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Journal of Combinatorial Theory, Series A 113 (2006) 1501–1515
Journal of Combinatorial Theory Series A
www.elsevier.com/locate/jcta

Combinatorial interpretations of the q-Faulhaber and q-Salié coefficients

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Received 14 June 2005

Available online 24 March 2006

Dedicated to Xavier Viennot on the occasion of his sixtieth birthday

Abstract

Recently, Guo and Zeng discovered two families of polynomials featuring in a q-analogue of Faulhaber's formula for the sums of powers and a q-analogue of Gessel–Viennot's formula involving Salié's coefficients for the alternating sums of powers. In this paper, we show that these are polynomials with symmetric, nonnegative integral coefficients by refining Gessel–Viennot's combinatorial interpretations.
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Keywords: q-Faulhaber coefficients; q-Salié coefficients; Complete symmetric functions; Lattice paths

1. Introduction

In the early seventeenth century, Faulhaber [2] considered the sums of powers $S_{m,n} = \sum_{k=1}^{n} k^m$ and provided formulas for the coefficients $f_{m,k}$ $(0 \leqslant m \leqslant 8)$ in

$$S_{2m+1,n} = \frac{1}{2}\sum_{k=1}^{m} f_{m,k}\big(n(n+1)\big)^{k+1}. \tag{1}$$

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[1] The author was supported by EC's IHRP Programme, within Research Training Network "Algebraic Combinatorics in Europe," grant HPRN-CT-2001-00272.

0097-3165/$ – see front matter © 2006 Elsevier Inc. All rights reserved.
doi:10.1016/j.jcta.2006.01.007

In 1989, Gessel and Viennot [4] studied the alternating sums $T_{m,n} = \sum_{k=1}^{n} (-1)^{n-k} k^m$ and showed that there exist integers $s_{m,k}$ such that

$$T_{2m,n} = \frac{1}{2} \sum_{k=1}^{m} s_{m,k} \big(n(n+1)\big)^k. \tag{2}$$

Furthermore, they proved that, up to some factors, the Faulhaber coefficients $f_{m,k}$ and the Salié coefficients $s_{m,k}$ count certain families of nonintersecting lattice paths. There is a huge literature on this subject. Faulhaber's work, including more generally r-fold sums of powers, was nicely exposed by Knuth [6]. For the study of polynomial relations between sums of powers functions, see Beardon [1].

Recall that a natural q-analogue of the nonnegative integer n is given by $[n] = \frac{1-q^n}{1-q}$ and the corresponding q-factorial is $[n]! = \prod_{k=1}^{n} [k]$. Recently, Guo and Zeng [5], continuing work of Schlosser [8], Warnaar [9] and Garrett and Hummel [3], have found interesting q-analogues of (1) and (2). More precisely, for $m, n \in \mathbb{N}$, setting

$$S_{m,n}(q) = \sum_{k=1}^{n} \frac{[2k]}{[2]} [k]^{m-1} q^{\frac{m+1}{2}(n-k)}, \tag{3}$$

$$T_{m,n}(q) = \sum_{k=1}^{n} (-1)^{n-k} [k]^m q^{\frac{m}{2}(n-k)}, \tag{4}$$

they proved the following results:

Theorem 1.1. *There exist polynomials $P_{m,k}$, $Q_{m,k}$, $G_{m,k}$ and $H_{m,k}$ in $\mathbb{Z}[q]$ such that*

$$S_{2m+1,n}(q) = \sum_{k=0}^{m} \big(-q^n\big)^{m-k} \frac{[k]!}{[m+1]!} P_{m,m-k}(q) \frac{([n][n+1])^{k+1}}{[2]}, \tag{5}$$

$$S_{2m,n}(q) = \big(1 - q^{n+1/2}\big) \sum_{k=1}^{m} \big(-q^n\big)^{m-k} \frac{(1-q^{1/2})^{m-k} Q_{m,m-k}(q^{1/2})}{\prod_{i=0}^{m-k} (1 - q^{m-i+1/2})} \frac{([n][n+1])^k}{[2]}, \tag{6}$$

$$T_{2m,n}(q) = \sum_{k=1}^{m} \big(-q^n\big)^{m-k} \frac{G_{m,m-k}(q)}{\prod_{i=0}^{m-k} (1 + q^{m-i})} \big([n][n+1]\big)^k, \tag{7}$$

$$\begin{aligned} T_{2m-1,n}(q) = {} & (-1)^{m+n} H_{m,m-1}\big(q^{1/2}\big) \frac{q^{(m-1/2)n}}{(1+q^{1/2})^m \prod_{i=0}^{m-1} (1 + q^{m-i-1/2})} \\ & + \frac{1 - q^{n+1/2}}{1 - q^{1/2}} \sum_{k=1}^{m} \big(-q^n\big)^{m-k} \frac{H_{m,m-k}(q^{1/2}) ([n][n+1])^{k-1}}{(1+q^{1/2})^{m-k+1} \prod_{i=0}^{m-k} (1 + q^{m-i-1/2})}. \end{aligned} \tag{8}$$

Comparing with (3) and (4), we have

$$f_{m,k} = (-1)^{m-k} \frac{k!}{(m+1)!} P_{m,m-k}(1),$$
$$s_{m,k} = (-1)^{m-k} 2^{k-m} G_{m,m-k}(1),$$

but the numbers corresponding to $Q_{m,k}(1)$ and $H_{m,k}(1)$ do not seem to be studied in the literature. The first values of $P_{m,k}$, $Q_{m,k}$, $G_{m,k}$ and $H_{m,k}$ are given in Tables 1–4, respectively.

Table 1
Values of $P_{m,k}(q)$ for $0 \leqslant m \leqslant 5$

$k \setminus m$	0	1	2	3	4	5
0	1	1	1	1	1	1
1			1	$2(q+1)$	$3q^2+4q+3$	$2(q+1)(2q^2+q+2)$
2				$2(q+1)$	$(q+1)(5q^2+8q+5)$	$(q+1)(9q^4+19q^3+29q^2+19q+9)$
3					$(q+1)(5q^2+8q+5)$	$2(q+1)^2(q^2+q+1)(7q^2+11q+7)$
4						$2(q+1)^2(q^2+q+1)(7q^2+11q+7)$

Table 2
Values of $Q_{m,k}(q)$ for $1 \leqslant m \leqslant 4$

$k \setminus m$	1	2	3	4
0	1	1	1	1
1		1	$2q^2+q+2$	$3q^4+2q^3+4q^2+2q+3$
2			$2q^2+q+2$	$(q^2+q+1)(5q^4+q^3+9q^2+q+5)$
3				$(q^2+q+1)(5q^4+q^3+9q^2+q+5)$

Table 3
Values of $G_{m,k}(q)$ for $1 \leqslant m \leqslant 5$

$k \setminus m$	1	2	3	4	5
0	1	1	1	1	1
1		2	$3(q+1)$	$4(q^2+q+1)$	$5(q+1)(q^2+1)$
2			$6(q+1)$	$2(q+1)(5q^2+7q+5)$	$5(q+1)(3q^4+4q^3+8q^2+4q+3)$
3				$4(q+1)(5q^2+7q+5)$	$5(q+1)^2(7q^4+14q^3+20q^2+14q+7)$
4					$10(q+1)^2(7q^4+14q^3+20q^2+14q+7)$

Table 4
Values of $H_{m,k}(q)$ for $1 \leqslant m \leqslant 4$

$k \setminus m$	1	2	3	4
0	1	1	1	1
1		2	$3q^2+2q+3$	$4q^4+3q^3+4q^2+3q+4$
2			$2(3q^2+2q+3)$	$10q^6+15q^5+30q^4+26q^3+30q^2+15q+10$
3				$2(10q^6+15q^5+30q^4+26q^3+30q^2+15q+10)$

We say that a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ of degree n has *symmetric coefficients* if $a_i = a_{n-i}$ for $0 \leqslant i \leqslant n$. The tables above suggest that the coefficients of the polynomials $P_{m,k}$, $Q_{m,k}$, $G_{m,k}$ and $H_{m,k}$ are nonnegative and symmetric. The aim of this paper is to prove this fact by showing that the coefficients count certain families of nonintersecting lattice paths.

2. Inverses of matrices

Recall that the *nth complete symmetric functions in r variables* $x_1, x_2, \ldots, x_r$ has the following generating function:

$$\sum_{n \geqslant 0} h_n(x_1, \ldots, x_r) z^n = \frac{1}{(1 - x_1 z)(1 - x_2 z) \cdots (1 - x_r z)}.$$

For $r, s \geqslant 0$, let $h_n(\{1\}^r, \{q\}^s)$ denote the nth complete symmetric functions in $r+s$ variables, of which r are specialized to 1 and the others to q, i.e.,

$$\sum_{n\geqslant 0} h_n\big(\{1\}^r, \{q\}^s\big) z^n = \frac{1}{(1-z)^r(1-qz)^s}. \tag{9}$$

By convention, $h_n(\{1\}^r, \{q\}^s) = 0$ if $r < 0$ or $s < 0$. For convenience, we also write $h_n(\{1, q\}^r)$ instead of $h_n(\{1\}^r, \{q\}^r)$. We need the following result.

Lemma 2.1. *For $(a, b) \in \{(0, 1), (1, 0), (1, 1)\}$, we have*

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0} h_{m-2k}\big(\{1\}^{k+a}, \{q\}^{k+b}\big)\left(\frac{q^l}{[l]^2}\right)^k z^m$$
$$= \frac{[l]^2}{[2l]}\begin{cases} \frac{[l+1]}{[l]-[l+1]z} - \frac{q[l-1]}{[l]-q[l-1]z} & \text{if } (a,b) = (1,1), \\ \frac{1}{[l]-[l+1]z} + \frac{q^l}{[l]-q[l-1]z} & \text{if } (a,b) = (1,0), \\ \frac{q^l}{[l]-[l+1]z} + \frac{1}{[l]-q[l-1]z} & \text{if } (a,b) = (0,1). \end{cases} \tag{10}$$

Proof. Using the definition (9) of the complete symmetric functions we have

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0} h_{m-2k}\big(\{1\}^{k+a}, \{q\}^{k+b}\big) x^k z^m = \sum_{k\geqslant 0} \frac{x^k z^{2k}}{(1-z)^{k+a}(1-qz)^{k+b}}$$
$$= \frac{1}{(1-z)^{a-1}(1-qz)^{b-1}} \frac{1}{(1-z)(1-qz) - xz^2}.$$

Setting $x = q^l/[l]^2$ a little calculation shows that the denominator of the second fraction factorizes:

$$\frac{1}{(1-z)(1-qz) - xz^2} = \frac{[l]^2}{([l] - qz[l-1])([l] - z[l+1])}.$$

The result then follows from the standard partial fraction decomposition. □

The following lemma might be interesting per se. When $q = 1$ it reduces to simple applications of the binomial theorem.

Lemma 2.2. *For $k, m \geqslant 1$, set*

$$c_{k,m}(q) := h_{2m-k}\big(\{1, q^2\}^{k-m+1}\big) + q h_{2m-k-1}\big(\{1, q^2\}^{k-m+1}\big),$$
$$g_{k,m}(q) := h_{2m-k}\big(\{1\}^{k-m+1}, \{q\}^{k-m}\big) + h_{2m-k}\big(\{1\}^{k-m}, \{q\}^{k-m+1}\big),$$
$$d_{k,m}(q) := g_{k,m}\big(q^2\big) + q g_{k-1,m-1}\big(q^2\big).$$

Let $X_n = \frac{[n][n+1]}{q^n}$. For $m, l \geqslant 1$, we have

$$X_l^{m+1} - X_{l-1}^{m+1} = \sum_k h_{m-2k}\big(\{1, q\}^{k+1}\big)[2l][l]^{2(m-k)} q^{-l(m-k+1)}, \tag{11}$$

$$\frac{1-q^{l+1/2}}{(1-q^{1/2})q^{l/2}}X_l^m - \frac{1-q^{l-1/2}}{(1-q^{1/2})q^{(l-1)/2}}X_{l-1}^m$$
$$= \sum_k c_{m,m-k}\big(q^{1/2}\big)[2l][l]^{2(m-k-1/2)}q^{-l(m-k+1/2)}, \tag{12}$$

$$X_l^m + X_{l-1}^m = \sum_k g_{m,m-k}(q)[l]^{2(m-k)}q^{-l(m-k)}, \tag{13}$$

$$\frac{1-q^{l+1/2}}{(1-q^{1/2})q^{l/2}}X_l^{m-1} + \frac{1-q^{l-1/2}}{(1-q^{1/2})q^{(l-1)/2}}X_{l-1}^{m-1}$$
$$= \sum_k d_{m,m-k}\big(q^{1/2}\big)[l]^{2(m-k-1/2)}q^{-l(m-k-1/2)}. \tag{14}$$

Proof. The proof rests on the previous lemma.

- Equating the coefficients of (10) in the case $(a, b) = (1, 1)$ yields that

$$\sum_k h_{m-2k}\big(\{1, q\}^{k+1}\big)q^{lk}[l]^{-2k}$$
$$= \frac{[l]}{[2l]}\bigg([l+1]\bigg(\frac{[l+1]}{[l]}\bigg)^m - q[l-1]\bigg(\frac{q[l-1]}{[l]}\bigg)^m\bigg).$$

  Multiplying this expression with $[2l]\frac{[l]^{2m}}{q^{l(m+1)}}$ we obtain (11).
- Since $c_{m,m-k}(q^{1/2}) = h_{m-2k}(\{1, q\}^{k+1}) + q^{1/2}h_{m-1-2k}(\{1, q\}^{k+1})$, Eq. (12) follows directly from the previous calculation.
- As $g_{m,m-k}(q) = h_{m-2k}(\{1\}^{k+1}, \{q\}^k) + h_{m-2k}(\{1\}^k, \{q\}^{k+1})$, applying Lemma 2.1 with $(a, b) = (1, 0), (0, 1)$, we get

$$\sum_{m\geqslant 0}\sum_{k\geqslant 0}\big(h_{m-2k}\big(\{1\}^{k+1}, \{q\}^k\big) + h_{m-2k}\big(\{1\}^k, \{q\}^{k+1}\big)\big)q^{lk}[l]^{-2k}z^m$$
$$= \frac{[l]^2}{[2l]}\bigg(\frac{1+q^l}{[l]-[l+1]z} + \frac{1+q^l}{[l]-q[l-1]z}\bigg).$$

  Multiplying the coefficient of z^m of this expression with $[l]^{2m}q^{-lm}$ we obtain (13).
- Since $d_{m,m-k}(q^{1/2}) = g_{m,m-k}(q) + q^{1/2}g_{m-1,m-k-1}(q)$, Eq. (14) follows directly from the previous calculation. □

The following is the main result of this section.

Theorem 2.3. *The inverses of the lower triangular matrices*

$$\big(h_{2m-k}\big(\{1, q\}^{k-m+1}\big)\big)_{0\leqslant k,m\leqslant n}, \qquad \big(c_{k,m}(q)\big)_{1\leqslant k,m\leqslant n}, \qquad \big(g_{k,m}(q)\big)_{1\leqslant k,m\leqslant n},$$
$$\big(d_{k,m}(q)\big)_{1\leqslant k,m\leqslant n}$$

are respectively the lower triangular matrices

$$\bigg((-1)^{k-m}\frac{[m]!}{[k+1]!}P_{k,k-m}(q)\bigg)_{0\leqslant k,m\leqslant n}, \tag{15}$$

$$\left((-1)^{k-m}\frac{(1-q)^{k-m+1}Q_{k,k-m}(q)}{\prod_{i=0}^{k-m}(1-q^{2k-2i+1})}\right)_{1\leqslant k,m\leqslant n}, \tag{16}$$

$$\left((-1)^{k-m}\frac{G_{k,k-m}(q)}{\prod_{i=0}^{k-m}(1+q^{k-i})}\right)_{1\leqslant k,m\leqslant n}, \tag{17}$$

$$\left((-1)^{k-m}\frac{H_{k,k-m}(q)}{(1+q)^{k-m+1}\prod_{i=0}^{k-m}(1+q^{2k-2i-1})}\right)_{1\leqslant k,m\leqslant n}. \tag{18}$$

Proof. Recall that $X_n=\frac{[n][n+1]}{q^n}$.

- Summing Eq. (11) over l from 1 to n and applying Eq. (3), we obtain

$$X_n^{m+1}=[2]\sum_{k=0}^{\lfloor m/2\rfloor}h_{m-2k}\big(\{1,q\}^{k+1}\big)S_{2m-2k+1,n}(q)q^{-n(m-k+1)}. \tag{19}$$

  Plugging (5) in Eq. (19), the right-hand side becomes

$$\sum_{k=0}^{\lfloor m/2\rfloor}\sum_{l=0}^{m-k}h_{m-2k}\big(\{1,q\}^{k+1}\big)(-1)^{m-k-l}\frac{[l]!}{[m-k+1]!}P_{m-k,m-k-l}(q)X_n^{l+1}.$$

  Comparing the coefficients of X_n^{l+1} we see that $(h_{2m-k}(\{1,q\}^{k-m+1}))_{0\leqslant k,m\leqslant n}$ and (15) are indeed inverses.
- Summing Eq. (12) over l from 1 to n and applying Eq. (3), we obtain

$$\frac{1-q^{n+1/2}}{(1-q^{1/2})q^{n/2}}X_n^m=[2]\sum_{k=0}^{\lfloor m/2\rfloor}c_{m,m-k}\big(q^{1/2}\big)S_{2m-2k,n}(q)q^{-n(m-k+1/2)}. \tag{20}$$

  Substituting (6) into (20) and dividing both sides by $\frac{1-q^{n+1/2}}{(1-q^{1/2})q^{n/2}}$, we get

$$X_n^m=\sum_{k=0}^{\lfloor m/2\rfloor}\sum_{l=1}^{m-k}c_{m,m-k}\big(q^{1/2}\big)(-1)^{m-k-l}\frac{(1-q^{1/2})^{m-k-l}Q_{m-k,m-k-l}\big(q^{1/2}\big)}{\prod_{i=0}^{m-k-l}(1-q^{m-k-i+1/2})}X_n^l.$$

  Comparing the coefficients of X_n^l, we see that $(c_{k,m}(q))_{1\leqslant k,m\leqslant n}$ and (16) are indeed inverses.
- Equation (13) may be written as

$$(-1)^{n-l}X_l^m-(-1)^{n-l+1}X_{l-1}^m=(-1)^{n-l}\sum_{k=0}^{\lfloor m/2\rfloor}g_{m,m-k}(q)\frac{(1-q^l)^{2m-2k}}{(1-q)^{2m-2k}}q^{-l(m-k)}. \tag{21}$$

  Summing Eq. (21) over l from 1 to n and applying Eq. (4), we obtain

$$X_n^m=\sum_{k=0}^{\lfloor m/2\rfloor}g_{m,m-k}(q)T_{2m-2k,n}(q)q^{-n(m-k)}. \tag{22}$$

Substituting (7) into (22), the right-hand side becomes

$$\sum_{k=0}^{\lfloor m/2\rfloor}\sum_{l=1}^{m-k} g_{m,m-k}(q)(-1)^{m-k-l}\frac{G_{m-k,m-k-l}(q)}{\prod_{i=0}^{m-k-l}(1+q^{m-k-i})}X_n^l. \tag{23}$$

Comparing the coefficients of X_n^l, we see that $(g_{k,m}(q))_{1\leqslant k,m\leqslant n}$ and (17) are inverse to each other.

- Equation (14) may be written as

$$(-1)^{n-l}\frac{1-q^{l+1/2}}{(1-q^{1/2})q^{l/2}}X_l^{m-1}-(-1)^{n-l+1}\frac{1-q^{l-1/2}}{(1-q^{1/2})q^{(l-1)/2}}X_{l-1}^{m-1}$$
$$=(-1)^{n-l}\sum_k d_{m,m-k}\big(q^{1/2}\big)[l]^{2(m-k-1/2)}q^{-l(m-k-1/2)}. \tag{24}$$

Summing Eq. (24) over l from 1 to n and applying Eq. (4), we obtain

$$\frac{1-q^{n+1/2}}{(1-q^{1/2})q^{n/2}}X_n^{m-1}=\sum_k d_{m,m-k}\big(q^{1/2}\big)T_{2m-2k-1,n}(q)q^{-n(m-k-1/2)},\quad m\geqslant 2. \tag{25}$$

Substituting (8) into (25) yields

$$\frac{1-q^{n+1/2}}{(1-q^{1/2})q^{n/2}}\left(X_n^{m-1}-\sum_k\sum_{l=1}^{m-k}\frac{(-1)^{m-k-l}d_{m,m-k}(q^{1/2})H_{m-k,m-k-l}(q^{1/2})X_n^{l-1}}{(1+q^{1/2})^{m-k-l+1}\prod_{i=0}^{m-k-l}(1+q^{m-k-i-1/2})}\right)$$
$$=(-1)^n\sum_k\frac{(-1)^{m-k}d_{m,m-k}(q^{1/2})H_{m-k,m-k-1}(q^{1/2})}{(1+q^{1/2})^{m-k}\prod_{i=0}^{m-k-1}(1+q^{m-k-i-1/2})}. \tag{26}$$

We now show that the right-hand side of (26) must vanish. Suppose $0<q<1$. Denote the left-hand side of (26) by L_n. If there exists an $n\in\mathbb{N}$ such that $L_n=0$ we are done. Suppose $L_n\neq 0$ for all $n\geqslant 1$, then L_n is a rational function in $t=q^{n/2}$ and can be written as

$$L_n=t^s f(t)\quad\text{with } t=q^{n/2},$$

where s is an integer and $f(t)$ a rational function with $f(0)\neq 0$. Since $f(q^{n/2})\neq 0$, the right-hand side of (26) implies that

$$f\big(q^{n/2}\big)f\big(q^{(n+1)/2}\big)<0,\quad \forall n\geqslant 1.$$

Taking the limit as $n\to\infty$ we get $(f(0))^2\leqslant 0$, which is impossible. Hence $L_n=0$ and (26) reduces to

$$X_n^{m-1}=\sum_k d_{m,m-k}\big(q^{1/2}\big)\sum_{l=1}^{m-k}\frac{(-1)^{m-k-l}H_{m-k,m-k-l}(q^{1/2})X_n^{l-1}}{(1+q^{1/2})^{m-k-l+1}\prod_{i=0}^{m-k-l}(1+q^{m-k-i-1/2})}. \tag{27}$$

Comparing the coefficients of X_n^{l-1} on both sides of (27), we see that $(d_{k,m}(q))_{1\leqslant k,m\leqslant n}$ and (18) are indeed inverses. □

The following easily verified result has been given by Gessel and Viennot [4].

Lemma 2.4. *Let $(A_{i,j})_{0\leqslant i,j\leqslant m}$ be an invertible lower triangular matrix, and let $(B_{i,j}) = (A_{i,j})^{-1}$. Then we have $B_{n,n} = A_{n,n}^{-1}$ and*

$$B_{n,k} = \frac{(-1)^{n-k}}{A_{k,k}A_{k+1,k+1}\cdots A_{n,n}} \det_{0\leqslant i,j\leqslant n-k-1}(A_{k+i+1,k+j}),$$

where $0 \leqslant n \leqslant m$ and $0 \leqslant k \leqslant n-1$.

Using the above lemma we derive immediately from Theorem 2.3 the following determinant formulas:

$$P_{m,k}(q) = \det_{0\leqslant i,j\leqslant k-1}\big(h_{m-k-i+2j-1}\big(\{1,q\}^{i-j+2}\big)\big), \tag{28}$$

$$Q_{m,k}(q) = \det_{0\leqslant i,j\leqslant k-1}\big(c_{m-k+i+1,m-k+j}(q)\big), \tag{29}$$

$$G_{m,k}(q) = \det_{0\leqslant i,j\leqslant k-1}\big(g_{m-k+i+1,m-k+j}(q)\big), \tag{30}$$

$$H_{m,k}(q) = \det_{0\leqslant i,j\leqslant k-1}\big(d_{m-k+i+1,m-k+j}(q)\big), \tag{31}$$

for $k \geqslant 1$.

3. Combinatorial interpretations

A *lattice path* or *path* $s_0 \to s_n$ is a sequence of points $(s_0, s_1, \ldots, s_n)$ in the plane $\mathbb{Z}^2$ such that $s_i - s_{i-1} = (1,0), (0,1)$ for all $i = 1, \ldots, n$. Let us assign a weight to each step (s_i, s_{i+1}) of $s_0 \to s_n$. We define the weight $N(s_0 \to s_n)$ of the path $s_0 \to s_n$ to be the product of the weights of its steps. Let $s_0 = (a,b)$ and $s_n = (c,d)$ such that $a \leqslant c$ and $b \leqslant d$. If we weight each vertical step with x-coordinate i by x_i and all horizontal steps by 1, then

$$N(s_0 \to s_n) = h_{d-b}(x_a, x_{a+1}, \ldots, x_c). \tag{32}$$

Now consider two sequences of lattice points $\mathbf{u} := (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} := (v_1, v_2, \ldots, v_n)$ such that for $i < j$ and $k < l$ any lattice path between u_i and v_l has a common point with any lattice path between u_j and v_k. Set

$$N(\mathbf{u},\mathbf{v}) := \sum N(u_1 \to v_1)\cdots N(u_n \to v_n),$$

where the sum is over all families of nonintersecting paths $(u_1 \to v_1, \ldots, u_n \to v_n)$.

The following remarkable result can be found in Gessel and Viennot [4]. For historical remarks see also Krattenthaler [7].

Theorem 3.1 *(Lindström–Gessel–Viennot). We have*

$$N(\mathbf{u},\mathbf{v}) = \det_{1\leqslant i,j\leqslant n}\big(N(u_j \to v_i)\big).$$

We are now ready to exhibit the combinatorial interpretation of the q-Faulhaber coefficients.

Theorem 3.2. *Let $\mathbf{u} = (u_0, \ldots, u_{k-1})$ and $\mathbf{v} = (v_0, \ldots, v_{k-1})$, where $u_i = (2i, -2i)$ and $v_i = (2i+3, m-k-i-1)$ for $0 \leqslant i \leqslant k-1$.*

(i) *The polynomial $P_{m,k}(q)$ is the sum of the weights of k-nonintersecting paths from $\mathbf{u}$ to $\mathbf{v}$, where a vertical step with an even x-coordinate has weight q, and all the other steps have weight* 1.

(ii) *The polynomial $Q_{m,k}(q)$ is the sum of the weights of k-nonintersecting paths from $\mathbf{u}$ to $\mathbf{v}$, where the weight of the individual steps is the same as before with the exception that q is replaced with q^2 and the vertical step starting from any u_j has weight q^2+q instead of q^2.*

Proof. For (i), by means of (32) we have

$$N(u_j \to v_i) = h_{m-k-i+2j-1}\big(\{1,q\}^{i-j+2}\big).$$

The result then follows from (28) and Theorem 3.1.

For (ii), assume that $u'_j = (2j+1, -2j)$ and $u''_j = (2j, 1-2j)$. The first step of a lattice path from u_j to v_i is either $u_j \to u'_j$ or $u_j \to u''_j$. As $N(u_j \to u'_j) = 1$, $N(u_j \to u''_j) = q^2+q$ and $h_n(x_1,\dots,x_{r-1}) + x_r h_{n-1}(x_1,\dots,x_r) = h_n(x_1,\dots,x_r)$, we have

$$\begin{aligned}
N(u_j \to v_i) &= N\big(u_j \to u'_j\big)N\big(u'_j \to v_i\big) + N\big(u_j \to u''_j\big)N\big(u''_j \to v_i\big) \\
&= N\big(u'_j \to v_i\big) + \big(q^2+q\big)N\big(u''_j \to v_i\big) \\
&= h_{m-k-i+2j-1}\big(\{1\}^{i-j+2}, \{q^2\}^{i-j+1}\big) \\
&\quad + \big(q^2+q\big)h_{m-k-i+2j-2}\big(\{1,q^2\}^{i-j+2}\big) \\
&= h_{m-k-i+2j-1}\big(\{1,q^2\}^{i-j+2}\big) + q h_{m-k-i+2j-2}\big(\{1,q^2\}^{i-j+2}\big).
\end{aligned}$$

The result then follows from (29) and Theorem 3.1. □

Corollary 3.3. *The polynomials $P_{m,k}(q)$ and $Q_{m,k}(q)$ have symmetric coefficients.*

Proof. A combinatorial way to see the symmetry of the coefficients of $P_{m,k}(q)$ is as follows: Modifying the weights in Theorem 3.2(i) such that vertical steps with an odd x-coordinate have weight q and all the others have weight 1 does not change the entries of the determinant in (28).

Now consider any given family of paths with weight q^w, when vertical steps with even x-coordinate have weight q. After the modification of the weights it will have weight $q^{\max - w}$, where max is the total number of vertical steps in such a family of paths, which implies the claim.

For the polynomials $Q_{m,k}$, we use the following alternative weight: vertical steps with odd x-coordinate have weight q^2, vertical steps with starting point u_j have weight $1+q$ and all the others have weight 1. □

When $k = m-1$, there is only one lattice path from $u_0 = (0,0)$ to $v_0 = (3,0)$, which has weight 1. This establishes the following result:

Corollary 3.4. *For $m \geqslant 2$, we have $P_{m,m-1}(q) = P_{m,m-2}(q)$ and $Q_{m,m-1}(q) = Q_{m,m-2}(q)$.*

For the combinatorial interpretation of the q-Salié coefficients, we need an auxiliary lemma:

Lemma 3.5. *Let $(A_{i,j})_{1\leqslant i,j\leqslant n}$ and $(B_{i,j})_{1\leqslant i,j\leqslant n}$ be two matrices. Then*

$$\det_{1\leqslant i,j\leqslant n}(A_{i,j}+B_{i,j}) = \sum_{I\subseteq\{1,\dots,n\}} \det_{1\leqslant i,j\leqslant n}\big(D_{ij}^{(I)}\big),$$

where

$$D_{ij}^{(I)} = \begin{cases} A_{i,j}, & \text{if } j \in I, \\ B_{i,j}, & \text{otherwise.} \end{cases}$$

Theorem 3.6. *Let* $\mathbf{u} = (u_0, \ldots, u_{k-1})$ *and* $\mathbf{v} = (v_0, \ldots, v_{k-1})$*, where* $u_i = (2i, -2i)$ *and* $v_i = (2i+2, m-k-i-1)$ *for* $0 \leqslant i \leqslant k-1$.

(i) *The polynomial* $G_{m,k}(q)$ *is the sum of the weights of* k*-nonintersecting lattice paths* $\mathbf{L}$ *from* $\mathbf{u}$ *to* $\mathbf{v}$ *with the weight of* $\mathbf{L}$ *being*

$$\sum_{I \subseteq \{0,1,\ldots,k-1\}} w_I(\mathbf{L}),$$

where w_I *is defined as follows: for each* $i \in I$*, vertical steps with* x*-coordinate* $2i-1$ *have weight* q*, and for any integer* $i \notin I$*, vertical steps with* x*-coordinate* $2i$ *have weight* q*. All other steps have weight* 1.

(ii) *The polynomial* $H_{m,k}(q)$ *is the sum of the weights of* k*-nonintersecting lattice paths* $\mathbf{L}$ *from* $\mathbf{u}$ *to* $\mathbf{v}$*, with the weight of* $\mathbf{L}$ *being*

$$\sum_{I \subseteq \{0,1,\ldots,k-1\}} \bar{w}_I(\mathbf{L}),$$

where $\bar{w}_I$ *is the same as* w_I*—replacing* q *with* q^2*—with the exception of vertical steps starting from one of the points* u_i*, which have an additional weight* q*. More precisely, if the weight of such a step would be* 1*, it has weight* $1+q$*, if its weight would be* q^2*, it has weight* q^2+q.

Proof. (i) We apply Lemma 3.5 to $\det_{0 \leqslant i,j \leqslant k-1}(g_{m-k+i+1,m-k+j}(q))$, where

$$g_{m-k+i+1,m-k+j}(q) = h_{m-k-i+2j-1}\bigl(\{1\}^{i-j+2}, \{q\}^{i-j+1}\bigr) + h_{m-k-i+2j-1}\bigl(\{1\}^{i-j+1}, \{q\}^{i-j+2}\bigr).$$

Suppose that $j \in I$ and $0 \leqslant i \leqslant k-1$. Then we have to show that $h_{m-k-i+2j-1}(\{1\}^{i-j+2}, \{q\}^{i-j+1})$ is the sum of the weights of lattice paths from u_j to v_i, where the vertical steps have the weight given in the claim. To this end, note that $h_{m-k-i+2j-1}(\{1\}^{i-j+2}, \{q\}^{i-j+1})$ counts lattice paths from u_j to v_i, when steps on $i-j+1$ given vertical lines have weight q, those steps on the remaining $i-j+2$ vertical lines have weight 1.

By the construction in the claim, steps on exactly one of the vertical lines with x-coordinates $2r-1$ and $2r$ have weight q. Since $j \in I$, steps on the vertical line with x-coordinate $2j$, i.e., with the x-coordinate of u_j, have weight 1.

Similarly, if $j \notin I$ we can verify that there are exactly $i-j+2$ vertical lines between u_j and v_i with steps thereon having weight q.

(ii) In the same way, we can show that for $j \in I$ and $0 \leqslant i \leqslant k-1$,

$$h_{m-k-i+2j-1}\bigl(\{1\}^{i-j+2}, \{q^2\}^{i-j+1}\bigr) + q h_{m-k-i+2j-2}\bigl(\{1\}^{i-j+2}, \{q^2\}^{i-j+1}\bigr)$$

is the sum of weights of lattice paths from u_j to v_i, where the vertical steps have the weight given in the claim. Meanwhile, for $j \notin I$ and $0 \leqslant i \leqslant k-1$,

$$h_{m-k-i+2j-1}\bigl(\{1\}^{i-j+1}, \{q^2\}^{i-j+2}\bigr) + q h_{m-k-i+2j-2}\bigl(\{1\}^{i-j+1}, \{q^2\}^{i-j+2}\bigr)$$

is the sum of weights of lattice paths from u_j to v_i. □

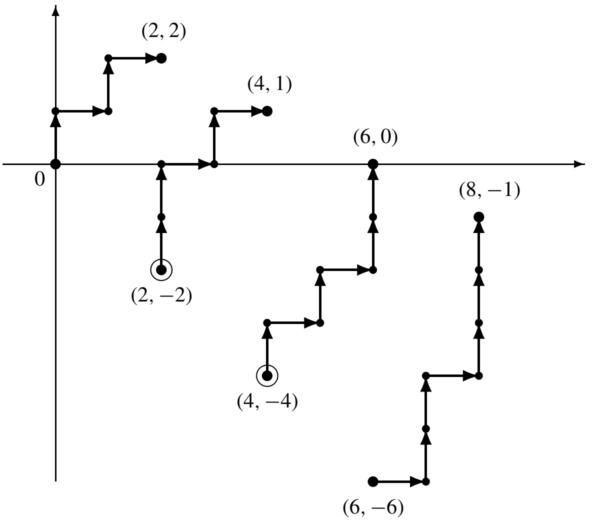

Fig. 1. An example for Theorem 3.6, where $I=\{1,2\}$, $w_I(\mathbf{L})=q^8$ and $\bar{w}_I(\mathbf{L})=q^{14}(q+q^2)(q+1)^2$.

As an illustration of the underlying configurations in Theorem 3.6, we give an example in Fig. 1 for $m=7$ and $k=4$.

Corollary 3.7. *The polynomials $G_{m,k}(q)$ and $H_{m,k}(q)$ have symmetric coefficients.*

Proof. A combinatorial way to see the symmetry of the coefficients of $G_{m,k}(q)$ is as follows: Modifying w_I such that for each $i\in I$, vertical steps with x-coordinate $2i$ have weight q, and for any integer $i\notin I$, vertical steps with x-coordinate $2i-1$ have weight 1 does not change the entries of the determinant in (30).

Now consider any given family of paths with weight q^w provided by Theorem 3.6(i). After the modification of the weights it will have weight $q^{\max-w}$, where max is the total number of vertical steps in such a family of paths, which implies the claim.

We omit the proof of the symmetry of the coefficients of $H_{m,k}(q)$. □

Corollary 3.8. *Let $\mathbf{u}=(u_0,\dots,u_{k-1})$ and $\mathbf{v}=(v_0,\dots,v_{k-1})$, where $u_i=(2i,-2i)$ and $v_i=(2i+2,m-k-i-1)$ for $0\leqslant i\leqslant k-1$.*

(i) *The polynomial $G_{m,k}(q)$ is the sum of the weights of k-nonintersecting lattice paths $\mathbf{L}$ from $\mathbf{u}$ to $\mathbf{v}$ with the weight of $\mathbf{L}$ being*

$$q^{\sigma_{2k}(\mathbf{L})}\prod_{i=0}^{k-1}\left(q^{\sigma_{2i-1}(\mathbf{L})}+q^{\sigma_{2i}(\mathbf{L})}\right),$$

where σ_j denotes the number of vertical steps with x-coordinate j.

(ii) *The polynomial $H_{m,k}(q)$ is the sum of the weights of k-nonintersecting lattice paths $\mathbf{L}$ from $\mathbf{u}$ to $\mathbf{v}$ with the weight of $\mathbf{L}$ being*

$$(1+q)^{f(\mathbf{L})}q^{2\sigma_{2k}(\mathbf{L})}\prod_{i=0}^{k-1}\left(q^{2\sigma_{2i-1}(\mathbf{L})}+q^{2\sigma_{2i}(\mathbf{L})-f_i(\mathbf{L})}\right),$$

where σ_j is as in (i) *and f (respectively f_i) denotes the number of vertical steps starting from* $\mathbf{u}$ (*respectively u_i*).

Proof. (i) By the definition of w_I, for $0 \leqslant i \leqslant k-1$, if $i \in I$, then vertical steps on the line with x-coordinates $2i-1$ have weight q and vertical steps on the line with x-coordinates $2i$ have weight 1; and if $i \notin I$, the case is just contrary. Note that steps on the vertical line with x-coordinates $2k$ always have weight q and steps on the vertical line with x-coordinates $2k-1$ always have weight 1. This implies that

$$\sum_{I\subseteq\{0,1,\ldots,k-1\}} w_I(\mathbf{L}) = q^{\sigma_{2k}(\mathbf{L})}\prod_{i=0}^{k-1}\bigl(q^{\sigma_{2i-1}(\mathbf{L})}+q^{\sigma_{2i}(\mathbf{L})}\bigr).$$

(ii) Notice that for $0 \leqslant i \leqslant k-1$, we have $f_i(\mathbf{L}) = 1$ if $\mathbf{L}$ contains a vertical step starting from the point u_i, and $f_i(\mathbf{L}) = 0$ otherwise. Similarly to (i), we have

$$\begin{aligned}\sum_{I\subseteq\{0,1,\ldots,k-1\}} \bar{w}_I(\mathbf{L}) &= q^{2\sigma_{2k}(\mathbf{L})}\prod_{i=0}^{k-1}\bigl(q^{2\sigma_{2i-1}(\mathbf{L})}(1+q)^{f_i(\mathbf{L})}+q^{2\sigma_{2i}(\mathbf{L})-2f_i(\mathbf{L})}\bigl(q^2+q\bigr)^{f_i(\mathbf{L})}\bigr)\\ &= (1+q)^{f(\mathbf{L})}q^{2\sigma_{2k}(\mathbf{L})}\prod_{i=0}^{k-1}\bigl(q^{2\sigma_{2i-1}(\mathbf{L})}+q^{2\sigma_{2i}(\mathbf{L})-f_i(\mathbf{L})}\bigr).\end{aligned}$$

This completes the proof. □

The computation of $G_{4,2}(q)$ is illustrated in Fig. 2, while the value of $H_{4,2}(q)$ as given in Table 4 is the sum of values in Table 5.

Remark. Since

$$\det_{1\leqslant i,j\leqslant n}(A_{i,j}+B_{i,j}) = \sum_{I\subseteq\{1,\ldots,n\}}\det_{1\leqslant i,j\leqslant n}\bigl(C_{ij}^{(I)}\bigr),$$

where

$$C_{ij}^{(I)} = \begin{cases} A_{i,j}, & \text{if } i \in I,\\ B_{i,j}, & \text{otherwise,}\end{cases}$$

we may also define w_I in Theorem 3.6(i) as follows: for each $i \in I$, vertical steps with x-coordinate $2i+3$ have weight q, and for any integer $i \notin I$, vertical steps with x-coordinate $2i+2$ have weight q. All other steps have weight 1. In this case, for each $i \in I$ and $0 \leqslant j \leqslant k-1$, we can show that $h_{m-k-i+2j-1}(\{1\}^{i-j+2},\{q\}^{i-j+1})$ is the sum of the weights of lattice paths from u_j to v_i. Moreover, there holds

$$\sum_{I\subseteq\{0,1,\ldots,k-1\}} w_I(\mathbf{L}) = q^{\sigma_0(\mathbf{L})}\prod_{i=1}^{k}\bigl(q^{\sigma_{2i}(\mathbf{L})}+q^{\sigma_{2i+1}(\mathbf{L})}\bigr).$$

Similarly, we may define $\bar{w}_I$ in Theorem 3.6(ii) as follows: for each $i \in I$, a vertical step toward the point v_i has weight $q+1$, vertical steps with x-coordinate $2i+3$ have weight q^2. For any integer $i \notin I$, a vertical step toward the point v_i has weight q^2+q, and vertical steps with x-coordinate $2i+2$ not toward v_i have weight q^2. All other steps have weight 1. In this case, we have

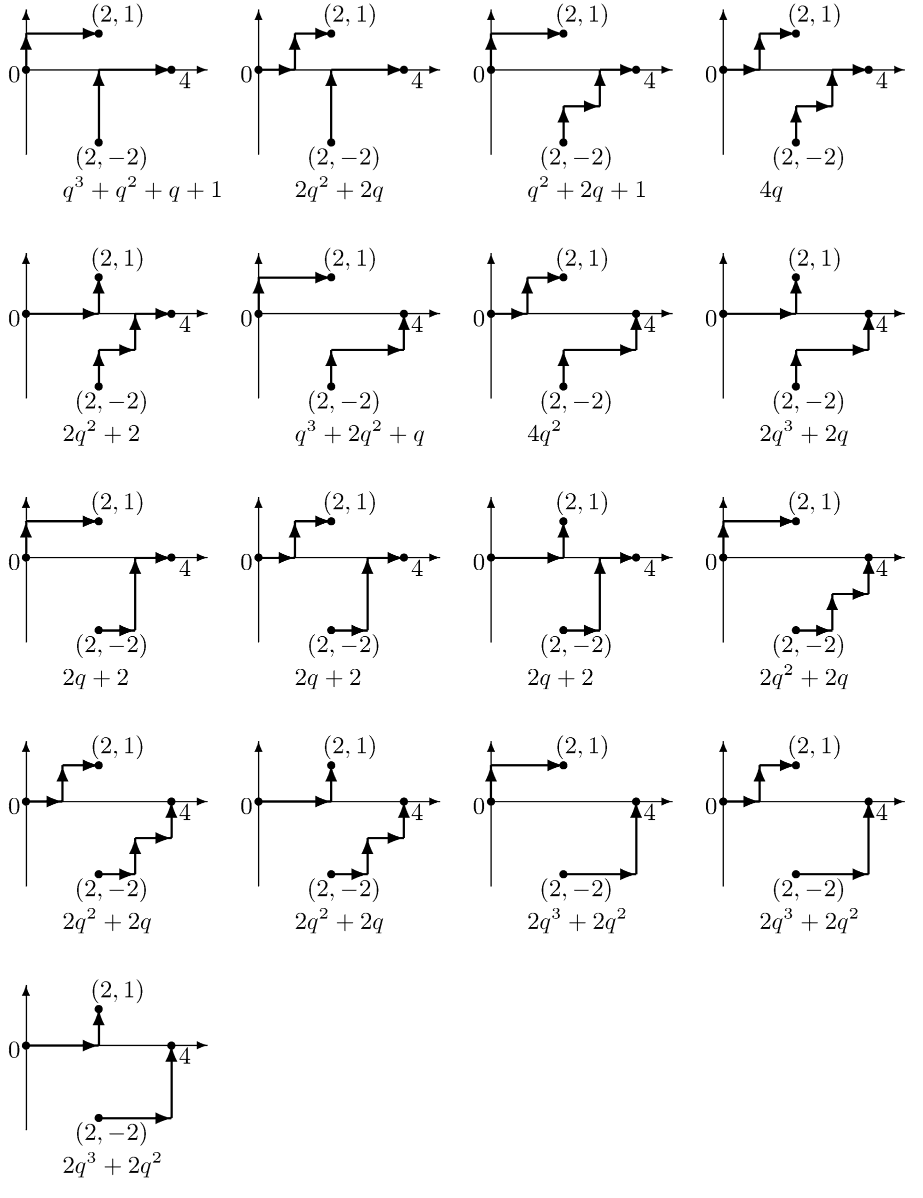

Fig. 2. An illustration for $G_{4,2}(q) = 10q^3 + 24q^2 + 24q + 10$.

Table 5
Values of $\sum_{I \subseteq \{0,1\}} \bar{w}_I(\mathbf{L})$ corresponding to Fig. 2

$(1+q)^3(1+q^3)$	$2q^2(1+q)^2$	$(1+q)^4$	$2q(1+q)^2$
$2(1+q)(1+q^3)$	$q^2(1+q)^4$	$2q^3(1+q)^2$	$2q^2(1+q)(1+q^3)$
$2(1+q)^2$	$2(1+q^2)$	$2(1+q^2)$	$2q^2(1+q)^2$
$2q^2(1+q^2)$	$2q^2(1+q^2)$	$2q^4(1+q)^2$	$2q^4(1+q^2)$
$2q^4(1+q^2)$			

$$\sum_{I\subseteq\{0,1,\dots,k-1\}} \bar{w}_I(\mathbf{L}) = (1+q)^{\bar{f}(\mathbf{L})} q^{2\sigma_0(\mathbf{L})} \prod_{i=1}^{k} \bigl(q^{2\sigma_{2i}(\mathbf{L})-\bar{f}_{i-1}(\mathbf{L})} + q^{2\sigma_{2i+1}(\mathbf{L})}\bigr),$$

where $\bar{f}$ (respectively $\bar{f}_i$) denotes the number of vertical steps ending in $\mathbf{v}$ (respectively v_i).

When $k = m-1$, there is only one lattice path from $u_0 = (0,0)$ to $v_0 = (2,0)$, which has weight 1. This establishes the following result.

Corollary 3.9. *For $m \geqslant 2$, we have $G_{m,m-1}(q) = 2G_{m,m-2}(q)$ and $H_{m,m-1}(q) = 2H_{m,m-2}(q)$.*

4. Open problems

We would like to point out three directions of possible further research: It appears that the polynomials $P_{m,k}$ and $G_{m,k}$ are *log-concave*. However, we have not pursued this question further. Note that the polynomials $Q_{m,k}$ and $H_{m,k}$ are not even *unimodal*.

Guo and Zeng gave in [5] even finer q-analogues of the polynomials considered here, replacing (3) and (4) by

$$S_{m,n,r}(q) = \sum_{k=1}^{n} \frac{[2rk]}{[2r]} [k]^{m-1} q^{\frac{m+2r-1}{2}(n-k)},$$

$$T_{m,n,r}(q) = \sum_{k=1}^{n} (-1)^{n-k} \frac{[(2r-1)k]}{[2r-1]} [k]^{m-1} q^{\frac{m}{2}(n-k)},$$

where $r \geqslant 1$. Although the coefficients of the corresponding polynomials $P_{m,k,r}$, $Q_{m,k,r}$, $G_{m,k,r}$ and $H_{m,k,r}$ are not positive anymore, one might hope for a refinement of Theorem 2.3.

Finally, we should point out that Gessel and Viennot [4] also presented nice generating functions for the coefficients $f_{m,k}$ and $s_{m,k}$, namely

$$\sum_{m,k} s_{m,k} t^k \frac{x^{2n}}{(2n)!} = \frac{\cosh\sqrt{1+4t}\,\frac{x}{2}}{\cosh\frac{x}{2}},$$

$$\sum_{m,k} f_{m,k} t^k \frac{x^{2n+1}}{(2n+1)!} = \frac{\cosh\sqrt{1+4t}\,\frac{x}{2} - \cosh\frac{x}{2}}{t\sinh\frac{x}{2}}.$$

It would be interesting to find the corresponding refinements of the above formulae.

5. Epilogue

One may wonder how these results were discovered. The truth is, that at first “only” formula (5) was known. Using this formula, Table 1 was computed. Then, in analogy to [4], the matrix

$$\left((-1)^{k-m} \frac{[m]!}{[k+1]!} P_{k,k-m}(q)\right)_{0\leqslant k,m\leqslant n}$$

was inverted and, since we were looking for a lattice path interpretation, the entry in row i and column j of the inverse matrix had to be the weighted number of lattice paths from u_j to v_i. This given, it was easy to find the correct weights. Finally, we read the proof given in [4] backwards, its first line corresponding to our Lemma 2.2.

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