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Serre's vanishing conjecture for Ext-groups

Izuru Mori[1]

Department of Mathematics, Syracuse University, Syracuse, NY 13244, USA

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Abstract

Let R be a noetherian commutative local ring, and M, N be finitely generated R-modules. Then a generalized form of Serre's Vanishing conjecture can be stated as follows: if

(1) length $(M \otimes_R N) < \infty$,
(2) $\mathrm{pd}(M), \mathrm{pd}(N) < \infty$, and
(3) $\dim M + \dim N < \dim R$,

then

$$\chi(M,N) := \sum_{i=0}^{\infty} (-1)^i \,\mathrm{length}(\mathrm{Tor}_i^R(M,N)) = 0.$$

It is known that Serre's Vanishing conjecture holds for a complete intersection ring R, but is not known for a Gorenstein ring R. We can make a similar conjecture replacing Tor by Ext, namely, if M and N satisfy the above three conditions, then

$$\xi(M,N) := \sum_{i=0}^{\infty} (-1)^i \,\mathrm{length}(\mathrm{Ext}_R^i(M,N)) = 0.$$

In this paper, we will prove that, over a Gorenstein ring R, the Tor-version of the Vanishing conjecture, the Ext-version of the Vanishing conjecture, and the commutativity of the intersection multiplicity defined in Mori and Smith (J. Pure Appl. Algebra 157 (2001) 279) are all equivalent. Further, we will prove that a certain Ext-version of the Vanishing conjecture holds for a large class of noncommutative projective schemes, typically including all commutative projective schemes over a field, by extending Bézout's Theorem.

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[1] Izuru Mori, Department of Mathematics, The University of Toledo, Toledo, OH 43606-3390, Tel.: +1-419-530-4932; Fax: +1-419-530-4720.

E-mail addresses: imori@syr.edu, imori@utnet.utoledo.edu (I. Mori).

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1. Introduction

Let R be a noetherian commutative local ring, and M,N be finitely generated R-modules. Then Serre [19] defined the intersection multiplicity of M and N by

$$\chi(M,N) := \sum_{i=0}^{\infty} (-1)^i \operatorname{length}(\operatorname{Tor}_i^R(M,N)).$$

Note that $\chi(M,N)$ is well defined if and only if

- $T^R(M,N) := \sup\{\operatorname{length}(\operatorname{Tor}_i^R(M,N))\} < \infty$, and
- $t^R(M,N) := \sup\{i \mid \operatorname{Tor}_i^R(M,N) \neq 0\} < \infty$.

Definition 1.1. We say that R satisfies T-Vanishing if $\chi(M,N) = 0$ for every pair of finitely generated R-modules M,N having the following property, which we denote Vt:

(1) $\operatorname{length}(M \otimes_R N) < \infty$ (which is equivalent to $T^R(M,N) < \infty$),
(2) $\operatorname{pd}(M), \operatorname{pd}(N) < \infty$ (which implies $t^R(M,N) < \infty$), and
(3) $\dim M + \dim N < \dim R$.

Serre's original Vanishing conjecture says that every regular ring R satisfies T-Vanishing, and Serre [19] actually proved it in the case that R contains a field. Later, Roberts [18] and Gillet–Soulé [8] independently proved that every complete intersection ring R satisfies T-Vanishing, but we do not know if every Gorenstein ring R satisfies T-Vanishing. On the other hand, it is known that T-Vanishing holds for graded modules over a graded ring by Peskine and Szpiro [17].

We can make a similar conjecture replacing Tor by Ext. Recall that the Euler form of M and N is defined by

$$\xi(M,N) := \sum_{i=0}^{\infty} (-1)^i \operatorname{length}(\operatorname{Ext}_R^i(M,N)).$$

Note that $\xi(M,N)$ is well defined if and only if

- $E_R(M,N) := \sup\{\operatorname{length}(\operatorname{Ext}_R^i(M,N))\} < \infty$, and
- $e_R(M,N) := \sup\{i \mid \operatorname{Ext}_R^i(M,N) \neq 0\} < \infty$.

Definition 1.2. We say that R satisfies E-Vanishing if $\xi(M,N) = 0$ for every pair of finitely generated R-modules M,N having the following property, which we denote Ve:

(1) $\operatorname{length}(M \otimes_R N) < \infty$ (which implies $E_R(M,N) < \infty$),
(2) $\operatorname{pd}(M), \operatorname{id}(N) < \infty$ (which implies $e_R(M,N) < \infty$), and
(3) $\dim M + \dim N < \dim R$.

Note that if R is a Gorenstein ring and M is a finitely generated R-module, then $\operatorname{pd}(M) < \infty$ if and only if $\operatorname{id}(M) < \infty$, so Property Vt and Property Ve are equivalent. (In this case, we simply call them Property V.) It is known that E-Vanishing holds for graded modules over a (not necessarily commutative) graded ring by Mori [14, Theorem 3.9].

In order to develop intersection theory for noncommutative schemes, we defined in [16] the new intersection multiplicity of M and N by

$$M \cdot N := (-1)^{\operatorname{codim} M}\xi(M,N).$$

This new intersection multiplicity is used in noncommutative algebraic geometry because $\operatorname{Ext}^i_R(M,N)$ makes sense for a noncommutative ring R and left R-modules M,N. An obvious question is when these two intersection multiplicities agree.

Definition 1.3. We say that R satisfies the E-T Formula if $M \cdot N = \chi(M,N)$ for every pair of finitely generated R-modules M,N having the following property, which we denote F:

(1) $\operatorname{length}(M \otimes_R N) < \infty$,
(2) $\operatorname{pd}(M), \operatorname{pd}(N) < \infty$, and
(3) $\dim M + \dim N \leqslant \dim R$.

Chan [5, Theorem 4] proved that every complete intersection ring R satisfies the E-T Formula, but we do not know if every Gorenstein ring R satisfies the E-T Formula. Note that if a Gorenstein ring R satisfies the E-T Formula, then T-Vanishing and E-Vanishing are equivalent. In fact, in this paper, we will prove:

Theorem 1.4. *Let R be a Gorenstein ring. Then R satisfies T-Vanishing if and only if R satisfies E-Vanishing.*

Although the new intersection multiplicity was originally defined in order to apply it to a noncommutative ring, the above theorem gives more reason to study E-Vanishing even for a commutative ring.

One of the desired properties of intersection multiplicity is the commutativity, which is obvious for Serre's intersection multiplicity. This paper was partly motivated by the question when the new intersection mulitiplicity is commutative.

Definition 1.5. We say that R satisfies the E-E Formula if $M \cdot N = N \cdot M$ for every pair of finitely generated R-modules M,N having Property F.

Clearly, if R satisfies the E-T Formula, then R satisfies the E-E Formula. Let R be a Gorenstein ring. Chan [5, Theorem 6] proved that if R satisfies the E-T Formula, then R satisfies T-Vanishing (hence E-Vanishing). In this paper, we will prove:

Theorem 1.6. *Let R be a Gorenstein ring. Then R satisfies the E-E Formula if and only if R satisfies E-Vanishing.*

We also study when $\chi(M,N)$ and $\xi(M,N)$ are well defined. In this paper, we will prove that if R has a dualizing complex, then the conditions $T^R(M,N) < \infty$, $E_R(M,N) < \infty$, and $E_R(N,M) < \infty$ are all equivalent. Avramov and Buchweitz [2] proved that the conditions $t^R(M,N) < \infty$, $e_R(M,N) < \infty$, and $e_R(N,M) < \infty$ are all equivalent for a complete intersection ring R, but we do not know it for a Gorenstein ring R. Combining these results, we have:

Corollary 1.7. *Let R be a complete intersection ring, and M,N be finitely generated R-modules. Then $\chi(M,N)$ is well defined if and only if $\xi(M,N)$ is well defined if and only if $\xi(N,M)$ is well defined.*

In the second half of the paper, we will prove that some versions of E-Vanishing hold for a large class of noncommutative projective schemes, typically including all commutative projective schemes over a field, by extending Bézout's Theorem.

2. Hyperhomological algebra

In this section, we will introduce terminology and notation, which we will use throughout the paper, and collect some elementary results on hyperhomological algebra. For a category $\mathscr{C}$, the notation $M \in \mathscr{C}$ means that M is an object in $\mathscr{C}$.

Let $\mathscr{C}$ be an abelian category and X, Y be cochain complexes of objects in $\mathscr{C}$. The ith cohomology of X is denoted by $h^i(X) \in \mathscr{C}$. We say that a cochain map $f : X \to Y$ is a quasi-isomorphism if the induced maps $h^i(f) : h^i(X) \to h^i(Y)$ are isomorphisms in $\mathscr{C}$ for all i. The derived category of $\mathscr{C}$ is denoted by $\mathscr{D}(\mathscr{C})$, so that a cochain map $f : X \to Y$ is a quasi-isomorphism if and only if it induces an isomorphism $f : X \to Y$ in $\mathscr{D}(\mathscr{C})$. In particular, $X \cong 0$ in $\mathscr{D}(\mathscr{C})$ if and only if $h^i(X) = 0$ for all i.

For $X \in \mathscr{D}(\mathscr{C})$, we define

$$\sup X = \sup\{i \mid h^i(X) \neq 0\}$$

and

$$\inf X = \inf\{i \mid h^i(X) \neq 0\}.$$

If $X \cong 0$ in $\mathscr{D}(\mathscr{C})$, then we define $\sup X = -\infty$ and $\inf X = \infty$.

A complex $X \in \mathscr{D}(\mathscr{C})$ is bounded above (resp. bounded below) if $\sup X < \infty$ (resp. $\inf X > -\infty$), and bounded if it is both bounded above and bounded below. The full subcategory of $\mathscr{D}(\mathscr{C})$ consisting of bounded (resp. bounded above, resp. bounded below) complexes is denoted by $\mathscr{D}^b(\mathscr{C})$ (resp. $\mathscr{D}^-(\mathscr{C})$, resp. $\mathscr{D}^+(\mathscr{C})$). For each integer n, the twist of X is denoted by $X[n] \in \mathscr{D}(\mathscr{C})$, so that $(X[n])^i = X^{n+i}$. Note that $\sup X[n] = \sup X - n$ and $\inf X[n] = \inf X - n$.

In the first half of the paper, we will focus on a noetherian commutative local ring $(R, \mathfrak{m}, k)$, that is, $\mathfrak{m}$ is the unique maximal ideal of R, and $k = R/\mathfrak{m}$ is the residue field of R. However, presumably all the statements can be modified for noetherian commutative connected algebra $(A, \mathfrak{m}, k)$, that is, $A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ is a graded algebra, $\mathfrak{m} = A_1 \oplus A_2 \oplus \cdots$ is the augmentation ideal of A, and $k = A_0 = A/\mathfrak{m}$ is a field. In fact, some statements can be modified even for a noetherian noncommutative connected algebra (see Section 5). I will leave the reader to make necessary modifications in each statememt for a connected algebra.

Let $(R, \mathfrak{m}, k)$ be a noetherian commutative local ring. The category of R-modules is denoted by $\operatorname{Mod} R$, and the full subcategory of $\operatorname{Mod} R$ consisting of finitely generated R-modules is denoted by $\operatorname{mod} R$. We denote $\mathscr{D}(R)=\mathscr{D}(\operatorname{Mod} R)$ for the derived category of the category of R-modules, and $\mathscr{D}_{fg}(R)$ for the full subcategory of $\mathscr{D}(R)$ consisting of complexes whose cohomologies are all finitely generated R-modules.

2.1. Derived functors

The right derived functor of

$$\operatorname{Hom}_R(-,-): \mathscr{D}(R) \times \mathscr{D}(R) \to \mathscr{D}(R)$$

is denoted by $\operatorname{RHom}_R(-,-)$, and its cohomologies are denoted by

$$\operatorname{Ext}^i_R(-,-) = h^i(\operatorname{RHom}_R(-,-)).$$

If $X \in \mathscr{D}^-_{fg}(R)$, $Y \in \mathscr{D}^+_{fg}(R)$, then $\operatorname{RHom}_R(X, Y) \in \mathscr{D}_{fg}(R)$.

The left derived functor of

$$- \otimes_R - : \mathscr{D}(R) \times \mathscr{D}(R) \to \mathscr{D}(R)$$

is denoted by $- \otimes^L_R -$, and its cohomologies are denoted by

$$\operatorname{Tor}^R_{-i}(-,-) = h^i(- \otimes^L_R -).$$

If $X, Y \in \mathscr{D}^-_{fg}(R)$, then $X \otimes^L_R Y \in \mathscr{D}_{fg}(R)$.

2.2. Homological dimensions [3, Proposition 5.5, Corollary 2.10.F]

Let $X \in \mathscr{D}(R)$.

(1) We define the projective dimension of X by

$$\operatorname{pd}(X) = \sup(\{\sup \operatorname{RHom}_R(X, M) \mid M \in \operatorname{Mod} R\}).$$

If $X \in \mathscr{D}^-_{fg}(R)$, then $\operatorname{pd}(X) = \sup \operatorname{RHom}_R(X, k) = -\inf(k \otimes^L_R X)$.

(2) We define the injective dimension of X by

$$\operatorname{id}(X) = \sup(\{\sup \operatorname{RHom}_R(M, X) \mid M \in \operatorname{Mod} R\}).$$

If $X \in \mathscr{D}^+_{fg}(R)$, then $\operatorname{id}(X) = \sup \operatorname{RHom}_R(k, X)$.

(3) We define the flat dimension of X by

$$\operatorname{fd}(X) = \sup(\{-\inf(N \otimes^L_R X) \mid N \in \operatorname{Mod} R\}).$$

If $X \in \mathscr{D}^-_{fg}(R)$, then $\operatorname{fd}(X) = -\inf(k \otimes^L_R X) = \operatorname{pd}(X)$.

(4) If R is a Gorenstein ring and $X \in \mathscr{D}^b_{fg}(R)$, then $\operatorname{pd}(X) < \infty$ if and only if $\operatorname{id}(X) < \infty$.

2.3. Bounds [7, Lemma 2.1]

(1) If $X \in \mathscr{D}^-(R)$ and $Y \in \mathscr{D}^+(R)$, then

$$\inf \mathrm{RHom}_R(X,Y) \geqslant \inf Y - \sup X,$$

$$\sup \mathrm{RHom}_R(X,Y) \leqslant \mathrm{pd}(X) + \sup Y,$$

$$\sup \mathrm{RHom}_R(X,Y) \leqslant -\inf X + \mathrm{id}(Y).$$

(2) If $X, Y \in \mathscr{D}^-(R)$, then

$$\sup(X \otimes_R^L Y) \leqslant \sup X + \sup Y,$$

$$\inf(X \otimes_R^L Y) \geqslant \inf X - \mathrm{fd}(Y),$$

$$\inf(X \otimes_R^L Y) \geqslant \inf Y - \mathrm{fd}(X).$$

(3) If $X, Y \in \mathscr{D}_{fg}^-(R)$, then

$$\sup(X \otimes_R^L Y) = \sup X + \sup Y.$$

2.4. Isomorphisms [7, Proposition 1.1], [3, Lemma 4.4]

For $X, Y, Z \in \mathscr{D}(R)$, the following canonical isomorphisms exist in $\mathscr{D}(R)$:
(1)

$$X \otimes_R^L Y \cong Y \otimes_R^L X,$$

$$(X \otimes_R^L Y) \otimes_R^L Z \cong X \otimes_R^L (Y \otimes_R^L Z),$$

$$\mathrm{RHom}_R(X \otimes_R^L Y, Z) \cong \mathrm{RHom}_R(X, \mathrm{RHom}_R(Y,Z)).$$

(2) If $X \in \mathscr{D}_{fg}^-(R)$, $Y \in \mathscr{D}^+(R)$, and $\mathrm{pd}(X) < \infty$ or $\mathrm{fd}(Z) < \infty$, then

$$\mathrm{RHom}_R(X,Y) \otimes_R^L Z \cong \mathrm{RHom}_R(X, Y \otimes_R^L Z).$$

(3) If $X \in \mathscr{D}_{fg}^-(R)$, $Y \in \mathscr{D}^b(R)$, and $\mathrm{pd}(X) < \infty$ or $\mathrm{id}(Z) < \infty$, then

$$X \otimes_R^L \mathrm{RHom}_R(Y,Z) \cong \mathrm{RHom}_R(\mathrm{RHom}_R(X,Y),Z).$$

2.5. Localization [3, Lemma 5.2]

Let $\mathfrak{p} \in \mathrm{Spec}\, R$.

(1) If $X, Y \in \mathscr{D}(R)$, then

$$(X \otimes_R^L Y)_{\mathfrak{p}} \cong X_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}}^L Y_{\mathfrak{p}}$$

in $\mathscr{D}(R_{\mathfrak{p}})$.

(2) If $X \in \mathscr{D}_{fg}^-(R)$, $Y \in \mathscr{D}^+(R)$, then

$$\mathrm{RHom}_R(X,Y)_{\mathfrak{p}} \cong \mathrm{RHom}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$$

in $\mathscr{D}(R_{\mathfrak{p}})$.

2.6. Support

For $X \in \mathscr{D}(R)$, we define the support of X by

$$\operatorname{Supp} X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X_\mathfrak{p} \ncong 0 \text{ in } \mathscr{D}(R_\mathfrak{p})\} = \bigcup_{i=-\infty}^{\infty} \operatorname{Supp} h^i(X)$$

and the dimension of X by

$$\dim X = \dim \operatorname{Supp} X.$$

Lemma 2.1. (1) *If* $X, Y \in \mathscr{D}_{fg}(R)$, *then*

$$\operatorname{Supp}(X \otimes_R^L Y) \subseteq \operatorname{Supp} X \cap \operatorname{Supp} Y.$$

(2) *If* $X, Y \in \mathscr{D}_{fg}^-(R)$, *then*

$$\operatorname{Supp}(X \otimes_R^L Y) = \operatorname{Supp} X \cap \operatorname{Supp} Y.$$

(3) *If* $X \in \mathscr{D}_{fg}^-(R), Y \in \mathscr{D}^+(R)$, *then*

$$\operatorname{Supp}(\operatorname{RHom}_R(X, Y)) \subseteq \operatorname{Supp} X \cap \operatorname{Supp} Y.$$

Proof. (1) and (3) are clear. Suppose that $X, Y \in \mathscr{D}_{fg}^-(R)$. If $\mathfrak{p}$ is in the support of X and Y, then $X_\mathfrak{p}, Y_\mathfrak{p} \ncong 0$ in $\mathscr{D}_{fg}^-(R_\mathfrak{p})$. Since

$$\sup(X \otimes_R^L Y)_\mathfrak{p} = \sup(X_\mathfrak{p} \otimes_{R_\mathfrak{p}}^L Y_\mathfrak{p}) = \sup X_\mathfrak{p} + \sup Y_\mathfrak{p} > -\infty,$$

$(X \otimes_R^L Y)_\mathfrak{p} \ncong 0$ in $\mathscr{D}(R_\mathfrak{p})$, so $\mathfrak{p}$ is in the support of $X \otimes_R^L Y$. $\square$

2.7. Annihilator

For $X \in \mathscr{D}(R)$, we define the annihilator of X by

$$\operatorname{Ann} X = \bigcap_{i=-\infty}^{\infty} \operatorname{Ann} h^i(X).$$

Lemma 2.2. *If* $X \in \mathscr{D}_{fg}^b(R)$, *then*

$$\operatorname{Supp} X = \operatorname{Spec}(R/\operatorname{Ann} X).$$

Proof. If $X \in \mathscr{D}_{fg}^b(R)$, then $h^i(X) \in \operatorname{mod} R$ for all i, so

$$\begin{aligned}\operatorname{Supp} X &= \bigcup_{i:\ \text{finite}} \operatorname{Supp} h^i(X) = \bigcup_{i:\ \text{finite}} \operatorname{Spec}(R/\operatorname{Ann} h^i(X)) \\ &= \bigcup_{i:\ \text{finite}} \mathscr{V}(\operatorname{Ann} h^i(X)) = \mathscr{V}\left(\bigcap_{i:\ \text{finite}} \operatorname{Ann} h^i(X)\right) \\ &= \mathscr{V}(\operatorname{Ann} X) = \operatorname{Spec}(R/\operatorname{Ann} X). \qquad \square\end{aligned}$$

Remark 2.3. Let $X \in \mathscr{D}(R)$. In general,

$$\operatorname{Ann} X \neq \{r \in R \mid rX \cong 0 \text{ in } \mathscr{D}(R)\}.$$

For example, if $R = k[[x]]$ and

$$X:\ 0 \to R \xrightarrow{x\cdot} R \to R/xR \to 0$$

is a complex of R-modules, then $h^i(X) = 0$ for all i, so $\operatorname{Ann} X = R$. On the other hand,

$$xX:\ 0 \to xR \xrightarrow{x\cdot} xR \to x(R/xR) = 0 \to 0$$

is not exact, so

$$x \notin \{r \in R \mid rX \cong 0 \text{ in } \mathscr{D}(R)\}.$$

2.8. Dualizing complexes [9, Chapter V]

A dualizing complex for R is a complex $D_R \in \mathscr{D}^b_{fg}(R)$ such that

- the canonical map $R \to \operatorname{RHom}_R(D_R, D_R)$ is an isomorphism in $\mathscr{D}(R)$, and
- $\operatorname{id}(D_R) < \infty$.

A dualizing complex D_R is normalized if $-\inf D_R = \dim R$. For a fixed dualizing complex D_R, we define the functor $\dagger : \mathscr{D}(R) \to \mathscr{D}(R)$ by

$$X^\dagger = \operatorname{RHom}_R(X, D_R).$$

We will list several properties of a dualizing complex D_R.

(1) A normalized dualizing complex is unique up to isomorphism in $\mathscr{D}(R)$ (if it exists).
(2) A homomorphic image of a Gorenstein ring R has a (normalized) dualizing complex.
(3) If R is a Cohen–Macaulay ring having a dualizing complex, then the canonical module $\omega_R \in \mathscr{D}^b_{fg}(R)$ is a dualizing complex, and $\omega_R[\dim R] \in \mathscr{D}^b_{fg}(R)$ is the normalized dualizing complex.
(4) A ring R is Gorenstein if and only if $R \in \mathscr{D}^b_{fg}(R)$ is a dualizing complex. In this case, $R[\dim R] \in \mathscr{D}^b_{fg}(R)$ is the normalized dualizing complex.
(5) The functor $\dagger$ defines a duality for $\mathscr{D}^b_{fg}(R)$, that is, for any $X \in \mathscr{D}^b_{fg}(R)$, $X^\dagger \in \mathscr{D}^b_{fg}(R)$ and $(X^\dagger)^\dagger \cong X$ in $\mathscr{D}(R)$.
(6) If D_R is normalized, then $k^\dagger \cong k$ in $\mathscr{D}(R)$.

3. Intersection multiplicity over a commutative local ring

In this section, we will define two notions of intersection multiplicity for complexes over a commutative local ring, and study when they are well defined.

In this section, let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring having a dualizing complex D_R.

For $X \in \mathscr{D}(R)$, we define the Euler characteristic of X by

$$\chi(X) := \sum_{i=-\infty}^{\infty} (-1)^i \operatorname{length} h^i(X).$$

Note that $\chi(X)$ is well defined if and only if

- $\sup\{\operatorname{length} h^i(X)\} < \infty$, and
- $X \in \mathscr{D}^b(R)$.

We denote $\mathscr{D}^{\mathfrak{m}}(R)$ for the full subcategory of $\mathscr{D}(R)$ consisting of complexes X such that $\operatorname{Supp} X \subseteq \{\mathfrak{m}\}$. If $X \in \mathscr{D}_{fg}(R)$, then the first condition $\sup\{\operatorname{length} h^i(X)\} < \infty$ is equivalent to $X \in \mathscr{D}^{\mathfrak{m}}(R)$, so $\chi(X)$ is well defined if and only if $X \in \mathscr{D}^{b,\mathfrak{m}}(R) = \mathscr{D}^b(R) \cap \mathscr{D}^{\mathfrak{m}}(R)$. Let $X, Y \in \mathscr{D}^b_{fg}(R)$. If $X \otimes^L_R Y \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, then we define

$$\chi(X, Y) := \chi(X \otimes^L_R Y) = \sum_{i=-\infty}^{\infty} (-1)^i \operatorname{length}(\operatorname{Tor}^R_i(X, Y)).$$

If $\operatorname{RHom}_R(X, Y) \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, then we define

$$\xi(X, Y) := \chi(\operatorname{RHom}_R(X, Y)) = \sum_{i=-\infty}^{\infty} (-1)^i \operatorname{length}(\operatorname{Ext}^i_R(X, Y)),$$

and

$$X \cdot Y := (-1)^{\operatorname{codim} X} \xi(X, Y) = (-1)^{\dim R - \dim X} \xi(X, Y).$$

Proposition 3.1. *Let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring. If $X, Y \in \mathscr{D}^b_{fg}(R)$, then*

$$\operatorname{Supp}(\operatorname{RHom}_R(X, Y)) = \operatorname{Supp} X \cap \operatorname{Supp} Y.$$

In particular, the following are equivalent:

(1) $X \otimes^L_R Y \in \mathscr{D}^{\mathfrak{m}}(R)$.
(2) $\operatorname{RHom}_R(X, Y) \in \mathscr{D}^{\mathfrak{m}}(R)$.
(3) $\operatorname{RHom}_R(Y, X) \in \mathscr{D}^{\mathfrak{m}}(R)$.

Proof. By Lemma 2.1(3),

$$\operatorname{Supp} X = \operatorname{Supp}(X^\dagger)^\dagger = \operatorname{Supp}(\operatorname{RHom}_R(\operatorname{RHom}_R(X, D_R), D_R))$$
$$\subseteq \operatorname{Supp}(\operatorname{RHom}_R(X, D_R)) \subseteq \operatorname{Supp} X,$$

so

$$\operatorname{Supp} X^\dagger = \operatorname{Supp}(\operatorname{RHom}_R(X, D_R)) = \operatorname{Supp} X.$$

Since $\operatorname{id}(D_R) < \infty$,

$$X \otimes^L_R Y^\dagger = X \otimes^L_R \operatorname{RHom}_R(Y, D_R) \cong \operatorname{RHom}_R(\operatorname{RHom}_R(X, Y), D_R) = \operatorname{RHom}_R(X, Y)^\dagger$$

in $\mathscr{D}_{fg}(R)$. By Lemma 2.1(2),

$$\begin{aligned}\mathrm{Supp}(\mathrm{RHom}_R(X,Y)) &= \mathrm{Supp}(\mathrm{RHom}_R(X,Y)^\dagger) = \mathrm{Supp}(X \otimes_R^L Y^\dagger)\\ &= \mathrm{Supp}\, X \cap \mathrm{Supp}\, Y^\dagger = \mathrm{Supp}\, X \cap \mathrm{Supp}\, Y. \qquad \Box\end{aligned}$$

Corollary 3.2. *Let* $(R,\mathfrak{m},k,D_R)$ *be a noetherian commutative local ring, and* $M,N \in \mathrm{mod}\, R$. *Then the following are equivalent*:

(1) $\mathrm{length}(M \otimes_R N) < \infty$.
(2) $T^R(M,N) < \infty$.
(3) $E_R(M,N) < \infty$.
(4) $E_R(N,M) < \infty$.

We will define a class of rings such that $e_R(M,N) < \infty$ if and only if $e_R(N,M) < \infty$.

Definition 3.3 (Huneke and Jorgensen [10, Definition 3.0]). A noetherian commutative local ring R is called an AB ring if

$$\sup\{e_R(M,N) \mid M,N \in \mathrm{mod}\, R \text{ such that } e_R(M,N) < \infty\} < \infty.$$

Every complete intersection ring is an AB ring by Huneke and Jorgensen [10, Corollary 3.4], but we do not know if every Gorenstein ring is an AB ring.

Corollary 3.4. *Let* $(R,\mathfrak{m},k)$ *be a noetherian commutative local ring, and* $M,N \in \mathrm{mod}\, R$.

(1) *If R is a Gorenstein AB ring, then* $\xi(M,N)$ *is well defined if and only if* $\xi(N,M)$ *is well defined.*
(2) *If R is a complete intersection ring, then* $\chi(M,N)$ *is well defined if and only if* $\xi(M,N)$ *is well defined if and only if* $\xi(N,M)$ *is well defined.*

Proof. (1) By Huneke and Jorgensen [10, Theorem 4.1], $e_R(M,N) < \infty$ if and only if $e_R(N,M) < \infty$ over a Gorenstein AB ring R. The result follows from Corollary 3.2.

(2) By [2], $t^R(M,N) < \infty$ if and only if $e_R(M,N) < \infty$ if and only if $e_R(N,M) < \infty$ over a complete intersection ring R. The result follows from Corollary 3.2. $\Box$

We do not know if $t^R(M,N) < \infty$, $e_R(M,N) < \infty$, and $e_R(N,M) < \infty$ are all equivalent over a Gorenstein ring R. However, we know that the following is true:

Proposition 3.5. *Let* $(R,\mathfrak{m},k,D_R)$ *be a noetherian commutative local ring, and* $X,Y \in \mathscr{D}^b_{fg}(R)$. *Then*

$$\sup(\mathrm{RHom}_R(X,Y^\dagger)) \leqslant -\inf(X \otimes_R^L Y) + \mathrm{id}(D_R), \text{ and}$$

$$\inf(X \otimes_R^L Y) \geqslant \inf D_R - \sup \mathrm{RHom}_R(X,Y^\dagger).$$

In particular, the following are equivalent:

(1) $X \otimes_R^L Y \in \mathscr{D}^b(R)$.
(2) $\mathrm{RHom}(X, Y^\dagger) \in \mathscr{D}^b(R)$.
(3) $\mathrm{RHom}(Y, X^\dagger) \in \mathscr{D}^b(R)$.

Proof. Since

$$\mathrm{RHom}_R(X, Y^\dagger) = \mathrm{RHom}_R(X, \mathrm{RHom}_R(Y, D_R)) \cong \mathrm{RHom}_R(X \otimes_R^L Y, D_R)$$

in $\mathscr{D}_{fg}(R)$,

$$\sup \mathrm{RHom}_R(X, Y^\dagger) = \sup \mathrm{RHom}_R(X \otimes_R^L Y, D_R) \leqslant -\inf(X \otimes_R^L Y) + \mathrm{id}(D_R).$$

Since $\mathrm{id}(D_R) < \infty$,

$$X \otimes_R^L Y \cong X \otimes_R^L (Y^\dagger)^\dagger = X \otimes_R^L \mathrm{RHom}_R(Y^\dagger, D_R) \cong \mathrm{RHom}_R(\mathrm{RHom}_R(X, Y^\dagger), D_R)$$

in $\mathscr{D}_{fg}(R)$. It follows that

$$\begin{aligned}\inf(X \otimes_R^L Y) &= \inf \mathrm{RHom}_R(\mathrm{RHom}_R(X, Y^\dagger), D_R)\\ &\geqslant \inf D_R - \sup \mathrm{RHom}_R(X, Y^\dagger). \qquad \square\end{aligned}$$

Corollary 3.6. *Let* $(R, \mathfrak{m}, k, D_R)$ *be a noetherian commutative local ring, and* $X, Y \in \mathscr{D}^b_{fg}(R)$. *Then the following are equivalent*:

(1) $X \otimes_R^L Y \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$.
(2) $\chi(X, Y)$ *is well defined.*
(3) $\xi(X, Y^\dagger)$ *is well defined.*
(4) $\xi(Y, X^\dagger)$ *is well defined.*

Remark 3.7. Let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring, and $X \in \mathscr{D}^b_{fg}(R)$. By Proposition 3.5,

$$\mathrm{id}(X) = \sup \mathrm{RHom}_R(k, X) \leqslant -\inf(k \otimes_R^L X^\dagger) + \mathrm{id}(D_R) = \mathrm{pd}(X^\dagger) + \mathrm{id}(D_R)$$

and

$$\mathrm{pd}(X) = -\inf(k \otimes_R^L X) \leqslant \sup \mathrm{RHom}_R(k, X^\dagger) - \inf D_R = \mathrm{id}(X^\dagger) - \inf D_R.$$

In particular, $\mathrm{id}(X^\dagger) < \infty$ if and only if $\mathrm{pd}(X) < \infty$.

Remark 3.8. Let R be a Gorenstein ring so that $R \in \mathscr{D}^b_{fg}(R)$ is a dualizing complex, and $M, N \in \mathrm{mod}\, R$. If N is maximal Cohen–Macaulay, then $N^* := \mathrm{Hom}_R(N, R) = N^\dagger$. Applying Proposition 3.5, we have

$$t^R(M, N) \leqslant e_R(M, N^*) \leqslant t^R(M, N) + \mathrm{id}(R).$$

In particular, $t^R(M, N) < \infty$ if and only if $e_R(M, N^*) < \infty$, recapturing [10, Theorem 2.1].

4. Vanishing conjecture for Gorenstein rings

In this section, we will study E-Vanishing for a Gorenstein ring. In particular, we will show that, over a Gorenstein ring, T-Vanishing, E-Vanishing and the E-E Formula are all equivalent.

In this section, let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring as in the previous section, but let D_R always be the normalized dualizing complex for R.

Lemma 4.1. *Let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring. If $X \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, then $X^\dagger \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$ and*

$$\chi(X^\dagger) = \chi(X).$$

Proof. If M is an R-module of length $M = m < \infty$, then there is a finite filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

such that $M_i/M_{i-1} \cong k$ for $i = 1, \ldots, m$, so

$$\begin{aligned}\xi(M, D_R) &= \sum_{i=1}^{m} \xi(M_i/M_{i-1}, D_R) = m\xi(k, D_R)\\ &= m\chi(\mathrm{RHom}_R(k, D_R)) = m\chi(k^\dagger)\\ &= m\chi(k) = \mathrm{length}\, M.\end{aligned}$$

For $X \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, there is a convergent spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_R(h^{-q}(X), D_R) \Rightarrow \mathrm{Ext}^{p+q}_R(X, D_R).$$

Since $X \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$ and $\mathrm{id}(D_R) < \infty$, the spectral sequence $\{E_2^{pq}\}$ is bounded and $\mathrm{length}\, E_2^{pq} < \infty$ for all $p, q \in \mathbb{Z}$, so

$$\begin{aligned}\chi(X^\dagger) &= \chi(\mathrm{RHom}_R(X, D_R))\\ &= \sum_i (-1)^i \,\mathrm{length}\, \mathrm{Ext}^i_R(X, D_R)\\ &= \sum_{p,q} (-1)^{p+q} \,\mathrm{length}\, \mathrm{Ext}^p_R(h^{-q}(X), D_R)\\ &= \sum_q (-1)^q \left(\sum_p (-1)^p \,\mathrm{length}\, \mathrm{Ext}^p_R(h^{-q}(X), D_R)\right)\\ &= \sum_q (-1)^q \xi(h^{-q}(X), D_R)\\ &= \sum_q (-1)^q \,\mathrm{length}\, h^{-q}(X)\\ &= \chi(X). \qquad \square\end{aligned}$$

Remark 4.2. We define a functor $*: \mathscr{D}(R) \to \mathscr{D}(R)$ by

$$X^* = \mathrm{RHom}_R(X, R).$$

If R is a Gorenstein ring of dimension d, then $D_R \cong R[d]$ in $\mathscr{D}(R)$, so

$$X^* = \mathrm{RHom}_R(X, R) \cong \mathrm{RHom}_R(X, D_R[-d]) \cong \mathrm{RHom}_R(X, D_R)[-d] \cong X^\dagger[-d]$$

in $\mathscr{D}(R)$. Applying Lemma 4.1 to a Gorenstein ring R and $X \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, we have

$$\chi(X^*) = \chi(X^\dagger[-d]) = (-1)^d \chi(X)$$

recapturing the result of Szpiro [21]. (Note that we do not need to assume that $\mathrm{pd}(X) < \infty$ for this result.)

Lemma 4.3. *Let* $(R, \mathfrak{m}, k, D_R)$ *be a noetherian commutative local ring, and* $X, Y \in \mathscr{D}^b_{fg}(R)$.

(1) *If* $X \otimes^L_R Y \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, *then*

$$\chi(X, Y) = \xi(X, Y^\dagger) = \xi(Y, X^\dagger).$$

(2) *If* $\mathrm{RHom}_R(X, Y) \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, *and* $\mathrm{pd}(X) < \infty$ *or* $\mathrm{pd}(Y) < \infty$, *then* $\chi(X^*, Y)$ *is well defined and*

$$\chi(X^*, Y) = \xi(X, Y).$$

(3) *If* $X \otimes^L_R Y \in \mathscr{D}^{b,\mathfrak{m}}_{fg}(R)$, *and* $\mathrm{pd}(X) < \infty$ *or* $\mathrm{id}(Y) < \infty$, *then* $\xi(X^*, Y)$ *is well defined and*

$$\xi(X^*, Y) = \chi(X, Y).$$

Proof. (1) Since we have an isomorphism

$$(X \otimes^L_R Y)^\dagger = \mathrm{RHom}_R(X \otimes^L_R Y, D_R) \cong \mathrm{RHom}(X, \mathrm{RHom}_R(Y, D_R)) = \mathrm{RHom}_R(X, Y^\dagger)$$

in $\mathscr{D}(R)$,

$$\begin{aligned}\chi(X, Y) &= \chi(X \otimes^L_R Y) = \chi((X \otimes^L_R Y)^\dagger)\\ &= \chi(\mathrm{RHom}_R(X, Y^\dagger)) = \xi(X, Y^\dagger)\end{aligned}$$

by Lemma 4.1. By symmetry,

$$\chi(X, Y) = \chi(Y, X) = \xi(Y, X^\dagger).$$

(2) If $\mathrm{pd}(X) < \infty$ or $\mathrm{pd}(Y) < \infty$, then we have an isomorphism

$$X^* \otimes^L_R Y = \mathrm{RHom}_R(X, R) \otimes^L_R Y \cong \mathrm{RHom}_R(X, R \otimes^L_R Y) = \mathrm{RHom}_R(X, Y)$$

in $\mathscr{D}(R)$, so

$$\chi(X^*, Y) = \chi(X^* \otimes_R^L Y) = \chi(\mathrm{RHom}_R(X, Y)) = \xi(X, Y).$$

(3) If $\mathrm{pd}(X) < \infty$ or $\mathrm{id}(Y) < \infty$, then we have an isomorphism

$$\mathrm{RHom}_R(X^*, Y) = \mathrm{RHom}_R(\mathrm{RHom}_R(X, R), Y) \cong X \otimes_R^L \mathrm{RHom}_R(R, Y) = X \otimes_R^L Y$$

in $\mathscr{D}(R)$, so

$$\xi(X^*, Y) = \chi(\mathrm{RHom}_R(X^*, Y)) = \chi(X \otimes_R^L Y) = \chi(X, Y). \qquad \square$$

In Section 1, we defined several statements such as T-Vanishing, E-Vanishing, the E-T Formula, and the E-E Formula. We can define similar statements for complexes replacing $M, N \in \mathrm{mod}\, R$ by $X, Y \in \mathscr{D}^b_{fg}(R)$ (and replacing the condition length$(M \otimes_R N) < \infty$ by $\mathrm{Supp}(X \otimes_R^L Y) \subseteq \{\mathfrak{m}\}$). For example, we say that R satisfies T-Vanishing for complexes if $\chi(X, Y) = 0$ for every pair $X, Y \in \mathscr{D}^b_{fg}(R)$ having Property Vt for complexes:

(1) $\mathrm{Supp}(X \otimes_R^L Y) \subseteq \{\mathfrak{m}\}$,
(2) $\mathrm{pd}(X), \mathrm{pd}(Y) < \infty$, and
(3) $\dim X + \dim Y < \dim R$.

Of course, a statement for complexes implies the corresponding statement for modules, but the converse is not clear because $\mathrm{pd}(X) < \infty$ does not usually imply $\mathrm{pd}(h^i(X)) < \infty$ for all i.

Theorem 4.4. *Let $(R, \mathfrak{m}, k, D_R)$ be a noetherian commutative local ring. Then R satisfies T-Vanishing for complexes if and only if R satisfies E-Vanishing for complexes.*

Proof. Let $X, Y \in \mathscr{D}^b_{fg}(R)$. Since

(1) $\mathrm{Supp}(X \otimes_R^L Y^\dagger) = \mathrm{Supp}\, X \cap \mathrm{Supp}\, Y^\dagger = \mathrm{Supp}\, X \cap \mathrm{Supp}\, Y = \mathrm{Supp}(X \otimes_R^L Y)$,
(2) $\mathrm{id}(Y^\dagger) < \infty$ if and only if $\mathrm{pd}(Y) < \infty$ by Remark 3.7, and
(3) $\dim Y^\dagger = \dim \mathrm{Supp}\, Y^\dagger = \dim \mathrm{Supp}\, Y = \dim Y$,

it follows that the pair $X, Y \in \mathscr{D}^b_{fg}(R)$ has Property Vt for complexes if and only if the pair $X, Y^\dagger \in \mathscr{D}^b_{fg}(R)$ has Property Ve for complexes. Since $(Y^\dagger)^\dagger \cong Y$ in $\mathscr{D}(R)$, the result follows from Lemma 4.3 (1). $\square$

Corollary 4.5. *Let $(R, \mathfrak{m}, k)$ be a noetherian commutative Gorenstein local ring. Then R satisfies T-Vanishing if and only if R satisfies E-Vanishing.*

Proof. Let $\dim R = d$ and $M, N \in \mathrm{mod}\, R$ having Property V. (Recall that Property Vt and Property Ve are equivalent over a Gorenstein ring.) If $N \cong R/(y_1, \ldots, y_{d-n})$

for an R-regular sequence $\{y_1,\dots,y_{d-n}\}$, then $\chi(M,N)=0$ by Lichtenbaum [13, Lemma 1], so

$$\xi(M,N)=\sum_{i=0}^{d}(-1)^i\chi(\mathrm{Ext}_R^i(M,R),N)=0$$

by Chan [5, Proposition 2]. By [6, Claim 2.6], we may assume that N is Cohen–Macaulay to show either statement. If $N\in \mathrm{mod}\,R$ is a Cohen–Macaulay module of dimension n, then $\mathrm{Ext}_R^i(N,R)=0$ for all $i\neq d-n$ and $\mathrm{Ext}_R^{d-n}(N,R)\in\mathrm{mod}\,R$ is again a Cohen–Macaulay module of dimension n by [4, Theorem 3.3.10]. Since $D_R\cong R[d]\in\mathscr{D}_{fg}^b(R)$ is the normalized dualizing complex, $N^\dagger\cong \mathrm{Ext}_R^{d-n}(N,R)[n]$ in $\mathscr{D}(R)$, hence the proof of Theorem 4.4 applies by replacing $X,Y,Y^\dagger\in\mathscr{D}_{fg}^b(R)$ by $M,N,\mathrm{Ext}_R^{d-n}(N,R)\in\mathrm{mod}\,R$. □

Over a Gorenstein ring, we have more formulas for χ and ξ.

Lemma 4.6. *Let $(R,\mathfrak{m},k)$ be a noetherian commutative Gorenstein local ring of dimension d, and $X,Y\in\mathscr{D}_{fg}^b(R)$ such that $\mathrm{Supp}(X\otimes_R^L Y)\subseteq\{\mathfrak{m}\}$, and $\mathrm{pd}(X)<\infty$ or $\mathrm{pd}(Y)<\infty$. Then*

(1) $\chi(X,Y)=(-1)^d\chi(X^*,Y^*)$.
(2) $\xi(X,Y)=(-1)^d\xi(Y,X)=(-1)^d\xi(X^*,Y^*)=\xi(Y^*,X^*)$.

Proof. Note that under the assumptions on X,Y, all χ and ξ above are well defined. Since $X^*\cong X^\dagger[-d]$ in $\mathscr{D}(R)$, results follow from easy computations using Lemma 4.3. □

Lemma 4.7. *Let $(R,\mathfrak{m},k)$ be a noetherian commutative Cohen–Macaulay local ring and $X\in\mathscr{D}_{fg}^b(R)$. For any integer $0\leqslant r\leqslant \mathrm{codim}\,X$, there is $K\in\mathrm{mod}\,R$ such that $\dim K=r$, $\mathrm{pd}(K)<\infty$, and $\mathrm{Supp}(K\otimes_R^L X)\subseteq\{\mathfrak{m}\}$.*

Proof. Recall that $\mathrm{Supp}\,X=\mathrm{Spec}(R/\mathrm{Ann}\,X)$ by Lemma 2.2. Let $d=\dim R$ and $m=\dim X=\dim(R/\mathrm{Ann}\,X)$. Applying [6, Claim 2.5] to $M=R/\mathrm{Ann}\,X$ and $N=R/\mathfrak{m}=k$, we can choose a system of parameters $\{x_1,\dots,x_m\}$ for M contained in $\mathrm{Ann}\,N=\mathfrak{m}$ such that $\{x_1,\dots,x_m\}$ is an R-regular sequence. Since R is Cohen–Macaulay, we can extend it to an R-regular sequence $\{x_1,\dots,x_m,x_{m+1},\dots,x_{d-r}\}$ for any integer $0\leqslant r\leqslant d-m$. Define

$$K=R/(x_1,\dots,x_m,x_{m+1},\dots,x_{d-r})\in\mathrm{mod}\,R.$$

Clearly, $\dim K=r$, and $\mathrm{pd}(K)<\infty$. Since

$$\begin{aligned}\dim\mathrm{Supp}(K\otimes_R^L X)&=\dim(\mathrm{Supp}\,K\cap\mathrm{Supp}\,X)\\&\leqslant\dim(\mathrm{Supp}(R/(x_1,\dots,x_m))\cap\mathrm{Supp}\,M)\\&=\dim\mathrm{Supp}(R/(x_1,\dots,x_m)\otimes_R M)\\&=\dim\mathrm{Supp}(M/(x_1,\dots,x_m)M)\end{aligned}$$

$$= \dim(M/(x_1,\dots,x_m)M)$$
$$= 0$$

it follows that $\operatorname{Supp}(K \otimes^L_R X) \subseteq \{\mathfrak{m}\}$. □

Theorem 4.8. *Let $(R,\mathfrak{m},k)$ be a noetherian commutative Gorenstein local ring. Then R satisfies E-Vanishing for complexes if and only if R satisfies the E-E Formula for complexes.*

Proof. Let $X, Y \in \mathscr{D}^b_{fg}(R)$ having Property F for complexes.

Since R is a Gorenstein ring,

$$\xi(X,Y) = (-1)^{\dim R}\xi(Y,X)$$

by Lemma 4.6(2). If R satisfies E-Vanishing for complexes, then

$$X \cdot Y = \begin{cases} (-1)^{\operatorname{codim} X}\xi(X,Y) = (-1)^{\operatorname{codim} Y}\xi(Y,X) & \text{if } \dim X + \dim Y = \dim R \\ 0 & \text{if } \dim X + \dim Y < \dim R \end{cases} = Y \cdot X,$$

so R satisfies the E-E Formula for complexes.

Conversely, suppose that R satisfies the E-E Formula for complexes. If $\dim X + \dim Y < \dim R$, then there is $K \in \operatorname{mod} R$ such that $\dim K = \dim X + 1 \leqslant \operatorname{codim} Y$, $\operatorname{pd}(K) < \infty$, and $\operatorname{Supp}(K \otimes^L_R Y) \subseteq \{\mathfrak{m}\}$ by Lemma 4.7. Since $\dim K + \dim Y \leqslant \dim R$,

$$\begin{aligned}\xi(X \oplus K, Y) &= \xi(X,Y) + \xi(K,Y) \\ &= (-1)^{\operatorname{codim} X} X \cdot Y + (-1)^{\operatorname{codim} K} K \cdot Y \\ &= (-1)^{\operatorname{codim} X} Y \cdot X + (-1)^{\operatorname{codim} K} Y \cdot K \\ &= (-1)^{\operatorname{codim} X + \operatorname{codim} Y}\xi(Y,X) + (-1)^{\operatorname{codim} K + \operatorname{codim} Y}\xi(Y,K).\end{aligned}$$

On the other hand, since

(1) $\operatorname{Supp}((X \oplus K) \otimes^L_R Y) = \operatorname{Supp}(X \otimes^L_R Y) \cup \operatorname{Supp}(K \otimes^L_R Y) \subseteq \{\mathfrak{m}\}$,
(2) $\operatorname{pd}(X \oplus K) < \infty$, and
(3) $\dim(X \oplus K) + \dim Y = \dim K + \dim Y \leqslant \dim R$,

the pair $X \oplus K, Y \in \mathscr{D}^b_{fg}(R)$ has Property F for complexes. It follows that

$$\begin{aligned}\xi(X \oplus K, Y) &= (-1)^{\operatorname{codim}(X \oplus K)}(X \oplus K) \cdot Y \\ &= (-1)^{\operatorname{codim} K} Y \cdot (X \oplus K) \\ &= (-1)^{\operatorname{codim} K + \operatorname{codim} Y}\xi(Y, X \oplus K) \\ &= (-1)^{\operatorname{codim} X + \operatorname{codim} Y + 1}\xi(Y,X) + (-1)^{\operatorname{codim} K + \operatorname{codim} Y}\xi(Y,K),\end{aligned}$$

so

$$X \cdot Y = Y \cdot X = (-1)^{\operatorname{codim} Y} \xi(Y, X) = 0,$$

hence R satisfies E-Vanishing for complexes. □

Corollary 4.9. *Let $(R, \mathfrak{m}, k)$ be a noetherian commutative Gorenstein local ring. Then R satisfies E-Vanishing if and only if R satisfies the E-E Formula.*

Proof. Replace $X, Y \in \mathscr{D}^b_{fg}(R)$ by $M, N \in \operatorname{mod} R$ in the above proof. □

In summary, over a Gorenstein ring, we have the following implications:

$$\begin{array}{ccc} \text{T-Vanishing} & \Leftarrow & \text{the E-T Formula} \\ \Updownarrow & & \Downarrow \\ \text{E-Vanishing} & \Leftrightarrow & \text{the E-E Formula} \end{array}$$

Question 4.10. Over a noetherian commutative Gorenstein local ring, are the above four conditions equivalent?

5. Noncommutative connected algebras

Let X be a noetherian scheme over a field k. We define the Euler form of coherent $\mathscr{O}_X$-modules $\mathscr{M}$ and $\mathscr{N}$ by

$$\xi(\mathscr{M}, \mathscr{N}) := \sum_{i=0}^{\infty} (-1)^i \dim_k \operatorname{Ext}^i_X(\mathscr{M}, \mathscr{N})$$

and the Euler characteristic of $\mathscr{M}$ by

$$\chi(\mathscr{M}) := \xi(\mathscr{O}_X, \mathscr{M}) = \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{H}^i(X, \mathscr{M}).$$

The following lemma is a fun exercise to prove:

Lemma 5.1. *Let X be a noetherian projective scheme of finite type over an algebraically closed field k. If $\mathscr{M}$ and $\mathscr{N}$ are coherent $\mathscr{O}_X$-modules such that* $\operatorname{Supp} \mathscr{M} \cap \operatorname{Supp} \mathscr{N}$ *consists of finitely many closed points of X, and $\mathscr{M}$ has a locally free resolution of finite length, then*

$$\xi(\mathscr{M}, \mathscr{N}) = \sum_{x \in X:\ \text{closed}} \xi(\mathscr{M}_x, \mathscr{N}_x),$$

where $\xi(\mathscr{M}_x, \mathscr{N}_x)$ is calculated over the local ring $\mathscr{O}_{X,x}$ for each $x \in X$.

Let X be a noetherian projective scheme of finite type over an algebraically closed field k, and $\mathscr{M}$ be a coherent $\mathscr{O}_X$-module. For a closed point $x \in X$, $\dim \mathscr{M}_x \leqslant \dim \mathscr{M}$, so if $\mathcal{O}_{X,x}$ satisfies E-Vanishing for every closed point $x \in X$, then X satisfies an appropriate version of E-Vanishing by the above Lemma.

In the second half of the paper, we will prove some versions of E-Vanishing for a large class of noncommutative projective schemes, typically including all commutative projective schemes over a field k. Since localization behaves rather badly for noncommutative schemes, our approach is not using local analysis.

In the rest of the paper, let $(A, \mathfrak{m}, k)$ be a connected algebra (not necessarily commutative).

The category of graded left A-modules and graded left A-module homomorphisms of degree 0 is denoted by $\mathrm{GrMod}\, A$. For $M, N \in \mathrm{GrMod}\, A$, the set of graded left A-module homomorphisms $M \to N$ of degree 0 is denoted by $\mathrm{Hom}_A(M, N)$, which has a natural k-vector space structure. The full subcategory of $\mathrm{GrMod}\, A$ consisting of finitely generated graded left A-modules is denoted $\mathrm{grmod}\, A$. The category of graded right A-modules is denoted by $\mathrm{GrMod}\, A^o$, where A^o is the opposite algebra of A. The category of graded A-A bimodules is denoted by $\mathrm{GrMod}\, A^e$, where $A^e = A \otimes A^o$. We write $k = A/\mathfrak{m}$ viewed as an object in $\mathrm{GrMod}\, A, \mathrm{GrMod}\, A^o$, or $\mathrm{GrMod}\, A^e$, depending on the context.

A graded k-vector space V is right bounded (resp. left bounded) if $V_i = 0$ for all $i \gg 0$ (resp. $i \ll 0$), and bounded if it is both right bounded and left bounded. We say that V is locally finite if $\dim_k V_i < \infty$ for all i. For each integer n, the shift of V is denoted by $V(n)$, so that $V(n)_i = V_{n+i}$.

For $M, N \in \mathrm{GrMod}\, A$, we define

$$\underline{\mathrm{Ext}}^i_A(M, N) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Ext}^i_A(M, N(n)),$$

which has a natural graded k-vector space structure for each i. For $M \in \mathrm{GrMod}\, A$ (resp. $M \in \mathrm{GrMod}\, A^o$), the Matlis dual of M is defined by $M' = \underline{\mathrm{Hom}}_k(M, k)$, which has a natural graded right (resp. left) A-module structure. If M is locally finite, then $M'' \cong M$ in $\mathrm{GrMod}\, A$ (resp. $\mathrm{GrMod}\, A^o$).

We denote $\mathscr{D}(A) = \mathscr{D}(\mathrm{GrMod}\, A)$ for the derived category of the category of graded left A-modules, and $\mathscr{D}_{fg}(A)$ (resp. $\mathscr{D}_{lf}(A)$) for the full subcategory of $\mathscr{D}(A)$ consisting of complexes whose cohomologies are all finitely generated (resp. locally finite) graded left A-modules.

For $V \in \mathrm{GrMod}\, k$ locally finite, we define the Hilbert series of V by

$$H_V(t) = \sum_{i=-\infty}^{\infty} (\dim_k V_i) t^i \in \mathbb{Z}[[t, t^{-1}]].$$

More generally, for $X \in \mathscr{D}^b_{lf}(A)$, we define the Hilbert series of X by

$$H_X(t) = \sum_{i=-\infty}^{\infty} (-1)^i H_{h^i(X)}(t) \in \mathbb{Z}[[t, t^{-1}]].$$

If $H_X(t)$ is a rational function over $\mathbb{C}$, for which we simply write $H_X(t) \in \mathbb{C}(t)$, then we define $\mathrm{GKdim}\, X$ to be the order of the pole of $H_X(t)$ at $t = 1$, and the multiplicity of X by

$$e(X) = \lim_{t \to 1} (1 - t)^{\mathrm{GKdim}\, X} H_X(t).$$

Remark 5.2. If $(A, \mathfrak{m}, k)$ is a left noetherian connected algebra and $M \in \operatorname{grmod} A$, then M is locally finite, and $H_M(t) \in \mathbb{Z}[[t]][t^{-1}]$. Moreover, if $H_M(t) \in \mathbb{C}(t)$, then $\operatorname{GKdim} M$ agrees with the standard definition of GKdimension of M (if it is finite). In particular, if A is commutative, then $H_M(t) \in \mathbb{C}(t)$ for any $M \in \operatorname{grmod} A$, and $\operatorname{GKdim} M = \dim M$ (the Krull dimension of M). On the other hand, for $X \in \mathscr{D}^b_{fg}(A)$, $\operatorname{GKdim} X$ and $\dim X$ (the Krull dimension) may not agree. For example, if

$$X : 0 \to A \xrightarrow{0} A \to 0$$

is a complex of graded A-modules, then $H_X(t) = H_A(t) - H_A(t) = 0$, so $\operatorname{GKdim} X = 0$ (by convention). However, for any $\mathfrak{p} \in \operatorname{Spec} A$,

$$X_{\mathfrak{p}} : 0 \to A_{\mathfrak{p}} \xrightarrow{0} A_{\mathfrak{p}} \to 0$$

is not zero in $\mathscr{D}(A_{\mathfrak{p}})$, so $\dim X = \dim \operatorname{Supp} X = \dim \operatorname{Spec} A = \dim A$.

If $X, Y \in \mathscr{D}^b_{lf}(A)$, then $X(n), X[n], X \oplus Y \in \mathscr{D}^b_{lf}(A)$, and

$$H_{X(n)}(t) = t^{-n} H_X(t),$$

$$H_{X[n]}(t) = (-1)^n H_X(t),$$

$$H_{X \oplus Y}(t) = H_X(t) + H_Y(t)$$

for all $n \in \mathbb{Z}$. It follows that if $H_X(t) \in \mathbb{C}(t)$, then

$$\operatorname{GKdim}(X(n)) = \operatorname{GKdim}(X[n]) = \operatorname{GKdim} X$$

and

$$e(X(n)) = (-1)^n e(X[n]) = e(X)$$

for all $n \in \mathbb{Z}$.

Let $(A, \mathfrak{m}, k)$ be a locally finite connected algebra. If F is a finitely generated free complex of finite length, that is,

$$F^i = \bigoplus_{j=1}^{q_i} A(-\ell_{ij})$$

for all $i \in \mathbb{Z}$, then we define the characteristic polynomial of F by

$$Q_F(t) := \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{q_i} t^{\ell_{ij}} \in \mathbb{Z}[t, t^{-1}],$$

so that

$$H_F(t) = \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{q_i} H_{A(-\ell_{ij})}(t) = \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{q_i} t^{\ell_{ij}} H_A(t) = Q_F(t) H_A(t).$$

Clearly $Q_{F(n)}(t) = t^{-n} Q_F(t)$ for all $n \in \mathbb{Z}$.

Let $(A, \mathfrak{m}, k)$ be a connected algebra. Since the Matlis dual $(-)' : \operatorname{GrMod} A \to \operatorname{GrMod} A^o$ is an exact functor, it extends to a functor $(-)' : \mathscr{D}(A) \to \mathscr{D}(A^o)$.

If $X \in \mathscr{D}^b_{lf}(A)$, then

- $X' \in \mathscr{D}^b_{lf}(A^o)$,
- $(X')' \cong X$ in $\mathscr{D}(A)$, and
- $H_{X'}(t) = H_X(t^{-1})$.

The local cohomology functor $\underline{\Gamma}_{\mathfrak{m}} : \operatorname{GrMod} A \to \operatorname{GrMod} A$, defined by

$$\underline{\Gamma}_{\mathfrak{m}}(-) = \lim_{n\to\infty} \underline{\operatorname{Hom}}_A(A/A_{\geqslant n}, -)$$

has the right derived functor $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}} : \mathscr{D}^+(A) \to \mathscr{D}^+(A)$, whose cohomologies are denoted by

$$\underline{\mathrm{H}}^i_{\mathfrak{m}}(-) = h^i(\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(-)).$$

Definition 5.3 (Jörgensen and Zhang [12]; Mori [14]). Let $(A, \mathfrak{m}, k)$ be a connected algebra and $X \in \mathscr{D}^b_{lf}(A)$. We say that X is rational, if

- $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X) \in \mathscr{D}^b_{lf}(A)$,
- $H_X(t), H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X)}(t) \in \mathbb{C}(t)$, and
- $H_X(t)$ and $H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(X)}(t)$ are equal as rational functions over $\mathbb{C}$.

The above notion of rationality was first introduced in [12] to extend Stanley's Theorem to a noncommutative connected algebra. It was also used to compute intersection multiplicity over a noncommutative connected algebra in [14,15]. A large class of noetherian connected algebras A including all noetherian commutative connected algebras has the property that every $X \in \mathscr{D}^b_{fg}(A)$ is rational by [12, Proposition 5.5].

Remark 5.4. Let $(A, \mathfrak{m}, k)$ be a connected algebra and $X \in \mathscr{D}^b_{lf}(A)$ such that $H_X(t)$ can be represented by a rational function $f(t) \in \mathbb{C}(t)$, which we denote $H_X(t) \sim f(t)$. In general, $f(t)$ does not uniquely determine $H_X(t)$ as a series. However, if we know that $H_X(t) \in \mathbb{Z}[[t]][t^{-1}]$ or $H_X(t) \in \mathbb{Z}[[t^{-1}]][t]$, then $f(t)$ uniquely determines $H_X(t)$ as a series. For example, suppose that $H_X(t) \sim 1/(1-t)^{d+1} \in \mathbb{C}(t)$ for some $d \in \mathbb{N}$. If we know that $H_X(t) \in \mathbb{Z}[[t]][t^{-1}]$, then

$$\frac{1}{(1-t)^{d+1}} \sim \sum_{i=0}^{\infty} \binom{d+i}{d} t^i = H_X(t).$$

On the other hand, if we know that $H_X(t) \in \mathbb{Z}[[t^{-1}]][t]$, then

$$\frac{1}{(1-t)^{d+1}} = \frac{(-t)^{-d-1}}{(1-t^{-1})^{d+1}} \sim (-1)^{d+1} \sum_{i=0}^{\infty} \binom{d+i}{d} t^{-d-1-i} = H_X(t).$$

6. Bézout's Theorem for noncommutative projective schemes

We first recall the definition of a noncommutative projective scheme [1]. Let $(A, \mathfrak{m}, k)$ be a left noetherian connected algebra. We denote $\operatorname{Tors} A$ to be the full subcategory of $\mathrm{GrMod}\,A$ consisting of direct limits of right bounded modules and $\mathrm{Tails}\,A = \mathrm{GrMod}\,A/\mathrm{Tors}\,A$ to be the quotient category. We write $\pi : \mathrm{GrMod}\,A \to \mathrm{Tails}\,A$ for the quotient functor and $\mathscr{M} = \pi M \in \mathrm{Tails}\,A$ for the image of $M \in \mathrm{GrMod}\,A$. We call the pair $(\mathrm{Tails}\,A, \mathscr{A})$ the noncommutative projective scheme associated to A. By Serre, if A is a noetherian commutative graded algebra generated in degree 1 over k, then $\mathrm{Tails}\,A$ is equivalent to the category of quasi-coherent sheaves over $\mathrm{Proj}\,A$ and $\mathscr{A}$ corresponds to the structure sheaf of $\mathrm{Proj}\,A$ under this equivalence. We denote $\mathrm{tors}\,A$ and $\mathrm{tails}\,A$ to be the full subcategories of $\mathrm{Tors}\,A$ and $\mathrm{Tails}\,A$ consisting of noetherian objects so that $\mathrm{tails}\,A = \mathrm{grmod}\,A/\mathrm{tors}\,A$. Note that, for $M, N \in \mathrm{grmod}\,A$, if $\mathscr{M} \cong \mathscr{N}$ in $\mathrm{tails}\,A$, then $M_{\geqslant n} \cong N_{\geqslant n}$ for $n \gg 0$, so

$$H_M(t) - H_N(t) \in \mathbb{Z}[t, t^{-1}].$$

The following χ condition, which is automatic for a noetherian commutative connected algebra, is essential in noncommutative projective geometry [1].

Definition 6.1. Let $(A, \mathfrak{m}, k)$ be a left noetherian connected algebra. We say that A satisfies χ if

$$\dim_k \underline{\mathrm{Ext}}^i_A(k, M) < \infty$$

for all $i \geqslant 0$ and all $M \in \mathrm{grmod}\,A$.

Let $(A, \mathfrak{m}, k)$ be a left noetherian connected algebra. For $\mathscr{M}, \mathscr{N} \in \mathrm{Tails}\,A$, the set of morphisms $\mathscr{M} \to \mathscr{N}$ in $\mathrm{Tails}\,A$ is denoted by $\mathrm{Hom}_{\mathscr{A}}(\mathscr{M}, \mathscr{N})$, which has a natural k-vector space structure. For each integer n, the shift functor $(n) : \mathrm{GrMod}\,A \to \mathrm{GrMod}\,A$ induces an autoequivalence $(n) : \mathrm{Tails}\,A \to \mathrm{Tails}\,A$. We define

$$\underline{\mathrm{Ext}}^i_{\mathscr{A}}(\mathscr{M}, \mathscr{N}) = \bigoplus_{n=-\infty}^{\infty} \mathrm{Ext}^i_{\mathscr{A}}(\mathscr{M}, \mathscr{N}(n)),$$

which has a natural graded k-vector space structure for each i. In particular, we write

$$\mathrm{H}^i(\mathscr{M}) = \mathrm{Ext}^i_{\mathscr{A}}(\mathscr{A}, \mathscr{M}) \quad \text{and} \quad \underline{\mathrm{H}}^i(\mathscr{M}) = \underline{\mathrm{Ext}}^i_{\mathscr{A}}(\mathscr{A}, \mathscr{M}).$$

We denote $\mathscr{D}(\mathscr{A}) = \mathscr{D}(\mathrm{Tails}\,A)$ for the derived category of the category $\mathrm{Tails}\,A$, and $\mathscr{D}_{fg}(\mathscr{A})$ for the full subcategory of $\mathscr{D}(\mathscr{A})$ consisting of complexes whose cohomologies are all in $\mathrm{tails}\,A$. Since the quotient functor $\pi : \mathrm{GrMod}\,A \to \mathrm{Tails}\,A$ is exact, it extends to a functor $\pi : \mathscr{D}(A) \to \mathscr{D}(\mathscr{A})$. As before, we write $\mathscr{X} = \pi X \in \mathscr{D}(\mathscr{A})$ for the image of $X \in \mathscr{D}(A)$.

The right derived functor of

$$\mathrm{Hom}_{\mathscr{A}}(-,-) : \mathscr{D}^-(\mathscr{A}) \times \mathscr{D}^+(\mathscr{A}) \to \mathscr{D}(k)$$

is denoted by $\mathrm{RHom}_{\mathscr{A}}(-,-)$, and its cohomologies are denoted by

$$\mathrm{Ext}^i_{\mathscr{A}}(-,-) = h^i(\mathrm{RHom}_{\mathscr{A}}(-,-)).$$

Let $X \in \mathcal{D}^b_{fg}(A)$. If $H_X(t) \in \mathbb{C}(t)$, then we define the dimension of $\mathcal{X} \in \mathcal{D}^b_{fg}(\mathcal{A})$ by

$$\dim \mathcal{X} = \operatorname{GKdim} X - 1$$

and the degree of $\mathcal{X} \in \mathcal{D}^b_{fg}(\mathcal{A})$ by

$$\deg \mathcal{X} = \begin{cases} e(X) & \text{if } \operatorname{GKdim} X \geqslant 1, \\ 0 & \text{if } \operatorname{GKdim} X = 0. \end{cases}$$

If $Y \in \mathcal{D}^b_{fg}(A)$ such that $\mathcal{X} \cong \mathcal{Y}$ in $\mathcal{D}^b_{fg}(\mathcal{A})$, then

$$\pi h^i(X) \cong h^i(\pi X) \cong h^i(\pi Y) \cong \pi h^i(Y)$$

in $\operatorname{tails} A$ for all i, so

$$H_X(t) - H_Y(t) = \sum_{i:\,\text{finite}} (-1)^i (H_{h^i(X)}(t) - H_{h^i(Y)}(t)) \in \mathbb{Z}[t, t^{-1}],$$

hence $\dim \mathcal{X}$ and $\deg \mathcal{X}$ are well defined.

For $\mathcal{X}, \mathcal{Y} \in \mathcal{D}^b_{fg}(\mathcal{A})$, we define

- the Euler form of $\mathcal{X}$ and $\mathcal{Y}$ by

$$\xi(\mathcal{X}, \mathcal{Y}) := \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \operatorname{Ext}^i_{\mathcal{A}}(\mathcal{X}, \mathcal{Y}),$$

- the Euler characteristic of $\mathcal{X}$ by

$$\chi(\mathcal{X}) := \xi(\mathcal{A}, \mathcal{X}) = \sum_{i=-\infty}^{\infty} (-1)^i \dim_k \mathrm{H}^i(\mathcal{X}),$$

  and
- the intersection multiplicity of $\mathcal{X}$ and $\mathcal{Y}$ by

$$\mathcal{X} \cdot \mathcal{Y} := (-1)^{\operatorname{codim} \mathcal{X}} \xi(\mathcal{X}, \mathcal{Y}) = (-1)^{\dim \mathcal{A} - \dim \mathcal{X}} \xi(\mathcal{X}, \mathcal{Y}).$$

Lemma 6.2. *If $(A, \mathfrak{m}, k)$ is a left noetherian rational connected algebra satisfying χ, then*

$$H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) \in \mathbb{Z}[[t]][t^{-1}].$$

(In particular, $H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) \sim H_A(t^{-1})$ but $H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) \neq H_A(t^{-1})$ as series if A is infinite dimensional over k.) If $H_A(t) = \sum_{i=0}^{\infty} a_n t^n$ and $H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) = \sum_{i=-\infty}^{\infty} k_n t^n$, then $\chi(\mathcal{A}(n))$ is well defined and

$$\chi(\mathcal{A}(n)) = a_n - k_{-n}$$

for all $n \in \mathbb{Z}$.

Proof. If A is a left noetherian connected algebra satisfying χ, then $\underline{\mathrm{H}}^i_{\mathfrak{m}}(A)$ are locally finite and right bounded for all $i \geqslant 0$ by Artin and Zhang [1, Proposition 3.1(3), Proposition 3.5(1), Corollary 3.6(3)], so

$$H_{\underline{\mathrm{H}}^i_{\mathfrak{m}}(A)}(t) \in \mathbb{Z}[[t^{-1}]][t].$$

If A is rational, then $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A) \in \mathscr{D}^b_{lf}(A)$, so

$$H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)}(t) = \sum_{i:\ \text{finite}} (-1)^i H_{\underline{\mathrm{H}}^i_{\mathfrak{m}}(A)}(t) \in \mathbb{Z}[[t^{-1}]][t].$$

Since we have an exact sequence

$$0 \to \underline{\mathrm{H}}^0_{\mathfrak{m}}(A) \to A \to \underline{\mathrm{H}}^0(\mathscr{A}) \to \underline{\mathrm{H}}^1_{\mathfrak{m}}(A) \to 0$$

and isomorphisms

$$\underline{\mathrm{H}}^i(\mathscr{A}) \cong \underline{\mathrm{H}}^{i+1}_{\mathfrak{m}}(A)$$

for all $i \geqslant 1$ by [1, Proposition 7.2(2)],

$$\begin{aligned}
\chi(\mathscr{A}(n)) &= \sum_{i=0}^{\infty} (-1)^i \dim_k \mathrm{H}^i(\mathscr{A}(n)) \\
&= \sum_{i=0}^{\infty} (-1)^i \dim_k \underline{\mathrm{H}}^i(\mathscr{A})_n \\
&= \dim_k A_n - \sum_{i=0}^{\infty} (-1)^i \dim_k \underline{\mathrm{H}}^i_{\mathfrak{m}}(A)_n \\
&= \dim_k A_n - \sum_{i=0}^{\infty} (-1)^i \dim_k \underline{\mathrm{H}}^i_{\mathfrak{m}}(A)'_{-n} \\
&= a_n - k_{-n}
\end{aligned}$$

for all $n \in \mathbb{Z}$. □

Remark 6.3. Let A be a noetherian connected algebra. If A has the balanced dualizing complex $D_A \in \mathscr{D}^b(A^e)$ in the sense of [22], then $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)' \cong D_A$ in $\mathscr{D}(A^e)$ by Yekutieli [22, Corollary 4.21]. Since $h^i(D_A)$ is finitely generated as a graded left and right A-module for every $i \in \mathbb{Z}$, it follows easily that

$$H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) = H_{D_A}(t) = \sum_{i:\ \text{finite}} (-1)^i H_{h^i(D_A)}(t) \in \mathbb{Z}[[t]][t^{-1}]$$

in this case.

For any integers $d \geqslant 0$ and n, we denote

$$\binom{n}{d} = \begin{cases} 1 & \text{if } d = 0, \\ \displaystyle\prod_{i=1}^{d} \frac{n-d+i}{i} & \text{if } d \geqslant 1. \end{cases}$$

We need the following technical Lemma.

Lemma 6.4. *For any integers $d \geqslant 0$, n and m,*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i+m}{d} = \begin{cases} (-1)^d & if\ n = d, \\ 0 & if\ n > d. \end{cases}$$

Proof. We will prove the formula by induction on $d \geqslant 0$. If $d = 0$, then

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i+m}{0} = \sum_{i=0}^{n}(-1)^i\binom{n}{i} = \begin{cases} 1=(-1)^0 & \text{if } n=0,\\ (1-1)^n = 0 & \text{if } n>0 \end{cases}$$

for any m.

For $d \geqslant 0$,

$$\begin{aligned}
&\sum_{i=0}^{n+1}(-1)^i\binom{n+1}{i}\binom{i+m}{d+1}\\
&\quad=\binom{n+1}{0}\binom{m}{d+1}+\sum_{i=1}^{n+1}(-1)^i\left[\binom{n}{i-1}+\binom{n}{i}\right]\binom{i+m}{d+1}\\
&\quad=\binom{m}{d+1}+\sum_{i=1}^{n+1}(-1)^i\binom{n}{i-1}\binom{i+m}{d+1}+\sum_{i=1}^{n+1}(-1)^i\binom{n}{i}\binom{i+m}{d+1}\\
&\quad=\sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}\binom{i+m+1}{d+1}+\binom{n}{0}\binom{m}{d+1}\\
&\qquad+\sum_{i=1}^{n}(-1)^i\binom{n}{i}\binom{i+m}{d+1}\\
&\quad=-\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i+m+1}{d+1}+\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i+m}{d+1}\\
&\quad=-\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left[\binom{i+m+1}{d+1}-\binom{i+m}{d+1}\right]\\
&\quad=-\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{i+m}{d}\\
&\quad=\begin{cases} -(-1)^d=(-1)^{d+1} & \text{if } n=d \text{ (i.e. } n+1=d+1),\\ 0 & \text{if } n>d \text{ (i.e. } n+1>d+1) \end{cases}
\end{aligned}$$

for any m by induction. □

Remark 6.5. If $m \geqslant d$, then one can prove the above formula by looking at the homogeneous pieces of a graded Koszul complex over the polynomial algebra $k[x_1,\dots,x_{d+1}]$ or the exterior algebra $\bigwedge(x_1,\dots,x_n)$. I thank the referee for his pointing this out.

Lemma 6.6. *Let* $(A, \mathfrak{m}, k)$ *be a left noetherian rational connected algebra satisfying* χ *such that* $H_A(t) \sim p(t)/(1-t)^{d+1}$ *for some* $p(t) \in \mathbb{Z}[t]$ *and some* $d \in \mathbb{N}$. *Then*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} \chi(\mathscr{A}(i+m)) = \begin{cases} (-1)^d p(1) & if\ n = d, \\ 0 & if\ n > d \end{cases}$$

for any $m \in \mathbb{Z}$.

Proof. We may write $p(t) = \sum_{j=0}^{r} p_j t^j \in \mathbb{Z}[t]$ such that $p_r \neq 0$ (and $p_0 = 1$). Let $H_A(t) = \sum_{i=0}^{\infty} a_i t^i$ and $H_{\mathrm{R}\Gamma_{\mathfrak{m}}(A)'}(t) = \sum_{i=-\infty}^{\infty} k_i t^i$. Since

$$\sum_{i=0}^{\infty} a_i t^i \sim p(t)/(1-t)^{d+1}$$

and $H_A(t) \in \mathbb{Z}[[t]][t^{-1}]$, it follows that

$$\sum_{i=0}^{\infty} a_i t^i = \sum_{j=0}^{r} p_j t^j \cdot \sum_{i=0}^{d} \binom{d+i}{d} t^i \in \mathbb{Z}[[t]][t^{-1}]$$

(see Remark 5.4), so $a_i = \sum_{j=0}^{r} b_{i,j}$ where

$$b_{i,j} = \begin{cases} 0 & \text{if } i < j, \\ p_j \binom{d+i-j}{d} & \text{if } i \geqslant j \end{cases}$$

for all $i \in \mathbb{Z}$ (by the convention $a_i = 0$ for $i < 0$). Since A is rational,

$$\sum_{i=-\infty}^{\infty} k_i t^i \sim H_A(t^{-1}) \sim p(t^{-1})/(1-t^{-1})^{d+1} = (-1)^{d+1} t^{d+1} p(t^{-1})/(1-t)^{d+1}.$$

Since $H_{\mathrm{R}\Gamma_{\mathfrak{m}}(A)'}(t) \in \mathbb{Z}[[t]][t^{-1}]$ by Lemma 6.2,

$$\sum_{i=-\infty}^{\infty} k_i t^i = (-1)^{d+1} t^{d+1} \sum_{j=0}^{r} p_j t^{-j} \cdot \sum_{i=0}^{\infty} \binom{d+i}{d} t^i,$$

so $k_i = (-1)^{d+1} \sum_{j=0}^{r} \ell_{i,j}$ where

$$\ell_{i,j} = \begin{cases} 0 & \text{if } i < d+1-j, \\ p_j \binom{d+i-(d+1-j)}{d} = p_j \binom{i+j-1}{d} & \text{if } i \geqslant d+1-j \end{cases}$$

for all $i \in \mathbb{Z}$. Since

$$\ell_{-i,j} = \begin{cases} p_j \binom{-i+j-1}{d} = (-1)^d p_j \binom{d+i-j}{d} & \text{if } i \leqslant -d-1+j, \\ 0 & \text{if } i > -d-1+j. \end{cases}$$

it follows that

$$b_{i,j}+(-1)^d\ell_{-i,j}=\begin{cases} p_j\binom{d+i-j}{d} & \text{if } i\leqslant -d-1+j,\\ 0 & \text{if } -d-1+j<i<j,\\ p_j\binom{d+i-j}{d} & \text{if } i\geqslant j\end{cases}$$

$$=p_j\binom{d+i-j}{d}$$

for all $i\in\mathbb{Z}$ and $0\leqslant j\leqslant r$. By Lemma 6.2,

$$\chi(\mathscr{A}(i))=a_i-k_{-i}=\sum_{j=0}^{r}b_{i,j}-(-1)^{d+1}\sum_{j=0}^{r}\ell_{-i,j}$$

$$=\sum_{j=0}^{r}\{b_{i,j}+(-1)^d\ell_{-i,j}\}=\sum_{j=0}^{r}p_j\binom{d+i-j}{d}$$

for all $i\in\mathbb{Z}$. By Lemma 6.4

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\chi(\mathscr{A}(i+m))=\sum_{i=0}^{n}(-1)^i\binom{n}{i}\left[\sum_{j=0}^{r}p_j\binom{d+i+m-j}{d}\right]$$

$$=\sum_{j=0}^{r}p_j\left[\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{d+i+m-j}{d}\right]$$

$$=\begin{cases}(-1)^d\sum_{j=0}^{r}p_j=(-1)^d p(1) & \text{if } n=d,\\ 0 & \text{if } n>d\end{cases}$$

for any $m\in\mathbb{Z}$. □

Let $(A,\mathfrak{m},k)$ be a left noetherian rational connected algebra satisfying χ. By Lemma 6.2, $\chi(\mathscr{A}(i))$ is well defined for every $i\in\mathbb{Z}$, so we define a map

$$\xi_{\mathscr{A}}(-,-):\mathbb{Z}[T,T^{-1}]\times\mathbb{Z}[T,T^{-1}]\to\mathbb{Z}$$

by

$$\xi_{\mathscr{A}}(T^i,T^j):=\xi(\mathscr{A}(-i),\mathscr{A}(-j))=\xi(\mathscr{A},\mathscr{A}(i-j))=\chi(\mathscr{A}(i-j))$$

for $i,j\in\mathbb{Z}$. Note that, for $f(T),g(T)\in\mathbb{Z}[T,T^{-1}]$,

$$\xi_{\mathscr{A}}(f(T),g(T))=\xi_{\mathscr{A}}(1,f(T^{-1})g(T)).$$

Lemma 6.7. *Let* $(A, \mathfrak{m}, k)$ *be a left noetherian rational connected algebra satisfying* χ. *If* F, G *are finitely generated free resolutions of* $M, N \in \operatorname{grmod} A$ *of finite length, then*

$$\xi(\mathscr{M}, \mathscr{N}) = \xi(\mathscr{F}, \mathscr{G}) = \xi_{\mathscr{A}}(Q_F(T), Q_G(T)).$$

Proof. Suppose that

$$\cdots \to 0 \to F^{-m} \to F^{-m+1} \to \cdots \to F^{-1} \to F^0 \to 0 \to \cdots,$$

$$\cdots \to 0 \to G^{-n} \to G^{-n+1} \to \cdots \to G^{-1} \to G^0 \to 0 \to \cdots$$

are finitely generated free resolutions of $M, N \in \operatorname{grmod} A$ of finite length. Let M^i, N^i be the ith syzygies of M, N so that $\mathscr{M}^0 = \mathscr{M}, \mathscr{N}^0 = \mathscr{N}$ and there are exact sequences

$$0 \to \mathscr{M}^{-i-1} \to \mathscr{F}^{-i} \to \mathscr{M}^{-i} \to 0,$$

$$0 \to \mathscr{N}^{-i-1} \to \mathscr{G}^{-i} \to \mathscr{M}^{-i} \to 0$$

in $\operatorname{tails} A$ for all $i \geqslant 0$. Since $\xi(\mathscr{F}^i, \mathscr{G}^j)$ are well defined for all $i, j \in \mathbb{Z}$ by Lemma 6.2, $\xi(\mathscr{F}^i, \mathscr{N}^{-n+1})$ are well defined and

$$\xi(\mathscr{F}^i, \mathscr{N}^{-n+1}) = \xi(\mathscr{F}^i, \mathscr{G}^{-n+1}) - \xi(\mathscr{F}^i, \mathscr{G}^{-n})$$

for all $i \in \mathbb{Z}$. Inductively, $\xi(\mathscr{F}^i, \mathscr{G})$ are well defined and

$$\xi(\mathscr{F}^i, \mathscr{G}) = \xi(\mathscr{F}^i, \mathscr{N}) = \xi(\mathscr{F}^i, \mathscr{N}^0) = \sum_j (-1)^j \xi(\mathscr{F}^i, \mathscr{G}^j)$$

for all $i \in \mathbb{Z}$. It follows that $\xi(\mathscr{M}^{-m+1}, \mathscr{G})$ is well defined and

$$\xi(\mathscr{M}^{-m+1}, \mathscr{G}) = \xi(\mathscr{F}^{-m+1}, \mathscr{G}) - \xi(\mathscr{F}^{-m}, \mathscr{G}).$$

Inductively, $\xi(\mathscr{F}, \mathscr{G})$ is well defined and

$$\xi(\mathscr{F}, \mathscr{G}) = \xi(\mathscr{M}, \mathscr{G}) = \xi(\mathscr{M}^0, \mathscr{G}) = \sum_i (-1)^i \xi(\mathscr{F}^i, \mathscr{G}) = \sum_{i,j} (-1)^{i+j} \xi(\mathscr{F}^i, \mathscr{G}^j).$$

If

$$F^i = \bigoplus_s A(-\ell_{is}), \qquad G^j = \bigoplus_t A(-\ell_{jt}),$$

then

$$\begin{aligned}\xi(\mathscr{M}, \mathscr{N}) = \xi(\mathscr{F}, \mathscr{G}) &= \sum_{i,j} (-1)^{i+j} \xi(\mathscr{F}^i, \mathscr{G}^j) \\ &= \sum_{i,j} (-1)^{i+j} \sum_{s,t} \xi(\mathscr{A}(-\ell_{is}), \mathscr{A}(-\ell_{jt}))\end{aligned}$$

$$= \sum_{i,s} (-1)^i \sum_{j,t} (-1)^j \xi_{\mathscr{A}}(T^{\ell_{is}}, T^{\ell_{jt}})$$

$$= \xi_{\mathscr{A}} \left(\sum_{i,s} (-1)^i T^{\ell_{i,s}}, \sum_{j,t} (-1)^j T^{\ell_{jt}} \right)$$

$$= \xi_{\mathscr{A}}(Q_F(T), Q_G(T)). \qquad \square$$

Theorem 6.8 (Bézout's Theorem for noncommutative projective schemes). *Let* $(A, \mathfrak{m}, k)$ *be a left noetherian rational connected algebra of* GKdim $A = d + 1$ *statisfying* χ *such that* $H_A(t) \sim p(t)/(1-t)^{d+1}$ *for some* $p(t) \in \mathbb{Z}[t]$. *If* $M, N \in \operatorname{grmod} A$ *such that* $\operatorname{pd}(M),\ \operatorname{pd}(N) < \infty$, *then*

$$\mathscr{M} \cdot \mathscr{N} = \mathscr{N} \cdot \mathscr{M} = \begin{cases} 0 & if\ \dim \mathscr{M} + \dim \mathscr{N} < \dim \mathscr{A}, \\ \deg \mathscr{M} \deg \mathscr{N} / \deg \mathscr{A} & if\ \dim \mathscr{M} + \dim \mathscr{N} = \dim \mathscr{A}. \end{cases}$$

Proof. If GKdim $M = 0$ or GK dim $N = 0$, then $\mathscr{M} \cong 0$ or $\mathscr{N} \cong 0$ in tails A. In this case,

$$\dim \mathscr{M} + \dim \mathscr{N} < \dim \mathscr{A} = d$$

and

$$\mathscr{M} \cdot \mathscr{N} = \mathscr{N} \cdot \mathscr{M} = 0 = \deg \mathscr{M} \deg \mathscr{N} / \deg \mathscr{A},$$

so assume that GKdim M, GKdim $N \geqslant 1$.

Since $\operatorname{pd}(M), \operatorname{pd}(N) < \infty, M, N$ have finitely generated free resolutions F, G of finite length. Since the dimension and the degree are preserved by shifting, and

$$\xi(\mathscr{F}(-i), \mathscr{G}(-i)) = \xi(\mathscr{F}, \mathscr{G})$$

for all $i \in \mathbb{Z}$, we may assume that $Q_F(t), Q_G(t) \in \mathbb{Z}[t]$ by shifting both F, G by the same enough high degree. In particular, we may write

$$Q_F(t) = m_p(1-t)^p + m_{p+1}(1-t)^{p+1} + \cdots,$$

$$Q_G(t) = n_q(1-t)^q + n_{q+1}(1-t)^{q+1} + \cdots$$

in $\mathbb{Z}[t]$ where $m_p, n_q \neq 0$. Since

$$H_A(t) \sim p(t)/(1-t)^{d+1}, \quad H_M(t) = Q_F(t)H_A(t),\ H_N(t) = Q_G(t)H_A(t),$$

it follows that

$$\dim \mathscr{A} = d, \quad \dim \mathscr{M} = d - p, \quad \dim \mathscr{N} = d - q$$

and

$$\deg \mathscr{A} = p(1), \quad \deg \mathscr{M} = m_p p(1), \quad \deg \mathscr{N} = n_q p(1).$$

By Lemma 6.6

$$\begin{aligned}\xi_{\mathscr{A}}((1-T)^s,(1-T)^t) &= \xi_{\mathscr{A}}(1,(1-T^{-1})^s(1-T)^t)\\ &= \xi_{\mathscr{A}}(1,(-T)^t(1-T^{-1})^{s+t})\\ &= (-1)^t\xi_{\mathscr{A}}(1,T^t(1-T^{-1})^{s+t})\\ &= (-1)^t\sum_{i=0}^{s+t}(-1)^i\binom{s+t}{i}\xi_{\mathscr{A}}(1,T^{t-i})\\ &= (-1)^t\sum_{i=0}^{s+t}(-1)^i\binom{s+t}{i}\chi(\mathscr{A}(i-t))\\ &= \begin{cases}(-1)^t(-1)^d p(1)=(-1)^{d-t}p(1) & \text{if } s+t=d,\\ 0 & \text{if } s+t>d.\end{cases}\end{aligned}$$

By Lemma 6.7,

$$\begin{aligned}\mathscr{M}\cdot\mathscr{N} &= (-1)^p\xi(\mathscr{F},\mathscr{G})\\ &= (-1)^p\xi_{\mathscr{A}}(Q_F(T),Q_G(T))\\ &= (-1)^p\sum_{s\geqslant p,t\geqslant q} m_s n_t \xi_{\mathscr{A}}((1-T)^s,(1-T)^t)\\ &= \begin{cases}0 & \text{if } p+q>d \text{ (i.e. } \dim\mathscr{M} \\ & \quad +\dim\mathscr{N}<\dim\mathscr{A})\\ m_p n_q p(1)=\deg\mathscr{M}\deg\mathscr{N}/\deg\mathscr{A} & \text{if } p+q=d \text{ (i.e. } \dim\mathscr{M}\\ & \quad +\dim\mathscr{N}=\dim\mathscr{A}).\end{cases}\end{aligned}$$

By symmetry,

$$\mathscr{N}\cdot\mathscr{M}=\begin{cases}0 & \text{if } \dim\mathscr{M}+\dim\mathscr{N}<\dim\mathscr{A},\\ \deg\mathscr{M}\deg\mathscr{N}/\deg\mathscr{A} & \text{if } \dim\mathscr{M}+\dim\mathscr{N}=\dim\mathscr{A}.\end{cases}\qquad\square$$

Remark 6.9. Let $(A,\mathfrak{m},k)$ be a left noetherian rational connected algebra satisfying χ as in Theorem 6.8. If $\operatorname{tails} A$ has finite homological dimension, that is,

$$\sup\{\sup \operatorname{RHom}_{\mathscr{A}}(\mathscr{M},\mathscr{N}) \mid \mathscr{M},\mathscr{N}\in \operatorname{tails} A\}<\infty$$

(for example, if A is commutative and $\operatorname{Proj} A$ is smooth), then $\xi(\mathscr{M},\mathscr{N})$ is well defined for any pair $\mathscr{M},\mathscr{N}\in \operatorname{tails} A$, so we can similarly show that

$$\xi(\mathscr{F},\mathscr{G})=\xi_{\mathscr{A}}(Q_F(T),Q_G(T))$$

for every pair of finitely generated free complexes F,G of finite length without being free resolutions of single modules. In this case, we can replace $M,N\in \operatorname{grmod} A$ by $X,Y\in\mathscr{D}^b_{fg}(A)$ in Theorem 6.8.

Example 6.10. A noetherian connected algebra A is called a quantum polynomial ring if

- $\operatorname{gldim} A = d+1 < \infty$,
- A satisfies χ, and
- $H_A(t) \sim 1/(1-t)^{d+1}$.

If A is a homomorphic image of a quantum polynomial ring, then A satisfies all conditions of Theorem 6.8 (see [12, Proposition 5.5]). In particular, if A itself is a quantum polynomial ring, then $\operatorname{pd}(M) < \infty$ for any $M \in \operatorname{grmod} A$ and $\deg \mathscr{A} = 1$, so we can recapture [16, Theorem 8.6].

Example 6.11. If A is a noetherian commutative graded algebra generated in degree 1, then A is a homomorphic image of a noetherian commutative polynomial ring generated in degree 1, which is a quantum polynomial ring, so Theorem 6.8 always applies to A.

Example 6.12. Let $A = k\langle x,y\rangle/(x^2y - yx^2, xy^2 - y^2x)$. Although A is a noetherian rational connected algebra of $\operatorname{gldim} A = 3$ satisfying χ, Theorem 6.8 does not apply to A because $H_A(t) \sim 1/(1-t^2)(1-t)^2$. In fact, if $M = A/Ax, N = M(-1) \in \operatorname{grmod} A$ so that $\operatorname{pd}(M) = \operatorname{pd}(N) = 1 < \infty$, then

$$\dim \mathscr{A} = 2, \quad \dim \mathscr{M} = \dim \mathscr{N} = 1$$

and

$$\deg \mathscr{A} = \deg \mathscr{M} = \deg \mathscr{N} = \tfrac{1}{2},$$

so

$$\deg \mathscr{M} \deg \mathscr{M}/\deg \mathscr{A} = \deg \mathscr{M} \deg \mathscr{N}/\deg \mathscr{A} = \tfrac{1}{2},$$

while $\mathscr{M} \cdot \mathscr{M} = 0$ and $\mathscr{M} \cdot \mathscr{N} = 1$ by Mori and Smith [16, Proposition 10.1].

7. Cohen–Macaulay projective schemes

In this last section, we will extend the results of the previous section to modules of finite injective (rather than projective) dimension in the Cohen–Macaulay case.

Definition 7.1. Let $(A, \mathfrak{m}, k)$ be a connected algebra and $M \in \operatorname{GrMod} A$. We define the depth of M by

$$\operatorname{depth} M = \inf \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(M) = \inf\{i \mid \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\}$$

and the local dimension of M by

$$\operatorname{ldim} M = \sup \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(M) = \sup\{i \mid \underline{\mathrm{H}}^i_{\mathfrak{m}}(M) \neq 0\}.$$

We say that M is Cohen–Macaulay if depth $M = \operatorname{ldim} M < \infty$, and A is Cohen–Macaulay if it is Cohen–Macaulay as a graded left A-module over itself. A noetherian connected algebra A is called balanced Cohen–Macaulay if both A and A^o are Cohen–Macaulay and satisfy χ.

If A is a balanced Cohen–Macaulay algebra of $\operatorname{ldim} A = d+1$, then A has the balanced dualizing complex $D_A \cong \mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)' \cong \omega_A[d+1]$ in $\mathscr{D}(A^e)$ where $\omega_A \in \operatorname{GrMod} A^e$. We call ω_A the balanced dualizing module for A. We refer to [15] for basic properties of a balanced Cohen–Macaulay algebra.

Lemma 7.2. *Let $(A, \mathfrak{m}, k)$ be a rational balanced Cohen–Macaulay algebra of* $\operatorname{ldim} A = d+1$ *and ω_A be the balanced dualizing module. If $\omega_{\mathscr{A}} = \pi\omega_A \in \operatorname{Tails} A$, then $\chi(\omega_{\mathscr{A}}(n))$, $\xi(\omega_{\mathscr{A}}, \omega_{\mathscr{A}}(n))$ are well defined, and*

(1) $\chi(\omega_{\mathscr{A}}(n)) = (-1)^d \chi(\mathscr{A}(-n))$,
(2) $\xi(\omega_{\mathscr{A}}, \omega_{\mathscr{A}}(n)) = \chi(\mathscr{A}(n))$

for all $n \in \mathbb{Z}$.

Proof. (1) Let $H_A(t) = \sum_{i=0}^{\infty} a_n t^n$ and $H_{\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)'}(t) = \sum_{i=-\infty}^{\infty} k_n t^n$ as in Lemma 6.2. Since $\mathrm{R}\underline{\Gamma}_{\mathfrak{m}}(A)' \cong \omega_A[d+1]$ in $\mathscr{D}(A^e)$, $H_{\omega_A}(t) = (-1)^{d+1} \sum_{i=-\infty}^{\infty} k_n t^n$. By Artin and Zhang [1, Proposition 7.2(2)], we have an exact sequence

$$0 \to \underline{\mathrm{H}}^0_{\mathfrak{m}}(\omega_A) \to \omega_A \to \underline{\mathrm{H}}^0(\omega_{\mathscr{A}}) \to \underline{\mathrm{H}}^1_{\mathfrak{m}}(\omega_A) \to 0$$

and isomorphisms

$$\underline{\mathrm{H}}^i(\omega_{\mathscr{A}}) \cong \underline{\mathrm{H}}^{i+1}_{\mathfrak{m}}(\omega_A)$$

for all $i \geqslant 1$. Since

$$\underline{\mathrm{H}}^{i+1}_{\mathfrak{m}}(\omega_A) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A' & \text{if } i = d \end{cases}$$

by Smith [20, Chapter 11, Lemma 5,6],

$$\begin{aligned} \chi(\omega_{\mathscr{A}}(n)) &= \sum_i (-1)^i \dim_k \mathrm{H}^i(\omega_{\mathscr{A}}(n)) = \sum_i (-1)^i \dim_k \underline{\mathrm{H}}^i(\omega_{\mathscr{A}})_n \\ &= \dim_k(\omega_A)_n + (-1)^d \dim_k A'_n = \dim_k(\omega_A)_n + (-1)^d \dim_k A_{-n} \\ &= (-1)^d(-k_n + a_{-n}) = (-1)^d \chi(\mathscr{A}(-n)) \end{aligned}$$

for all $n \in \mathbb{Z}$ by Lemma 6.2.

(2) By Yekutieli and Zhang [23, Theorem 4.2], the associated noncommutative projective scheme $(\operatorname{Tails} A, \mathscr{A})$ is classical Cohen–Macaulay of cohomological dimension d with the dualizing sheaf $\omega_{\mathscr{A}} = \pi\omega_A \in \operatorname{tails} A$, that is,

$$\operatorname{Ext}^i_{\mathscr{A}}(-, \omega_{\mathscr{A}}) \cong \mathrm{H}^{d-i}(-)'$$

for all $i \in \mathbb{Z}$. By (1),

$$\begin{aligned}\xi(\omega_{\mathscr{A}}, \omega_{\mathscr{A}}(n)) = \xi(\omega_{\mathscr{A}}(-n), \omega_{\mathscr{A}}) &= \sum_i (-1)^i \dim_k \operatorname{Ext}^i_{\mathscr{A}}(\omega_{\mathscr{A}}(-n), \omega_{\mathscr{A}}) \\ &= \sum_i (-1)^i \dim_k \mathrm{H}^{d-i}(\omega_{\mathscr{A}}(-n))' \\ &= (-1)^d \sum_i (-1)^i \dim_k \mathrm{H}^i(\omega_{\mathscr{A}}(-n)) \\ &= (-1)^d \chi(\omega_{\mathscr{A}}(-n)) = \chi(\mathscr{A}(n))\end{aligned}$$

for all $n \in \mathbb{Z}$. □

Let $(A, \mathfrak{m}, k)$ be a balanced Cohen–Macaulay algebra and ω_A be the balanced dualizing module. If J is a finitely generated ω_A-free complex of finite length, that is,

$$J^i = \bigoplus_{j=1}^{r_i} \omega_A(-\ell_{ij}),$$

then we define the ω_A-characteristic polynomial of J by

$$R_J(t) = \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{r_i} t^{\ell_{ij}} \in \mathbb{Z}[t, t^{-1}],$$

so that

$$\begin{aligned}H_J(t) &= \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{r_i} H_{\omega_A}(-\ell_{ij})(t) = \sum_{i=-\infty}^{\infty} (-1)^i \sum_{j=1}^{r_i} t^{\ell_{ij}} H_{\omega_A}(t) \\ &= R_J(t) H_{\omega_A}(t).\end{aligned}$$

Clearly $R_{J(n)}(t) = t^{-n} R_J(t)$ for all $n \in \mathbb{Z}$.

By Lemma 7.2(1), $\chi(\omega_{\mathscr{A}}(i))$ is well defined for every $i \in \mathbb{Z}$, so we define a map

$$\xi_{\mathscr{A}, \omega_{\mathscr{A}}}(-,-) : \mathbb{Z}[T, T^{-1}] \times \mathbb{Z}[T, T^{-1}] \to \mathbb{Z}$$

by

$$\xi_{\mathscr{A}, \omega_{\mathscr{A}}}(T^i, T^j)) := \xi(\mathscr{A}(-i), \omega_{\mathscr{A}}(-j)) = \xi(\mathscr{A}, \omega_{\mathscr{A}}(i-j)) = \chi(\omega_{\mathscr{A}}(i-j))$$

for $i, j \in \mathbb{Z}$. Similarly, by Lemma 7.2(2), $\xi(\omega_{\mathscr{A}}, \omega_{\mathscr{A}}(i))$ is well defined for every $i \in \mathbb{Z}$, so we define a map

$$\xi_{\omega_{\mathscr{A}}}(-,-) : \mathbb{Z}[T, T^{-1}] \times \mathbb{Z}[T, T^{-1}] \to \mathbb{Z}$$

by

$$\xi_{\omega_{\mathscr{A}}}(T^i, T^j) := \xi(\omega_{\mathscr{A}}(-i), \omega_{\mathscr{A}}(-j)) = \xi(\omega_{\mathscr{A}}, \omega_{\mathscr{A}}(i-j)) = \chi(\mathscr{A}(i-j))$$

for $i, j \in \mathbb{Z}$.

Lemma 7.3. *Let* $(A, \mathfrak{m}, k)$ *be a balanced Cohen–Macaulay algebra and* ω_A *be the balanced dualizing module.*

(1) *If* F *is a finitely generated free resolutions of* $M \in \operatorname{grmod} A$ *of finite length and* J *is a finitely generated* ω_A*-free resolutions of* $N \in \operatorname{grmod} A$ *of finite length, then*

$$\xi(\mathcal{M}, \mathcal{N}) = \xi(\mathcal{F}, \mathcal{J}) = \xi_{\mathcal{A}, \omega_{\mathcal{A}}}(Q_F(T), R_J(T)).$$

(2) *If* J, K *are finitely generated* ω_A*-free resolutions of* $M, N \in \operatorname{grmod} A$ *of finite length, then*

$$\xi(\mathcal{M}, \mathcal{N}) = \xi(\mathcal{J}, \mathcal{K}) = \xi_{\omega_{\mathcal{A}}}(R_J(T), R_K(T)).$$

Proof. Similar to Lemma 6.7. □

Theorem 7.4 (Bézout's Theorem for Cohen–Macaulay projective schemes). *Let* $(A, \mathfrak{m}, k)$ *be a rational balanced Cohen–Macaulay algebra of* $\operatorname{GKdim} A = \operatorname{ldim} A = d+1$ *such that* $H_A(t) \sim p(t)/(1-t)^{d+1}$ *for some* $p(t) \in \mathbb{Z}[t]$*. If* $M, N \in \operatorname{grmod} A$ *such that either*

(1) $\operatorname{pd}(M), \operatorname{pd}(N) < \infty$,
(2) $\operatorname{id}(M), \operatorname{id}(N) < \infty$, *or*
(3) $\operatorname{pd}(M), \operatorname{id}(N) < \infty$,

then

$$\mathcal{M} \cdot \mathcal{N} = \begin{cases} 0 & if \ \dim \mathcal{M} + \dim \mathcal{N} < \dim \mathcal{A}, \\ \deg \mathcal{M} \deg \mathcal{N} / \deg \mathcal{A} & if \ \dim \mathcal{M} + \dim \mathcal{N} = \dim \mathcal{A}. \end{cases}$$

Proof. By Mori [15, Theorem 5.8], if $\operatorname{id}(M) < \infty$, then M has a finitely generated ω_A-free resolution J of finite length. Using Lemmas 7.2, 6.6 and 7.3, the proof is similar to that of Theorem 6.8 (but more complicated), and left to the reader. □

Note that if A is a reasonably nice balanced Cohen–Macaulay algebra such as PI, then $\operatorname{GKdim} A = \operatorname{ldim} A$ by Jörgensen [11, Theorem 5.2].

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