 \\
1 & 0 & 4 & 5 & 0 \\
3 & 0 & 0 & 1 & 6 \\
6 & 0 & 0 & 0 & 4
\end{pmatrix}$$

has the non-trivial even diagonals $\pi_1 = (123)$, $\pi_2 = (134)$, $\pi_3 = (145)$ and $\pi_4 = (12345)$, and

$$M := \frac{1}{2} H(f) = \begin{pmatrix}
5 & 1 & 0 & 2 \\
1 & 5 & 1 & 1 \\
0 & 1 & 5 & 2 \\
2 & 1 & 2 & 5
\end{pmatrix} \quad \text{with} \quad \det M = 390 \quad \text{and} \quad b = \frac{1}{10} \begin{pmatrix}
12 \\
22 \\
27 \\
23
\end{pmatrix}.$$ 

The vector $y = M^{-1}b = \frac{1}{10} \begin{pmatrix}
1 \\
3 \\
4 \\
2
\end{pmatrix}$ yields exactly the original representation of $A$ in $\Delta_n$.

**Example 4.2.** Let $A \in \Delta_4$ be positive. Then the whole alternating group consists in diagonals of $A$ that are numbered by $\pi_1 = 1_{S_n}$, $\pi_2 = (123)$, $\pi_3 = (132)$, $\pi_4 = (124)$, $\pi_5 = (142)$, $\pi_6 = (134)$, $\pi_7 = (143)$, $\pi_8 = (234)$, $\pi_9 = (243)$,
\[ \pi_{10} = (12)(34), \pi_{11} = (13)(24), \pi_{12} = (14)(23). \] Then

\[
\frac{1}{2} H(f) = \begin{pmatrix}
4 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 4 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
1 & 1 & 0 & 4 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 1 & 1 & 1 & 1 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 4 & 1 & 1 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 4 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 4 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 4
\end{pmatrix}
\]

with rank \( H(f) = 10. \)

Thus the affine subspace \( S \) has dimension 2. For

\[
A = \frac{1}{6} \begin{pmatrix}
1 & 3 & 1 & 1 \\
3 & 1 & 1 & 1 \\
1 & 1 & 3 & 1 \\
1 & 1 & 1 & 3
\end{pmatrix}
\quad \text{with} \quad b = \frac{1}{6} (8, 8, 8, 8, 4, 4, 4, 4, 4, 4)'
\]

the elements of \( S = \{ x \in \mathbb{R}^{12} \mid H(f)x = 2b \} \) have the form\(^1\)

\[
\left( t + \frac{1}{6}, \frac{1}{6} + s, \frac{1}{6} - s - t, \frac{1}{6} - s - t, \frac{1}{6} + s, s, -s - t, -s - t, s, t + \frac{1}{6}, t, t \right)
\]

with \( (s, t) \in \mathbb{R}^2 \). Looking for those elements that are non-negative and have coordinate sum 1, we note that for \( s = t = 0 \) the vector

\[
y := \left( \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, 0, 0, 0, 0, \frac{1}{6}, 0, 0 \right)
\]

fulfills the requirements and yields \( A \in \Delta_4 \) with

\[
A = \frac{1}{6} (1_{S_n} + (123) + (132) + (124) + (142) + (12)(34)).
\]

**Example 4.3.** With the matrix \( H(f) \) from the preceding examples, for

\[
B = \frac{1}{10} \begin{pmatrix}
1 & 7 & 1 & 1 \\
7 & 1 & 1 & 1 \\
1 & 1 & 7 & 1 \\
1 & 1 & 1 & 7
\end{pmatrix}
\quad \text{with} \quad b = \frac{1}{10} (16, 16, 16, 16, 4, 4, 4, 4, 16, 4, 4)'
\]

the elements of \( S = \{ x \in \mathbb{R}^{12} \mid H(f)x = 2b \} \) have the form\(^2\)

\[
\left( t + \frac{3}{10}, \frac{3}{10} + s, -s - t + \frac{1}{10}, -s - t + \frac{1}{10}, \frac{3}{10} + s, s, -s - t - \frac{1}{5}, -s - t - \frac{1}{5}, s, t + \frac{3}{10}, t, t \right)
\]

with \( (s, t) \in \mathbb{R}^2 \). But it is impossible to find \( t \geq 0, s \geq 0 \) with \( -s - t - \frac{1}{5} \geq 0 \). This shows that \( S \) does not contain non-negative elements and \( B \not\in \Delta_4 \).

---

\(^1\) Calculated with Maple.
\(^2\) Calculated with Maple.
Example 4.4. Choose again the matrix $B$ from Example 4.3. With the numbering $\pi_1 = (123)$, $\pi_2 = (132)$, $\pi_3 = (124)$, $\pi_4 = (142)$, $\pi_5 = (134)$, $\pi_6 = (143)$, $\pi_7 = (234)$, $\pi_8 = (243)$, $\pi_9 = (12)(34)$, $\pi_{10} = (13)(24)$, $\pi_{11} = (14)(23)$ the matrix $H(g)$ reads

$$H(g) = 2 \begin{pmatrix}
6 & 3 & 3 & 2 & 2 & 3 & 3 & 2 & 4 & 4 & 4 \\
3 & 6 & 2 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 4 \\
3 & 2 & 6 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 4 \\
2 & 3 & 3 & 6 & 2 & 3 & 3 & 2 & 4 & 4 & 4 \\
2 & 3 & 3 & 2 & 6 & 3 & 3 & 2 & 4 & 4 & 4 \\
3 & 2 & 2 & 3 & 3 & 6 & 2 & 3 & 4 & 4 & 4 \\
3 & 2 & 2 & 3 & 3 & 2 & 6 & 3 & 4 & 4 & 4 \\
2 & 3 & 3 & 2 & 2 & 3 & 3 & 6 & 4 & 4 & 4 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 & 4 & 4 & 8 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 4 & 8 & 8 \\
\end{pmatrix}.$$ 

The rank of $H(g)$ amounts to 9, and $R$ is an affine subspace of $\mathbb{R}^{11}$ of dimension 2, while

$$c = \left(3, 3, 3, 3, \frac{18}{10}, \frac{18}{10}, \frac{18}{10}, \frac{4}{10}, \frac{28}{10}, \frac{28}{10}\right)^t.$$ 

The elements of $R = \{ x \in \mathbb{R}^{11} \mid H(g)x = 2c \}$ have the form

$$\left(\frac{3}{10} + t, \frac{1}{10} - t - s, \frac{1}{10} - t - s, \frac{3}{10} + t, t, -\frac{1}{5} - t - s, -\frac{1}{5} - t - s, t, \frac{3}{10} + s, s, s \right)$$

with $(s, t) \in \mathbb{R}^2$. As above, $R \cap F^p = \emptyset$ and $B \not\in \Delta_4$. Moreover, for $s = 0$ and $t = -\frac{3}{10}$

$$y = \left(0, \frac{2}{5}, \frac{2}{5}, 0, -\frac{3}{10}, \frac{1}{5}, \frac{1}{10}, -\frac{3}{10}, \frac{3}{10}, 0, 0 \right)$$

and $g(y) = 0$,

showing that $B$ belongs to the linear span of $A_4$: 

$$B = \frac{2}{5}(123) + \frac{2}{5}(124) + \frac{-3}{10}(134) + \frac{1}{10}(145) + \frac{1}{10}(234) + \frac{-3}{10}(243) + \frac{3}{10}(12)(34) + \frac{3}{10}1_{S_6}.$$ 

Example 4.5. Omitting the condition $\pi_m \prec A$ leads nevertheless to linear even representations. Choose

$$A = \frac{1}{n+1}ee' + (12)$$

that is an even doubly-stochastic matrix for $n = 5$, since

$$A = \frac{1}{6} \left((123) + (132) + (12)(45) + (14)(35) + (15)(42) + (25)(34)\right).$$

But we also have

$$A = \frac{1}{6} \left(1_{S_6} + (12345) + (13524) + (14253) + (15432) + (12)\right)$$

yielding that the transposition $(12)$ belongs to the linear span of $A_5$,


$$- 1_{S_6} - (12345) - (13524) - (14253) - (15432)$$

\[\text{Calculated with Maple.}\]
and
\[
\min_{x \in \mathbb{R}^n} \left\| (12) - \sum_{\pi \in A_n} x_{\pi} \pi \right\| = 0.
\]

**Example 4.6.** The control of the positivity of the coefficient vector \( y \) is indispensable. The matrix

\[
A = \frac{1}{5} \left( 2(123) + 2(134) + 2(145) - (12345) \right) = \frac{1}{5} \begin{pmatrix}
0 & 1 & 2 & 2 & 0 \\
0 & 4 & 1 & 0 & 0 \\
2 & 0 & 2 & 1 & 0 \\
2 & 0 & 0 & 2 & 1 \\
1 & 0 & 0 & 0 & 4
\end{pmatrix}
\]

lies in the linear span of \( A_5 \) with \( y^t e_4 = 1 \) and \( f(y) = 0 \). But \( A \) is not even, because for \( \pi = 1_{S_n} \) and for \( k = 1 \), Mirsky’s condition (1) is not satisfied.

**References**