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Space–time fractional Schrödinger equation with time-independent potentials

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A B S T R A C T

We develop a space–time fractional Schrödinger equation containing Caputo fractional derivative and the quantum Riesz fractional operator from a space fractional Schrödinger equation in this paper. By use of the new equation we study the time evolution behaviors of the space–time fractional quantum system in the time-independent potential fields and two cases that the order of the time fractional derivative is between zero and one and between one and two are discussed respectively. The space–time fractional Schrödinger equation with time-independent potentials is divided into a space equation and a time one. A general solution, which is composed of oscillatory terms and decay ones, is obtained. We investigate the time limits of the total probability and the energy levels of particles when time goes to infinity and find that the limit values not only depend on the order of the time derivative, but also on the sign (positive or negative) of the eigenvalues of the space equation. We also find that the limit value of the total probability can be greater or less than one, which means the space–time fractional Schrödinger equation describes the quantum system where the probability is not conservative and particles may be extracted from or absorbed by the potentials. Additionally, the non-Markovian time evolution laws of the space–time fractional quantum system are discussed. The formula of the time evolution of the mechanical quantities is derived and we prove that there is no conservative quantities in the space–time fractional quantum system. We also get a Mittag–Leffler type of time evolution operator of wave functions and then establish a Heisenberg equation containing fractional operators.

1. Introduction

The history of the fractional calculus [1–4], dating back to the 17th century, is almost as long as that of the integer-order calculus. During the initial stage of the foundation of fractional calculus, its theory and application were made very slow progress due to without supporting of physics and mechanics. And this situation had not changed until the end of 1970s Mandelbrot [5] proposed that there is a lot of fractional dimension in nature and technology in which the phenomenon of self-similarity between entirety and part exists and there is a close connection between fractional Brownian motion and Riemann–Liouville fractional calculus. From then on, the fractional calculus has been used successfully to study many complex systems (or named complex phenomena). It has many important applications in various fields of science and engineering and the fractional differential equations become very popular for describing anomalous transport, diffusion–reaction processes, super-slow relaxation, etc. [6–10] (and the references therein).

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The classical Hamiltonian (or Lagrangian) mechanics is formulated in terms of derivative of integer order. This technique suggests advanced methods for the analysis of conservative systems, while the physical world is rather non-conservative because of friction [11]. The account of frictional forces in physical models increases the complexity in the mathematics needed to deal with them. The fractional Hamiltonian (and Lagrangian) equations of motion for the non-conservative systems were introduced into consideration by Riewe [12,13]. Recently, some papers on the fractional calculus applied in the classical mechanics appear, and the fractional Hamiltonian mechanics [14–16] and the fractional variational calculus are constructed for the classical mechanics [17,18]. We know that the Schrödinger equation in the quantum physics can be reformulated by use of the Hamiltonian canonical equations of motion in the classical mechanics. Muslih et al. [19] studied the fractional path-integral quantization of classical fields and derived a fractional Schrödinger equation containing partial left and right Riemann–Liouville fractional derivatives using the fractional canonical equations of motion. We as researchers naturally ask the questions: "How does the quantum world change if we make changes to the equations that describe it (i.e. generalize the derivative operators to become fractional derivative operators)?," "Do these changes shed any light on our current understanding?," and "Will the modified equations predict any new phenomena?" There is a physical reason for the merger of fractional calculus with quantum mechanics. The Feynman path integral formulation of quantum mechanics is based on a path integral over Brownian paths. In diffusion theory, this can also be done to generate the standard diffusion equation; however, there are examples of many phenomena that are only properly described when non-Brownian paths are considered. When this is done, the resulting diffusion equation has factional derivatives [7,8]. Due to the strong similarity between the Schrödinger equation and the standard diffusion equation one might expect modifications to the Schrödinger equation generated by considering non-Brownian paths in the path integral derivation. This gives the time-factional, space-fractional, and space–time-factional Schrödinger equation [21–23].

In quantum physics, the famous Schrödinger equation is given by (in one dimension)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t),$$

where $\psi(x,t)$ and $V(x,t)$ denote the wave function and the potential function, respectively. Feynman and Hibbs [20] reformulated the Schrödinger equation by use of a path integral approach considering the Gaussian probability distribution. Following them, Laskin [23–26] generalized the Feynman path integral to Lévy one, and developed a space fractional Schrödinger equation. The Lévy stochastic process is a natural generalization of the Gaussian process or the Wiener stochastic process and is characterized by the Lévy index α, $0 < \alpha \leqslant 2$ (when $\alpha = 2$, we have the Gaussian process). Laskin constructed the fractional quantum mechanics using the Lévy path integral and showed some properties of the space fractional quantum system. Afterwards, Guo and Xu [27], Dong and Xu [28] studied the space fractional Schrödinger equation with some specific potential fields and drove the progress of the fractional quantum mechanics.

The standard Schrödinger equation and the space fractional one both obey the Markovian evolution law. When considering non-Markovian evolution, just similar to introducing the time fractional diffusion equation to describe sub- or super-diffusion behavior [7,8], Naber [21] introduced the Caputo fractional derivative [2–4] instead of the first-order derivative over time to the standard Schrödinger equation to describe non-Markovian evolution in quantum physics and formulated a time fractional Schrödinger equation. The Hamiltonian for the time fractional quantum system was found to be non-Hermitian and not local in time. Naber solved the time fractional Schrödinger equation for a free particle and for a potential well. Probability and the resulting energy levels are found to increase over time to limiting values depending on the order of the time derivative. More recently, Wang and Xu [22] established a fractional Schrödinger equation with both space and time fractional derivatives from the standard Schrödinger equation and solved the generalized Schrödinger equation for a free particle and for an infinite rectangular potential well. Thus far, the fractional quantum system has been basically constructed and theoretically describes more extensive fractal [5] phenomena in quantum physics.

The authors of this paper develop a space–time fractional Schrödinger equation based on the space fractional Schrödinger equation. This space–time fractional Schrödinger equation is of minor difference from the one given by Wang and Xu [22] but formally better combines Naber's work with Laskin's. The space fractional Schrödinger equation [23] obtained by Laskin reads (in one dimension)

$$i\hbar\frac{\partial\psi(x,t)}{\partial t} = H_\alpha\psi(x,t), \tag{1.1}$$

where $\psi(x,t)$ is the time-dependent wave function, and H_α $(1 < \alpha \leqslant 2)$ is the fractional Hamiltonian operator given by

$$H_\alpha = D_\alpha\left(-\hbar^2\Delta\right)^{\alpha/2} + V(x,t). \tag{1.2}$$

Here D_α with physical dimension $[D_\alpha] = \mathrm{erg}^{1-\alpha} \times \mathrm{cm}^\alpha \times \mathrm{sec}^{-\alpha}$ is dependent on α [$D_\alpha = 1/(2m)$ for $\alpha = 2$, m denotes the mass of a particle] and $(-\hbar^2\Delta)^{\alpha/2}$ is the quantum Riesz fractional operator [2,4,23] defined by

$$\left(-\hbar^2\Delta\right)^{\alpha/2}\psi(x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}\mathrm{d}p\, e^{ipx/\hbar}|p|^\alpha\int_{-\infty}^{+\infty}e^{-ipx/\hbar}\psi(x,t)\,\mathrm{d}x. \tag{1.3}$$

Note that by use of the method of dimensional analysis we have given a specific expression of D_α in [28] as $D_\alpha = \bar{c}^{2-\alpha}/(\alpha m^{\alpha-1})$, where $\bar{c}$ denotes the characteristic velocity of the non-relativistic quantum system. Let us introduce the Planck units [29]

$$L_p = \sqrt{\frac{G\hbar}{c^3}}, \qquad T_p = \sqrt{\frac{G\hbar}{c^5}}, \qquad M_p = \sqrt{\frac{\hbar c}{G}}, \qquad E_p = M_p c^2, \tag{1.4}$$

where L_p, T_p, M_p, E_p are the Plank length, time, mass, and energy, and G and c are the gravitational constant and the speed of light in the vacuum, respectively. Then, using these Planck units, we can get the space fractional Schrödinger equation in the dimensionless form as

$$iT_p \frac{\partial \psi(x,t)}{\partial t} = \frac{D_\alpha T_p^{2-2\alpha}}{M_p^{1-\alpha} E_p^\alpha L_p^{2-2\alpha}} \left(-\hbar^2 \Delta\right)^{\alpha/2} \psi(x,t) + \frac{V(x,t)}{E_p} \psi(x,t). \tag{1.5}$$

Substituting the Caputo fractional derivative (the order is denoted by β and $0 < \beta < 2$ is considered in this paper) for the first-order derivative over time, Eq. (1.5) can be fractionalized as

$$(iT_p)^\beta D_t^\beta \psi(x,t) = \frac{D_\alpha T_p^{2-2\alpha}}{M_p^{1-\alpha} E_p^\alpha L_p^{2-2\alpha}} \left(-\hbar^2 \Delta\right)^{\alpha/2} \psi(x,t) + \frac{V(x,t)}{E_p} \psi(x,t). \tag{1.6}$$

This is a space–time fractional Schrödinger equation obtained from the space fractional Schrödinger equation. Here, it should be noted that Naber [21] has given a superficial and a physical reason to raise the power of the imaginary unit i to the order of the time derivative and all of the complex numbers in this paper are taken the principal value with the arguments θ satisfying $-\pi \leqslant \theta < \pi$.

In this paper, we focus on the time evolution properties of the space–time fractional Schrödinger equation with time-independent potential functions. The solutions to the space–time fractional Schrödinger equation are given and the time evolution law of the space–time fractional quantum system is investigated.

This paper is organized as follows. Sections 2 and 3 deal with the space–time fractional Schrödinger equation for $0 < \beta < 1$. The space–time fractional Schrödinger equation with time-independent potential is solved in Section 2. The equation is divided into a space equation and a time one, and then the general solution containing a time-dependent Mittag–Leffler function [3] is obtained. We find that the sign (positive or negative) of the eigenvalue of the space equation determines the consequences of the time evolution of the total probability and the energy in the space–time fractional quantum system: The total probability and the energy for a particle of any states in any time-independent potential fields are proved to reach a limiting value depending on the order of the time derivative when the eigenvalue of the space equation is positive and the limiting value is zero when the eigenvalue is negative. The space–time fractional Schrödinger equation for a free particle and a δ-potential well are solved as examples. Section 3 presents the time evolution law of the space–time fractional quantum system. The formula of the time evolution of mechanical quantities is derived in Section 3.1. When studying the time evolution of wave functions, a time evolution operator of Mittag–Leffler type is obtained in Section 3.2. The formula of the time limit of the total probability is proved again with the help of the time evolution operator. In Section 3.3, by use of the time evolution operator, we develop a Heisenberg equation, which contains fractional operators. In Section 4, the space–time fractional Schrödinger equation for $1 < \beta < 2$ is discussed in detail and some properties different from the case of $0 < \beta < 1$ are revealed. Our conclusions are given in Section 5.

2. Solutions to the space–time fractional Schrödinger equation

When the potential function is time-independent, with the help of Eq. (1.4), Eq. (1.6) can be rewritten as

$$(i\hbar)^\beta D_t^\beta \psi(x,t) = \mathscr{H}_\alpha \psi(x,t), \tag{2.1}$$

where

$$\mathscr{H}_\alpha = \frac{\hbar^\beta}{E_p T_p^\beta} \left(D_\alpha \left(-\hbar^2 \Delta\right)^{\alpha/2} + V(x,t)\right). \tag{2.2}$$

Here we should note that $\mathscr{H}_\alpha$ is not the Hamiltonian of the quantum system, but it is still Hermitian because it is the same as the Hamiltonian in the space fractional quantum mechanics [25] except for a positive product factor. When $\beta = 1$, the space–time fractional quantum system reduces to the space fractional one and $\mathscr{H}_\alpha$ reduces to the Hamiltonian H_α of the system correspondingly [25]. So we can call $\mathscr{H}_\alpha$ to be the pseudo-Hamiltonian of the system here.

Since $\mathscr{H}_\alpha$ is time-independent, Eq. (2.1) can be solved by separation of variables. By assuming

$$\psi(x.t) = f(t)\phi(x), \tag{2.3}$$

Eq. (2.1) can be divided into the following two equations:

$$\mathscr{H}_\alpha \phi(x) = \lambda \phi(x), \tag{2.4}$$

$$(i\hbar)^\beta D_t^\beta f(t) = \lambda f(t), \tag{2.5}$$

where λ is the eigenvalue of the operator $\mathscr{H}_\alpha$.

Eq. (2.4) can be solved after the fashion of same way as used in the space fractional quantum mechanics. We can assume that there exist a series of eigenvalues λ_n $(n = 0, 1, 2, \ldots)$ for Eq. (2.4), and the corresponding orthonormal eigenfunctions are $\phi_n(x)$, $n = 0, 1, 2, \ldots$. It is necessary to note that λ_n are just the energy eigenvalues, of the space fractional Schrödinger equation with the same potential, multiplied by $\hbar^\beta (E_p T_p^\beta)^{-1}$. Assuming $f(0) = 1$ and taking Laplace transform to Eq. (2.5) yields

$$(i\hbar)^\beta \left(p^\beta \hat{f}(p) - p^{\beta-1}\right) = \lambda \hat{f}(p). \tag{2.6}$$

So we have

$$\hat{f}(p) = \frac{p^{\beta-1}}{p^\beta - \lambda (i\hbar)^{-\beta}}. \tag{2.7}$$

Expanding the right side of Eq. (2.7) to a series form, after inverting the Laplace transform term by term [30], we can get

$$f(t) = E_\beta\left(\lambda(-it/h)^\beta\right), \tag{2.8}$$

where $E_\beta(\cdot)$ is the Mittag–Leffler function [3] defined by

$$E_\beta(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\beta n + 1)}.$$

Therefore, we can get a series of solutions to the space–time fractional Schrödinger equation (2.1) as

$$\psi_n(x,t) = E_\beta\left(\lambda_n(-it/h)^\beta\right)\phi_n(x), \quad n = 0, 1, 2, \ldots. \tag{2.9}$$

Then, the general solution can be written as

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n E_\beta\left(\lambda_n(-it/h)^\beta\right)\phi_n(x), \tag{2.10}$$

where a_n can be any complex numbers and the condition $\sum_{n=0}^{\infty} |a_n|^2 = 1$ is required to guarantee the wave function is normalized when $t = 0$. Thus, $|a_n|^2$ represents the probability to find that the system is in state $\psi_n(x,t)$.

To study the properties of the wave functions, let us give an another form of the Mittag–Leffler function in Eq. (2.8). In fact, we can use the following formula to invert the Laplace transform of Eq. (2.7):

$$f(t) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \hat{f}(p) e^{pt} \, \mathrm{d}p. \tag{2.11}$$

We can calculate the integral in Eq. (2.11) using contour integration method and residue theorem [31] in complex analysis. Since the integrand has a branch point at $p = 0$, the usual Bromwich contour cannot be used. The contour can be chosen like this [21,32]: A branch cut along the negative Real(p) should be made. That is, a cut from $-\infty$ into and then around the origin in a clockwise sense and then back out to $-\infty$. The usual Bromwich contour is continued after the cut. The poles of the integrand are

$$p_k = \hbar^{-1} \lambda^{1/\beta} \cdot e^{\mathrm{i}(2k\pi/\beta - \pi/2)}, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{2.12}$$

In the domain surrounded by the contour, the arguments of the poles should satisfy

$$-\pi < \arg p_k < \pi, \quad k = 0, \pm 1, \pm 2, \ldots. \tag{2.13}$$

Therefore, in the case of $0 < \beta < 1$, there is only one pole ($p_0 = \hbar^{-1}\lambda^{1/\beta} e^{-\pi i/2}$) to be considered to calculate the contour integration. Finally, we can get

$$f(t) = E_\beta\left(\lambda(-it/h)^\beta\right) = \frac{e^{-i\lambda^{1/\beta} t/\hbar}}{\beta} - F_\beta(\rho, t), \tag{2.14}$$

where

$$F_\beta(\rho, t) = \frac{\rho \sin(\pi\beta)}{\pi} \int_0^\infty \frac{e^{-rt} r^{\beta-1}}{r^{2\beta} - 2\rho r^\beta \cos(\pi\beta) + \rho^2} \, \mathrm{d}r, \tag{2.15}$$

and $\rho = \lambda\hbar^{-\beta}e^{-i\pi\beta/2}$. Some properties of $F_\beta(\rho,t)$ have been given by Naber in Ref. [21]. Additionally, $F_\beta(\rho,t)$ has the following two estimation formulas:

$$\left|F_\beta(\rho,t)\right| < M^{-1}|\rho|t^{-\beta}\Gamma(\beta), \tag{2.16}$$

$$\left|\frac{\partial}{\partial t}F_\beta(\rho,t)\right| < M^{-1}|\rho|t^{-(\beta+1)}\Gamma(\beta+1), \tag{2.17}$$

where $M = \min_{r\geqslant 0}\{|r^{2\beta} - 2\rho r^\beta \cos\pi\beta + \rho^2|\} > 0$. We note that in the above part an assumption that λ is positive is made. When $\lambda < 0$, some differences appear: In course of calculating the integral in Eq. (2.11), we can choose the same contour as before but the pole should be taken to $p_0 = \hbar^{-1}|\lambda|^{1/\beta}e^{(\pi/\beta-\pi/2)i}$ when $2/3 < \beta < 1$, and there is no poles surrounded by the contour when $0 < \beta \leqslant 2/3$. Therefore, we have

$$f(t) = \begin{cases} \beta^{-1}e^{|\lambda|^{1/\beta}te^{(\pi/\beta-\pi/2)i}/\hbar} - F_\beta(\rho,t), & \frac{2}{3} < \beta < 1, \\ -F_\beta(\rho,t), & 0 < \beta \leqslant \frac{2}{3}. \end{cases} \tag{2.18}$$

In the following part, we mainly consider the case of $\lambda > 0$ and the results for $\lambda < 0$ will be given by notes. Replacing the Mittag–Leffler functions in Eq. (2.10) by use of Eq. (2.14) yields

$$\psi(x,t) = \psi_S(x,t) + \psi_D(x,t), \tag{2.19}$$

where

$$\psi_S(x,t) = \frac{1}{\beta}\sum_{n=0}^{\infty} a_n\phi_n(x)e^{-i\lambda_n^{1/\beta}t/\hbar}, \tag{2.20}$$

$$\psi_D(x,t) = -\sum_{n=0}^{\infty} a_n\phi_n(x)F_\beta(\rho_n,t). \tag{2.21}$$

Here, $\psi_S(x,t)$ and $\psi_D(x,t)$ are called the oscillatory term and the decay one, respectively. When t goes to infinite, $\psi_S(x,t)$ oscillates rapidly and $\psi_D(x,t)$ approaches to zero. When $\beta = 1$, the decay term $\psi_D(x,t)$ vanishes. Note that when $\lambda < 0$ the solution can also be expanded to be an oscillatory term plus a decay one, but only the decay term exists when $0 < \beta \leqslant 2/3$.

According to the statistical explanation of the wave function, the total probability to find a particle in the state $\psi(x,t)$ at time t is

$$P(t) = \int_{-\infty}^{+\infty} \left|\psi(x,t)\right|^2 dx = \sum_{n=0}^{\infty} |a_n|^2 \left|E_\beta\left(\lambda_n(-it/h)^\beta\right)\right|^2, \tag{2.22}$$

which contains the following special cases:

$$P_n(t) = \int_{-\infty}^{+\infty} \left|\psi_n(x,t)\right|^2 dx = \left|E_\beta\left(\lambda_n(-it/h)^\beta\right)\right|^2, \quad n = 0,1,2,\ldots. \tag{2.23}$$

Taking account of Eqs. (2.14) and (2.16), the following limits hold:

$$\lim_{t\to+\infty} P(t) = \frac{1}{\beta^2}\sum_{n=0}^{\infty}|a_n|^2 = \frac{1}{\beta^2}, \qquad \lim_{t\to+\infty} P_n(t) = \frac{1}{\beta^2}. \tag{2.24}$$

Thus, the limits of the total probabilities are greater than one, which can be viewed that particles are created (extracted from the potential field) as time goes ahead. So the probability in the space–time fractional quantum system is not conservative. This result is the basic characteristics of all of the time fractional quantum system in the time-independent potential fields no matter whether the space term is fractional or not. Some examples with specific potentials can be found in [21,22] as special cases.

With the help of the energy operator, $E = i\hbar\frac{\partial}{\partial t}$, in the standard quantum mechanics, the energy levels $E_n(t)$ of the states $\psi_n(x,t)$ can be calculated

$$E_n(t) = \int_{-\infty}^{+\infty} \psi_n^*(x,t)i\hbar\frac{\partial}{\partial t}\psi_n(x,t)\,dx = i\hbar E_\beta^*\left(\lambda_n(-it/h)^\beta\right)\frac{\partial}{\partial t}E_\beta\left(\lambda_n(-it/h)^\beta\right). \tag{2.25}$$

Similarly, the energy of a particle in the state $\psi(x,t)$ is

$$E(t) = \int_{-\infty}^{+\infty} \psi^*(x,t)i\hbar\frac{\partial}{\partial t}\psi(x,t)\,dx = i\hbar\sum_{n=0}^{\infty}|a_n|^2 E_\beta^*\left(\lambda_n(-it/h)^\beta\right)\frac{\partial}{\partial t}E_\beta\left(\lambda_n(-it/h)^\beta\right). \tag{2.26}$$

Here, $E(t)$ should be interpreted as the weighted average of the energy of every energy eigenstates with the weighting factor being $|a_n|^2$.

Considering Eqs. (2.14) and (2.17), we obtain

$$\lim_{t\to+\infty} E_n(t) = \lambda_n^{1/\beta}/\beta^2, \qquad \lim_{t\to+\infty} E(t) = \sum_{n=0}^{\infty} |a_n|^2 \lambda_n^{1/\beta}/\beta^2. \tag{2.27}$$

So the energy levels come to limiting values in the end of the time evolution. In [21,22], some special cases can be found. Note that for $\lambda < 0$ ($\lambda_n < 0$), recalling Eqs. (2.16)–(2.18), and considering an inequality, $\cos(\pi/\beta - \pi/2) < 0$ for $2/3 < \beta < 1$, we can draw a conclusion that the time limits of the total probability (see (2.24)) and the energy levels (see (2.27)) are all zero, which means particles are completely absorbed by the potential in the end. Therefore, the consequences of the time evolution of the total probability and the energy levels have essential differences between the case that $\lambda > 0$ and $\lambda < 0$.

To end this section, let us solve the space–time fractional Schrödinger equation for a free particle and a δ-potential well as examples.

For a free particle [33], the space equation reads

$$\frac{D_\alpha \hbar^\beta}{E_p T_p^\beta}\left(-\hbar^2\Delta\right)^{\alpha/2}\phi(x) = \lambda\phi(x), \tag{2.28}$$

which has a solution

$$\phi(x) = C \cdot e^{ipx/\hbar},$$

where C is a constant and p denotes the momentum of the particle, and the eigenvalue is

$$\lambda = \frac{D_\alpha \hbar^\beta}{E_p T_p^\beta}|p|^\alpha. \tag{2.29}$$

Then, with the help of Eqs. (2.3) and (2.14), the plane wave solution for a free particle can be written as

$$\psi(x,t) = C \cdot E_\beta\left(\lambda(-it/h)^\beta\right)\exp(ipx/\hbar) = \psi_S(x,t) + \psi_D(x,t), \tag{2.30}$$

where

$$\psi_S(x,t) = \frac{C}{\beta}\exp\left\{i\frac{px}{\hbar} - i\left(\frac{D_\alpha}{E_p}\right)^{1/\beta}\frac{|p|^\alpha t}{T_p}\right\}, \tag{2.31}$$

$$\psi_D(x,t) = -C\exp(ipx/\hbar)F_\beta(\rho,t). \tag{2.32}$$

Therefore, they are the oscillatory term and the decay one, respectively. When $\beta = 1$, the decay term vanishes. Taking Eqs. (1.4) and (2.29) into account, Eq. (2.30) becomes

$$\psi(x,t) = C \cdot \exp\left(i\frac{px}{\hbar} - i\frac{D_\alpha |p|^\alpha t}{\hbar}\right),$$

which is just the plane wave solution for a free particle in the space fractional quantum mechanics [26].

A δ-potential well [28,34] is defined by $V(x) = -\gamma\delta(x)$ ($\gamma > 0$), where $\delta(x)$ denotes the Dirac delta function. The space equation reads

$$\frac{D_\alpha \hbar^\beta}{E_p T_p^\beta}\left[\left(-\hbar^2\Delta\right)^{\alpha/2} - \gamma\delta(x)\right]\phi(x) = \lambda\phi(x). \tag{2.33}$$

With the help of the results about the space fractional Schrödinger equation with δ-potential in [28], Eq. (2.33) can be easily solved. We have the unique eigenvalue

$$\lambda = -\frac{\hbar^\beta}{E_p T_p^\beta}\left(\sin(\pi/\alpha)\hbar\alpha D_\alpha^{1/\alpha}\gamma^{-1}\right)^{\alpha/(1-\alpha)},$$

and the corresponding eigenfunction, expressed in terms of H function [35], is

$$C \cdot H_{2,3}^{2,1}\left[|x|\left(\frac{D_\alpha \hbar^\alpha}{-\lambda}\right)^{-1/\alpha}\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2})\\ (0,1),(1-\frac{1}{\alpha},\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2})\end{matrix}\right],$$

where C is a constant. So the wave function is given by

$$\psi(x,t) = C \cdot H_{2,3}^{2,1}\left[|x|\left(\frac{D_\alpha \hbar^\alpha}{-\lambda}\right)^{-1/\alpha}\,\middle|\,\begin{matrix}(1-\frac{1}{\alpha},\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2})\\ (0,1),(1-\frac{1}{\alpha},\frac{1}{\alpha}),(\frac{1}{2},\frac{1}{2})\end{matrix}\right]E_\beta\left(\lambda(-it/h)^\beta\right).$$

Using the above solution, we can construct the even parity state for the δ-potential well and the odd parity state also does not exist here (for further details, see [28]). The energy level for the even parity state can be calculated by Eq. (2.25). Because λ is negative, we know the time limit of the energy level is zero.

3. Time evolution law

3.1. Time evolution law of mechanical quantities

Considering an identity of Caputo derivatives for $0 < \beta < 1$ [3,21],

$$D_t^{1-\beta} D_t^{\beta} y(t) = \frac{d}{dt} y(t) - \frac{[D_t^{\beta} y(t)]_{t=0}}{t^{1-\beta}\Gamma(\beta)}, \tag{3.1}$$

Eq. (2.1) can be converted into

$$\frac{\partial}{\partial t}\psi(x,t) = (i\hbar)^{-\beta}\mathscr{H}_\alpha\big(D_t^{1-\beta}\psi(x,t)\big) + g(t), \tag{3.2}$$

where $g(t) = [t^{1-\beta}\Gamma(\beta)]^{-1}[D_t^{\beta}\psi(x,t)]_{t=0}$, and the right side of Eq. (3.2) is the Hamiltonian of the space–time fractional quantum system.

Now, we can study the time evolution of a mechanical quantity F in the space–time fractional quantum system. The average value of the mechanical quantity F in the state $\psi(x,t)$ is given by

$$\overline{F} = \big(\psi(x,t), F\psi(x,t)\big). \tag{3.3}$$

With the help of Eq. (3.2), we have

$$\begin{aligned}\frac{d}{dt}\overline{F} &= \left(\frac{\partial}{\partial t}\psi(x,t), F\psi(x,t)\right) + \left(\psi(x,t), F\frac{\partial}{\partial t}\psi(x,t)\right) + \left(\psi(x,t), \frac{\partial F}{\partial t}\psi(x,t)\right) = 2\,\mathrm{Re}\left\{\left(\frac{\partial}{\partial t}\psi, F\psi\right)\right\} + \overline{\frac{\partial F}{\partial t}} \\ &= 2\,\mathrm{Re}\big\{\big((i\hbar)^{-\beta}\mathscr{H}_\alpha\big(D_t^{1-\beta}\psi\big) + g(t), F\psi\big)\big\} + \overline{\frac{\partial F}{\partial t}},\end{aligned} \tag{3.4}$$

where $\mathrm{Re}\{\cdot\}$ denotes the real part of a complex number.

The right side of Eq. (3.2) is time-dependent and non-local in time [21], so in general, $d\overline{F}/dt$ cannot be identically zero for all ψ. It means there are no conservative mechanical quantities of motion for the space–time fractional quantum system.

When $\alpha = 2$ and $\beta = 1$, $\mathscr{H}_\alpha$ reduces to the standard Hamiltonian H. Then, Eq. (3.4) becomes

$$\frac{d}{dt}\overline{F} = 2\,\mathrm{Re}\left\{\left(\frac{1}{i\hbar}H\psi, F\psi\right)\right\} + \overline{\frac{\partial F}{\partial t}} = \frac{1}{i\hbar}(\psi, FH\psi) - \frac{1}{i\hbar}(\psi, HF\psi) + \overline{\frac{\partial F}{\partial t}} = \frac{1}{i\hbar}\overline{[F,H]} + \overline{\frac{\partial F}{\partial t}}, \tag{3.5}$$

which accords with the standard quantum mechanics.

When F is independent of time, Eq. (3.4) gives

$$\frac{d}{dt}\overline{F} = 2\,\mathrm{Re}\big\{\big((i\hbar)^{-\beta}\mathscr{H}_\alpha\big(D_t^{1-\beta}\psi\big) + g(t), F\psi\big)\big\}. \tag{3.6}$$

Furthermore, we assume that F and $\mathscr{H}_\alpha$ commute with each other, that is, $[F, \mathscr{H}_\alpha] = 0$, which implies that there exist a complete sets of common eigenfunctions $\phi_k(x)$ for F and $\mathscr{H}_\alpha$, and $\mathscr{H}_\alpha\phi_k(x) = \lambda_k\phi_k(x)$, $F\phi_k(x) = F_k\phi_k(x)$, $k = 0, 1, 2, \ldots.$ Therefore, the state function $\psi(x,t)$ can be expanded as

$$\psi(x,t) = \sum_{k=0}^{\infty} a_k E_\beta\big(\lambda_k(-it/h)^\beta\big)\phi_k(x), \tag{3.7}$$

where $a_k = (\phi_k(x), \psi(x,0))$, $k = 0, 1, 2, \ldots.$

Recalling Eq. (3.3), the average value of F in the state can be evaluated

$$\overline{F} = \left(\sum_{k=0}^{\infty} a_k E_\beta\big(\lambda_k(-it/h)^\beta\big)\phi_k(x), F\sum_{k=0}^{\infty} a_k E_\beta\big(\lambda_k(-it/h)^\beta\big)\phi_k(x)\right) = \sum_{k=0}^{\infty} F_k|a_k|^2\big|E_\beta\big(\lambda_k(-it/h)^\beta\big)\big|^2. \tag{3.8}$$

So,

$$\frac{d}{dt}\overline{F} = \sum_{k=0}^{\infty} F_k|a_k|^2\frac{d}{dt}\big|E_\beta\big(\lambda_k(-it/h)^\beta\big)\big|^2, \tag{3.9}$$

which means that F is not a conservative quantity during the motion of the quantum system. However, if $\beta = 1$, Eq. (3.9) comes to $d\overline{F}/dt = 0$ and then F is conservative.

3.2. Time evolution operator of wave functions

We mark the wave function at the initial time and time t by $\psi(0)$ and $\psi(t)$ respectively and assume

$$\psi(t) = U(t,0)\psi(0), \tag{3.10}$$

where $U(t,0)$ is called the time evolution operator [36,37], from initial time to time t, of the wave function. In view of the fact that $\psi(t)$ should satisfy Eq. (2.1), we can immediately derive an equation for $U(t,0)$,

$$(i\hbar)^\beta D_t^\beta U(t,0) = \mathscr{H}_\alpha U(t,0). \tag{3.11}$$

Moreover, taking $t = 0$ in Eq. (3.10), we can get the initial condition for $U(t,0)$:

$$U(t,0)|_{t=0} = U(0,0) = 1. \tag{3.12}$$

Combining Eqs. (3.11) and (3.12) and using Laplace transform, the expression of $U(t,0)$ is obtained

$$U(t,0) = E_\beta\big((-it/h)^\beta \mathscr{H}_\alpha\big). \tag{3.13}$$

Therefore, the time evolution operator is of Mittag–Leffler type for the wave functions of the space–time fractional quantum system in the time-independent potential fields, while the one in the standard quantum mechanics is of exponential type.

It is obvious that $U(t,0)$ has the following properties:

1. $U^\dagger(t,0) = U^*(t,0);$ (3.14)
2. $U^\dagger(t,0)U(t,0) = U(t,0)U^\dagger(t,0) = \big|E_\beta\big((-it/h)^\beta \mathscr{H}_\alpha\big)\big|^2;$ (3.15)
3. $\lim_{t\to+\infty} U^\dagger(t,0)U(t,0) = 1/\beta^2;$ (3.16)
4. $D_t^\beta U(t,0)|_{t=0} = (i\hbar)^{-\beta}\mathscr{H}_\alpha;$ (3.17)
5. $U(t,0)$ commutes with $\mathscr{H}_\alpha$, namely, $[U, \mathscr{H}_\alpha] = 0.$ (3.18)

Here $U^\dagger(t,0)$ denotes the conjugate transpose of $U(t,0)$. The second property tells us that $U(t,0)$ is not a unitary operator, which differs from the behavior of the time evolution operator in the standard quantum mechanics. Besides, since the Hamiltonian of this quantum system is time-dependent and non-local in time [21], the composition formula, that is, $U(t_2,t_0) = U(t_2,t_1)U(t_1,t_0)$ $(t_2 > t_1 > t_0)$ is not valid here.

By use of the time evolution operator, we can also prove Eq. (2.22) about the probability limit. In fact, if the initial wave function $\psi(0)$ is normalized, according to the third property of $U(t,0)$, we have

$$\lim_{t\to+\infty} P(t) = \lim_{t\to+\infty} \int_{-\infty}^{+\infty} \psi^*(t)\psi(t)\,dx = \lim_{t\to+\infty} \int_{-\infty}^{+\infty} \psi^*(0)U^\dagger(t,0)U(t,0)\psi(0)\,dx = \frac{1}{\beta^2}\int_{-\infty}^{+\infty} \psi^*(0)\psi(0)\,dx = \frac{1}{\beta^2}. \tag{3.19}$$

3.3. Heisenberg equation of motion

Using the identity (3.1) of Caputo derivative and considering Eq. (3.17), Eq. (3.11) changes into

$$\frac{\partial}{\partial t}U(t,0) = \frac{\mathscr{H}_\alpha}{(i\hbar)^\beta}\left[D_t^{1-\beta}U(t,0) + \frac{1}{t^{1-\beta}\Gamma(\beta)}\right]. \tag{3.20}$$

To continue, we define a variable of a mechanical quantity F through

$$F_H(t) = U^\dagger(t,0)FU(t,0), \tag{3.21}$$

and denote $U^\dagger(t,0)U(t,0)$ by $\mathcal{U}$, the inverse operator of which is signed by $\mathcal{U}^{-1}$. The subscript H of $F_H(t)$ denotes the *Heisenberg picture*.

In the Heisenberg picture, recalling Eqs. (3.3) and (3.10), the average value of F in state $\psi(t)$ can be written as

$$\overline{F} = \big(U(t,0)\psi(0), FU(t,0)\psi(0)\big) = \big(\psi(0), F_H(t)\psi(0)\big). \tag{3.22}$$

Therefore, instead of studying the behavior of $\overline{F}$ in the changeable state $\psi(t)$ in the Schrödinger picture, we can study the behavior of $F_H(t)$ in the fixed state $\psi(0)$ in the Heisenberg picture.

With the help of Eqs. (3.20) and (3.21), the time rate of $F_H(t)$ can be calculated as follows:

$$\frac{d}{dt}F_H(t) = \left(\frac{\partial}{\partial t}U^\dagger(t,0)\right)FU(t,0) + U^\dagger(t,0)F\frac{\partial}{\partial t}U(t,0) + U^\dagger(t,0)\frac{\partial F}{\partial t}U(t,0) = \frac{1}{\hbar^\beta}\big[\mathcal{H}F_H(t) + F_H(t)\mathcal{H}^\dagger\big] + \left(\frac{\partial F}{\partial t}\right)_H, \tag{3.23}$$

where

$$\mathcal{H} = \frac{A}{t^{1-\beta}\Gamma(\beta)} + B, \quad A = i^{\beta}\mathscr{H}_\alpha \mathcal{U}^{-1}U, \quad B = i^{\beta}\mathscr{H}_\alpha\big(D_t^{1-\beta}U^{\dagger}\big)\mathcal{U}^{-1}U,$$

and

$$\left(\frac{\partial F}{\partial t}\right)_H = U^{\dagger}\frac{\partial F}{\partial t}U.$$

Eq. (3.23) is just the Heisenberg equation of motion for the space–time fractional quantum system. When $\alpha = 2$ and $\beta = 1$, we have $\mathcal{H} = iH$, and Eq. (3.23) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}F_H(t) = \frac{1}{i\hbar}\big[F_H(t), H\big] + \left(\frac{\partial F}{\partial t}\right)_H,$$

which accords with the standard quantum mechanics [36,37].

4. Some properties when $1 < \beta < 2$

From now on, we will consider the space–time fractional Schrödinger equation with the order of the time derivative being between one and two and all the notations here will have the same physical meanings mentioned before. In this case, after defining $\mathrm{d}f(t)/\mathrm{d}t|_{t=0} = f_1$, the Laplace transform of Eq. (2.5) should be

$$(i\hbar)^{\beta}\big(p^{\beta}\hat{f}(p) - p^{\beta-1} - f_1 p^{\beta-2}\big) = \lambda\hat{f}(p), \tag{4.1}$$

which has a solution

$$\hat{f}(p) = \frac{p^{\beta-1} + f_1 p^{\beta-2}}{p^{\beta} - \lambda(i\hbar)^{-\beta}}. \tag{4.2}$$

Expanding Eq. (4.2) to a series form, after inverting the Laplace transform term by term [30], we can get

$$f(t) = E_{\beta}\big(\lambda(-it/h)^{\beta}\big) + tf_1 E_{\beta,2}\big(\lambda(-it/h)^{\beta}\big), \tag{4.3}$$

where $E_{\mu,\nu}(\cdot)$ is the generalized Mittag–Leffler function [3] defined by

$$E_{\mu,\nu}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\mu n + \nu)}.$$

Then, the wave functions can be written as

$$\psi_n(x,t) = \big[E_{\beta}\big(\lambda_n(-it/h)^{\beta}\big) + tf_1 E_{\beta,2}\big(\lambda_n(-it/h)^{\beta}\big)\big]\phi_n(x), \quad n = 0, 1, 2, \ldots, \tag{4.4}$$

and the corresponding energy levels are

$$E_n(t) = i\hbar\big[E_{\beta}\big(\lambda_n(-it/h)^{\beta}\big) + tf_1 E_{\beta,2}\big(\lambda_n(-it/h)^{\beta}\big)\big]^* \frac{\partial}{\partial t}\big[E_{\beta}\big(\lambda_n(-it/h)^{\beta}\big) + tf_1 E_{\beta,2}\big(\lambda_n(-it/h)^{\beta}\big)\big]. \tag{4.5}$$

So the general wave function for the particle is given by

$$\psi(x,t) = \sum_{n=0}^{\infty} a_n \psi_n(x,t), \tag{4.6}$$

with the energy

$$E(t) = \sum_{n=0}^{\infty} |a_n|^2 E_n(t). \tag{4.7}$$

In a manner similar to that of the case $0 < \beta < 1$, with the help of formula (2.11), the inverse Laplace transform of Eq. (4.2) can be written as

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{p^{\beta-1} + f_1 p^{\beta-2}}{p^{\beta} - \lambda(i\hbar)^{-\beta}}e^{pt}\,\mathrm{d}p = I_1 + f_1 I_2, \tag{4.8}$$

where

$$I_1 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{p^{\beta-1}}{p^{\beta} - \lambda(i\hbar)^{-\beta}}e^{pt}\,\mathrm{d}p, \qquad I_2 = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{p^{\beta-2}}{p^{\beta} - \lambda(i\hbar)^{-\beta}}e^{pt}\,\mathrm{d}p.$$

The two integrals can be calculated by contour integral method choosing the same contour as mentioned in Section 2 and the poles of the integrands of I_1 and I_2 are the same to each other. But we should note that the poles are somewhat different from those in Section 2. Indeed, taking account of Eqs. (2.12) and (2.13), we can conclude that there is only one pole both for the integrands of I_1 and I_2 when $1 < \beta \leqslant \frac{4}{3}$ and the pole is

$$p_0 = \hbar^{-1}\lambda^{1/\beta}e^{-i\pi/2}.$$

There are two poles for them when $\frac{4}{3} < \beta < 2$ and the poles are

$$p_0 = \hbar^{-1}\lambda^{1/\beta}e^{-i\pi/2} \quad \text{and} \quad p_1 = \hbar^{-1}\lambda^{1/\beta}e^{i(2\pi/\beta - \pi/2)}.$$

After some calculations, we finally get

$$f(t) = \frac{e^{-i\lambda^{1/\beta}t/\hbar}}{\beta} - F_\beta(\rho,t) + f_1\left[\frac{i\hbar e^{-i\lambda^{1/\beta}t/\hbar}}{\beta\lambda^{1/\beta}} - \mathcal{F}_\beta(\rho,t)\right], \quad \text{when } 1 < \beta \leqslant \frac{4}{3}; \tag{4.9}$$

$$f(t) = \frac{1}{\beta}\left[e^{-i\lambda^{1/\beta}t/\hbar} + e^{\lambda^{1/\beta}t\sin(\frac{2\pi}{\beta})/\hbar}e^{-i\lambda^{1/\beta}t\cos(\frac{2\pi}{\beta})/\hbar} - \beta F_\beta(\rho,t)\right] + \frac{f_1}{\beta\lambda^{1/\beta}}\left\{i\hbar\left[e^{-i\lambda^{1/\beta}t/\hbar} + e^{\lambda^{1/\beta}t\sin(\frac{2\pi}{\beta})/\hbar}e^{-i[\lambda^{1/\beta}t\cos(\frac{2\pi}{\beta})/\hbar + 2\pi/\beta]}\right] - \beta\lambda^{1/\beta}\mathcal{F}_\beta(\rho,t)\right\}, \quad \text{when } \frac{4}{3} < \beta < 2, \tag{4.10}$$

where $F_\beta(\rho,t)$ and ρ have the same forms as we defined in Section 2, and

$$\mathcal{F}_\beta(\rho,t) = \frac{\rho\sin[\pi(\beta-1)]}{\pi}\int_0^\infty \frac{e^{-rt}r^{\beta-2}}{r^{2\beta} - 2\rho r^\beta\cos(\pi\beta) + \rho^2}\,dr. \tag{4.11}$$

$\mathcal{F}_\beta(\rho,t)$ also has the following two estimation formulas:

$$\left|\mathcal{F}_\beta(\rho,t)\right| < |\rho| t^{1-\beta}\Gamma(\beta-1)/M; \tag{4.12}$$

$$\left|\frac{\partial}{\partial t}\mathcal{F}_\beta(\rho,t)\right| < |\rho| t^{-\beta}\Gamma(\beta)/M. \tag{4.13}$$

Here, the definition of M has been given in Section 2. Note that the above results are based on the assumption that $\lambda > 0$, and if $\lambda < 0$, there is only one pole, $p_0 = \hbar^{-1}|\lambda|^{1/\beta}e^{(\pi/\beta - \pi/2)i}$, to be considered to calculated Eq. (4.8). After some calculations, we obtain

$$f(t) = \beta^{-1}e^{|\lambda|^{1/\beta}e^{(\pi/\beta-\pi/2)i}t/\hbar} - F_\beta(\rho,t) + f_1\left[\frac{i\hbar e^{|\lambda|^{1/\beta}e^{(\pi/\beta-\pi/2)i}t/\hbar - i\pi/\beta}}{\beta|\lambda|^{1/\beta}} - \mathcal{F}_\beta(\rho,t)\right]. \tag{4.14}$$

Here it should be noticed that with the help of Eqs. (4.9), (4.10), and (4.14), the wave functions (4.4) and (4.6) can easily be written to be oscillatory terms plus decay ones.

When studying the time limits of the total probability and the energy levels for particles, taking account of Eqs. (4.9), (4.10), (4.12), (4.13) and the inequality $\sin\frac{2\pi}{\beta} < 0$ for $1 < \beta < 2$, we get

$$\lim_{t\to+\infty} P_n(t) = \frac{1}{\beta^2}\left[1 + 2\frac{\hbar}{\lambda_n^{1/\beta}}\left(-\mathrm{Im}\{f_1\}\right) + |f_1|^2\left(\frac{\hbar}{\lambda_n^{1/\beta}}\right)^2\right], \tag{4.15}$$

$$\lim_{t\to+\infty} E_n(t) = \frac{\lambda^{1/\beta}}{\beta^2}\left[1 + 2\frac{\hbar}{\lambda_n^{1/\beta}}\left(-\mathrm{Im}\{f_1\}\right) + |f_1|^2\left(\frac{\hbar}{\lambda_n^{1/\beta}}\right)^2\right], \tag{4.16}$$

$$\lim_{t\to+\infty} P(t) = \sum_{n=0}^\infty |a_n|^2 \lim_{t\to+\infty} P_n(t), \qquad \lim_{t\to+\infty} E(t) = \sum_{n=0}^\infty |a_n|^2 \lim_{t\to+\infty} E_n(t), \tag{4.17}$$

where $\mathrm{Im}\{f_1\}$ denotes the imaginary part of f_1. Note that if $\lambda < 0$ ($\lambda_n < 0$), the time limits of both the total probability and the energy levels are infinities, which means the potentials release particles and the energy of the system gets larger and larger as time progresses.

From Eqs. (4.15)–(4.17), we can conclude that as time evolves the total probability and the energy levels go to certain limiting values but may increase or decrease because of the existence of the terms $-\mathrm{Im}\{f_1\}$ and $|f_1|^2$. The increase of the total probability can be viewed as particles are created (extracted from the potential) and the decrease of that may be regarded as particles are absorbed by the potential. To distinguish whether particles are created or absorbed, we need to know whether the limiting value of the total probability is greater or less than one.

Letting $f_1 = x + iy$ and replacing $\hbar\lambda_n^{-1/\beta}$ by ϑ_n, from Eq. (4.15), we can get

$$\lim_{t\to+\infty} P_n(t) = \frac{1}{\beta^2}\left[\vartheta_n^2 x^2 + (\vartheta_n y - 1)^2\right] \geqslant 0. \tag{4.18}$$

With the help of the above equation, after introducing some notations: $x_n = \vartheta_n^{-1}\sqrt{\beta^2 - (\vartheta_n y_n - 1)^2}$, $y_n = \vartheta_n^{-1} - \beta$ and $Y_n = \vartheta_n^{-1} + \beta$, we can conclude that:

(1) If $y > Y_n$, or $y < y_n$, or $|x| > x_n$ with $y_n \leqslant y \leqslant Y_n$, there holds $\lim_{t\to+\infty} P_n(t) > 1$;
(2) If $|x| < x_n$ with $y_n \leqslant y \leqslant Y_n$, there holds $\lim_{t\to+\infty} P_n(t) < 1$. Additionally, if $f_1 = i\vartheta_n^{-1}$, we have $\lim_{t\to+\infty} P_n(t) = 0$, which implies particles are absorbed completely in the end by the potential;
(3) If $|x| = x_n$ with $y_n \leqslant y \leqslant Y_n$, there holds $\lim_{t\to+\infty} P_n(t) = 1$, which means the probability is conservative.

The behavior of $P(t)$ is more complex, but from the above conclusions, we can still know that:

(1) If $y > \kappa_1$, or $y < \zeta_2$, or $|x| > \chi_1$ with $\zeta_1 < y < \kappa_2$, there holds $\lim_{t\to+\infty} P(t) > 1$;
(2) If $|x| < \chi_2$ with $\zeta_1 < y < \kappa_2$, there holds $\lim_{t\to+\infty} P(t) < 1$. But $\lim_{t\to+\infty} P(t)$ will never be zero, as long as the state $\psi(x,t)$ is the superposition of more than two different non-degenerate energy eigenstates.

Here, $\chi_1 = \sup_{n=0}^{+\infty}\{x_n\}$, $\chi_2 = \inf_{n=0}^{+\infty}\{x_n\}$, $\kappa_1 = \sup_{n=0}^{+\infty}\{Y_n\}$, $\kappa_2 = \inf_{n=0}^{+\infty}\{Y_n\}$, $\zeta_1 = \sup_{n=0}^{+\infty}\{y_n\}$, and $\zeta_2 = \inf_{n=0}^{+\infty}\{y_n\}$, with $\sup_{n=0}^{+\infty}\{\cdot\}$ and $\inf_{n=0}^{+\infty}\{\cdot\}$ denoting the superior limit and the inferior one of an array, respectively.

Considering an identity for Caputo fractional derivatives and integrals [3,21] for $1 < \beta < 2$,

$$I_t^{\beta-1} D_t^{\beta} y(t) = \frac{d}{dt} y(t) - \frac{dy(t)}{dt}\bigg|_{t=0}, \tag{4.19}$$

Eq. (2.1) can be converted into

$$\frac{\partial}{\partial t}\psi(x,t) = (i\hbar)^{-\beta}\mathscr{H}_\alpha\big(I_t^{\beta-1}\psi(x,t)\big) + \psi_t'(0), \tag{4.20}$$

where $\psi_t'(0) = \partial\psi(x,t)/\partial t|_{t=0}$.

Then in the same way as used in Section 3.1, the time evolution formula for a mechanical quantity F is obtained

$$\frac{d}{dt}\overline{F} = 2\,\mathrm{Re}\big\{\big((i\hbar)^{-\beta}\mathscr{H}_\alpha\big(I_t^{\beta-1}\psi\big) + \psi_t'(0), F\psi\big)\big\} + \overline{\frac{\partial F}{\partial t}}. \tag{4.21}$$

The left side of Eq. (4.20) is also non-local in time, so similar to the case that $0 < \beta < 1$, there are no conservative mechanical quantities of motion for this quantum system.

The time evolution operator $U(t,0)$ of the wave function still satisfies Eqs. (3.11) and (3.12). To obtain the specific expression of $U(t,0)$, we need a complementary initial condition as

$$\frac{\partial}{\partial t}U(t,0)\bigg|_{t=0} = U_1. \tag{4.22}$$

We can prove that U_1 should be equal to f_1. In fact,

$$\frac{\partial\psi(t)}{\partial t}\bigg|_{t=0} = \frac{\partial U(t,0)}{\partial t}\bigg|_{t=0}\psi(0) = U_1\psi(0).$$

Additionally, from Eqs. (4.4) and (4.6), $\frac{\partial\psi(t)}{\partial t}|_{t=0}$ can also be calculated as

$$\frac{\partial\psi(t)}{\partial t}\bigg|_{t=0} = f_1\psi(0).$$

So, $U_1 = f_1$ holds.

Combining Eqs. (3.11), (3.12) and (4.22), using Laplace transform, $U(t,0)$ can be derived as

$$U(t,0) = E_\beta\big((-it/h)^\beta\mathscr{H}_\alpha\big) + tU_1E_{\beta,2}\big((-it/h)^\beta\mathscr{H}_\alpha\big). \tag{4.23}$$

The first, fourth and fifth properties of the time evolution operator for $0 < \beta < 1$ still hold here but the second and third properties should be changed to

$$U^\dagger(t,0)U(t,0) = U(t,0)U^\dagger(t,0) = \big|E_\beta\big((-it/h)^\beta\mathscr{H}_\alpha\big) + tU_1E_{\beta,2}\big((-it/h)^\beta\mathscr{H}_\alpha\big)\big|^2, \tag{4.24}$$

$$\lim_{t\to+\infty} U^\dagger(t,0)U(t,0) = \frac{1}{\beta^2}\left[1 + 2\frac{\hbar}{\mathscr{H}_\alpha^{1/\beta}}\big(-\mathrm{Im}\{f_1\}\big) + |f_1|^2\left(\frac{\hbar}{\mathscr{H}_\alpha^{1/\beta}}\right)^2\right]. \tag{4.25}$$

Here, we note that in a manner similar to what we have done in Section 3.2 it is easy to prove the formula of the time limits of the total probabilities (see Eq. (4.15)) by virtue of Eq. (4.25).

We can make use of Eq. (4.19) to rewrite Eq. (3.11) as

$$\frac{\partial}{\partial t}U(t,0)=\frac{\mathscr{H}_\alpha}{(i\hbar)^\beta}I_t^{\beta-1}U(t,0)+U_1. \tag{4.26}$$

Then, recalling Eqs. (3.21) and (3.23), the Heisenberg equation for $1<\beta<2$ is obtained

$$\frac{\mathrm{d}}{\mathrm{d}t}F_H(t)=\frac{1}{\hbar^\beta}\Big[\bar{\mathcal{H}}F_H(t)+F_H(t)\bar{\mathcal{H}}^\dagger\Big]+\left(\frac{\partial F}{\partial t}\right)_H, \tag{4.27}$$

where

$$\bar{\mathcal{H}}=S+\hbar^\beta T,\quad S=i^\beta\mathscr{H}_\alpha\big(I_t^{\beta-1}U^\dagger\big)\mathcal{U}^{-1}U,\quad T=U_1^\dagger\mathcal{U}^{-1}U.$$

5. Conclusions

A space–time fractional Schrödinger equation containing Caputo fractional derivative and the quantum Riesz fractional operator is constructed in this paper. The space–time fractional Schrödinger equation with time-independent potential function for the order of the time derivative being between zero and two is studied. The equation is divided into a space equation and a time one and the general solution containing Mittag–Leffler functions is obtained. By use of an another form of the Mittag–Leffler functions, the wave functions are found composed of oscillatory terms and decay ones and the time limits of the total probability and the energy levels are discussed. With the help of the properties of fractional operators, we study the time evolution laws of the space–time fractional quantum system. The time evolution formulas of mechanical quantities are obtained (see Eqs. (3.4) and (4.21)) and from them we find that there is no conservative mechanical quantities of motion for the space–time fractional quantum system. A Mittag–Leffler type of time evolution operator of wave functions and a Heisenberg equation different from the standard quantum mechanics are also given by us (see Eqs. (3.13), (3.23), (4.23) and (4.27)). All of these results are the generalization of those in the standard quantum mechanics.

On studying the time evolution of the space–time fractional quantum system in the time-independent potential fields, we find that the time evolution properties of the quantum system not only depend on the order of time fractional derivative, but also are affected by the sign of the eigenvalue of the space equation. When the eigenvalue of the space equation is positive, the total probability and the energy levels reach some limiting values as time evolutes and the limiting value of the total probability may be less or greater than one, which means the potential may absorb or release particles but the absorbing or releasing behavior becomes weaker enough as time progresses so that the quantum system approaches some steady states with fixed non-zero particle probability and energy levels. Moreover, the limiting value of the total probability can never be zero when $0<\beta<1$ (β denotes the order of the time fractional derivative, as mentioned before) but may be zero when $1<\beta<2$ in some special cases (see the conclusions given in Section 4, on p. 1015 of this paper). Therefore, when the eigenvalue of the space equation is positive, only for $1<\beta<2$, the particles in the space–time fractional quantum system may be absorbed completely by the potential. When the eigenvalue of the space equation is negative, the time limits of the total probability and the energy levels are zeros when $0<\beta<1$ or infinities when $1<\beta<2$, which means the potential absorbs particles completely when $0<\beta<1$ but releases particles all the time when $1<\beta<2$ and the quantum system will never come to a steady state in the latter case.

From the conclusions given before, we know that the basic characteristics of all of the time fractional quantum systems in the time-independent potential fields is that the probability is not conservative no matter whether the space term is fractional or not. In other words, the introduction of the time fractional derivative to the Schrödinger equation causes non-conservation of probability.

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