Automorphism Groups of Cubic Surfaces

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The automorphism group of a generic cubic surface is trivial [M. Kostabashi, J. Algebra 116, 130–142]. In this article, we determine the automorphism group of each nonsingular cubic surface.

1. INTRODUCTION

In the previous paper [H0], we determined the automorphism groups of nonsingular quartic del Pezzo surfaces. Here we shall consider the automorphism groups of nonsingular cubic surfaces. B. Segre gave the list of nontrivial automorphism groups of cubic surfaces [S, pp. 147–152]. But his list is incomplete and descriptions of some groups are incorrect (Remark 5.4). He treats equations of cubic surfaces and his arguments are geometric. We do not treat any equation and our method is computational.

A cubic surface has 27 lines on it [H a, Chap. V, Sect. 4]. The configuration of the lines is investigated by many authors (cf. [H e, M, S]). There is a set of six mutually skew lines and there is a morphism to the projective plane which contracts the six lines. An ordered set of six mutually skew lines is called a marking of the cubic surface. The six contracted points are in a general position (i.e., not on a conic and no three collinear). Thus marked cubic surfaces are parameterized by an open set $U$ of the product $\mathbb{P}^2 \times \mathbb{P}^2$ (see Sect. 2). Fix a model of the configuration of 27 lines on a generic cubic surface and consider the automorphism group $G$ of the configuration. It is well known that $G$ is isomorphic to the Weyl group $W(E_6)$ of type $E_6$ [M, Chap. IV]. We shall introduce a natural action of $G$ on the moduli space $U$ (see Sect. 3). For a point in $U$, the automorphism group of the associated cubic surface is isomorphic to the stabilizer group of the point (Theorem 3.2). The action of $G$ on $U$ has been introduced by
I. Naruki [N, Sect. 1]. But he did not proceed on this line to determine automorphism groups of cubic surfaces, since Segre has already produced the list, I guess.

The Weyl group \( W(E_6) \) is generated by a set of six reflections. We shall give such reflections and express the actions of these reflections (Lemma 4.3). All the isomorphic classes of automorphism groups of cubic surfaces are given in Sect. 5 (Theorem 5.3). The key is to obtain the table in Proposition 5.2. The algorithm is very simple but the calculations by hand are too hard. We should use Mathematica on a UNIX machine. In the final section we describe a sketch of the procedure to get the table.

2. CONFIGURATION OF LINES AND MARKINGS

We shall fix an algebraically closed field of characteristic \( = 0 \) or \( > 5 \) as a ground field. By a cubic surface, we shall mean a nonsingular cubic hypersurface in \( \mathbb{P}^3 \). A surface \( S \) is a cubic surface if and only if \( S \) is obtained from the projective plane \( \mathbb{P}^2 \) by blowing up six points in a general position (i.e., not on a conic and no three collinear). If \( S \) is thus obtained, say \( \pi: \mathbb{P}^2 \rightarrow S \) is the blowing up and \( P_i \) (\( i = 1, \ldots, 6 \)) the base points, the six lines \( E_i = \pi^{-1}(P_i) \) (\( i = 1, \ldots, 6 \)) are mutually skew. An ordered set of six mutually skew lines is called a marking of the cubic surface. We say \((S, M)\) is a marked cubic surface if \( S \) is a cubic surface and \( M \) is a marking of \( S \). Marked cubic surfaces \((S, M)\) and \((S', M')\) are equivalent if there is an isomorphism \( \sigma: S \cong S' \) such that \( \sigma M = M' \).

Every marked cubic surface is obtained as above [Ha, Chap. V, Sect. 4]. For six points \( P_i \) (\( i = 1, \ldots, 6 \)) on \( \mathbb{P}^2 \) in a general position, there is a unique linear automorphism \( f \) of \( \mathbb{P}^2 \) such that \( f(P_1) = (1:0:0) \), \( f(P_2) = (0:1:0) \), \( f(P_3) = (0:0:1) \), and \( f(P_4) = (1:1:1) \). The points \( f(P_i) \) (\( i = 1, \ldots, 6 \)) are also in a general position. Define the open set \( U \) in the product \( \mathbb{P}^2 \times \mathbb{P}^2 \):

\[
U = \left\{ ((x_1:x_2:x_3), (y_1:y_2:y_3)) \left| gh \prod_i x_i y_i \prod_{i<j} (x_i - x_j)(y_j - y_i) \times \prod_{i<j} (x_i y_j - x_j y_i) \neq 0 \right. \right\},
\]

where \( g = x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2) \) and \( h = x_1 y_1(x_2 y_3 - x_3 y_2) + x_2 y_2(x_3 y_1 - x_1 y_3) + x_3 y_3(x_1 y_2 - x_2 y_1) \). Put \( P_1 = (1:0:0), P_2 = (0:1:0), P_3 = (0:0:1), \) and \( P_4 = (1:1:1) \). Then for a pair

\[
P = (P_5, P_6) \in \mathbb{P}^2 \times \mathbb{P}^2,
\]

the six points \( P_i \) (\( i = 1, \ldots, 6 \)) are in a general position if and only if \( P \in U \). For a point \( P = (P_5, P_6) \in U \), let \( \pi_P: \mathbb{P}^2 \rightarrow \mathbb{P}^2 \) be the blowing up
that the marked cubic surfaces \( S \) uniquely determined by the marking \( gM \), and thus of \( k \). Let \( \Gamma \) be a unique bijection \( \Gamma = (E_i, E_j, L_{ij}) \) of the configuration of lines on a generic marked cubic surface as indicated above. That is to say, the elements of \( \Gamma \) are labeled as \( \Gamma = (E_i, E_j, L_{ij}) \) and \( M = (E_1, \ldots, E_6) \) is the marking. Thus there is a unique bijection \( f_\Gamma : \Gamma \to \Gamma_s \) for the marked cubic surface \( (S_p, M_p) \) arising from a point \( P \in U \) such that \( f_\Gamma \) preserves the marking and the relations. Let \( G \) be the automorphism group of the configuration \( \Gamma \). Then \( G \) is isomorphic to the Weyl group \( W(E) \) of type \( E_6 \) [M, Chap. IV]. An automorphism \( g \in G \) is uniquely determined by the marking \( gM = (gE_1, \ldots, gE_6) \) (or equivalently \( g^{-1}M \)). In the following section, we shall define a natural action of \( G \) on \( U \).

### 3. Action of the Weyl Group on the Moduli of Marked Cubic Surfaces

As was seen in the previous section, the open set \( U \) of the product \( \mathbb{P}^2 \times \mathbb{P}^2 \) is a moduli of marked cubic surfaces. Now fix a model of the configuration \( \Gamma = (E_i, E_j, L_{ij}) \) of lines on a generic marked cubic surface and the marking \( M = (E_1, \ldots, E_6) \). For a point \( P = (P_1, P_6) \in U \), there is a unique bijection \( f_\Gamma : \Gamma \to \Gamma_s \) such that \( f_\Gamma M = M_p \) and \( f_\Gamma \) preserves the relations of lines. Let \( G = \text{Aut} \Gamma \) be the automorphism group of the configuration. For an automorphism \( g \in G \), \( f_p(g^{-1}M) \) is another marking of the cubic surface \( S_p \). Therefore, there is a unique point \( g(P) \in U \) such that the marked cubic surfaces \( (S_p, f_p(g^{-1}M)) \) and \( (S_{g(P)}, M_{g(P)}) \) are equivalent. That is to say, there is a unique isomorphism \( \alpha_{g, P} : S_p \cong S_{g(P)} \) such that \( \alpha_{g, P}(f_p(g^{-1}M)) = M_{g(P)} = f_{g(P)}M \). Since these maps preserve the relations, one sees that \( \alpha_{g, P} \circ f_p = f_{g(P)} \circ g \) as maps of \( \Gamma \to \Gamma_{S_{g(P)}} \). Conversely assume that there is an isomorphism \( \alpha : S_p \cong S_Q \) for a point \( Q \in U \) such that \( \alpha \circ f_p = f_Q \circ g \). Then \( g(P) = Q \) and \( g \) gives the equivalence of \( (S_p, f_p(g^{-1}M)) \) and \( (S_Q, M_Q) \).

**Proposition 3.1.** The map \( G \times U \to U, (g, P) \to g(P) \) is an action of \( G \) on \( U \).
Proof. For \(g, h \in G\) and \(P \in U\), since \(\sigma_{g, p} \circ f_p = f_{g(p)} \circ g\) and \(\sigma_{h, g(p)} \circ f_{g(p)} = f_{h(g(p))} \circ h\), \(\sigma_{h, g(p)} \circ \sigma_{g, p} \circ f_p = \sigma_{h, g(p)} \circ f_{g(p)} \circ g = f_{h(g(p))} \circ h \circ g\). Therefore \((hg)(P) = h(g(P))\).

Theorem 3.2 (cf. [N, Proposition 1.1]).

1. For \(P, Q \in U\), \(S_p\) is isomorphic to \(S_Q\) as varieties if and only if there exists a \(g \in G\) such that \(g(P) = Q\).

2. For \(P \in U\), \(Aut S_p \cong G_p\), where \(G_p = \{g \in G \mid g(P) = P\}\).

Proof. (1) If \(g(P) = Q\) for a \(g \in G\), since \((S_p, M_p)\) and \((S_Q, M_Q)\) are equivalent, \(S_p\) is isomorphic to \(S_Q\). Conversely assume that there is an isomorphism \(\sigma : S_p \cong S_Q\). Then \(\sigma M_p\) is another marking of \(S_Q\). Since \(f_Q^{-1} \sigma M_p\) is another marking of \(\Gamma\), there is a unique automorphism \(g \in G\) such that \(f_Q^{-1} \sigma M_p = gM\). Therefore \(g(P) = Q\) as required.

(2) By the bijection \(f_p : \Gamma \to \Gamma_s\), there is a canonical inclusion \(Aut S_p \hookrightarrow G = Aut \Gamma\). (2) follows from (1).

4. GENERATORS OF THE WEYL GROUP

Let \(\Gamma = \{E_i, F_i, L_{ij}\}\) be a model of the configuration of lines of a generic marked cubic surface, and let \(M = (E_1, \ldots, E_6)\) be the marking. Let \(\Sigma_6\) be the symmetric group of degree 6. By the representation of \(\Gamma\), we can view \(\Sigma_6\) as a subgroup of \(G = Aut \Gamma\). Let \(r \in G\) be the automorphism defined by \((rE_1, \ldots, rE_6) = (L_{23}, L_{13}, L_{12}, E_4, E_5, E_6)\).

Lemma 4.1. Put \((i, j, k) = (1, 2, 3)\) and \((l, m, n) = (4, 5, 6)\). Then \(rE_i = L_{jk}\), \(rE_i = E_i\), \(rF_i = F_i\), \(rF_i = L_{mn}\), \(rL_{ij} = E_k\), \(rL_{lm} = F_n\), and \(rL_{ii} = L_{ii}\).

Proof. Since \(r\) preserves the relations, the proof is straightforward.

Proposition 4.2. \(\Sigma_6\) and \(r\) generate \(G\).

Proof. Denote by \(G'\) the subgroup generated by \(\Sigma_6\) and \(r\). For \(g \in G\) and \(\sigma \in \Sigma_6\), put \(E'_i = gE_i\) \((i = 1, \ldots, 6)\). Then \(g \sigma E_i = E'_{\sigma(i)}\). Therefore it suffices to prove that, for any marking \(M' = (E'_1, \ldots, E'_6)\), there is a \(g \in G'\) such that \((E_1', \ldots, E_6') = (gE_1, \ldots, gE_6)\) as sets. There are just 72 sets of 6 mutually skew lines. They are listed in the following:

1. \(\{E_1, E_2, E_3, E_4, E_5, E_6\}\);
2. \(\{F_1, F_2, F_3, F_4, F_5, F_6\}\);
3. \(\{L_{jk}, L_{ik}, L_{ij}, E_1, E_2, E_3\}\);
4. \(\{L_{jk}, L_{ik}, L_{ij}, F_1, F_2, F_3\}\);
5. \(\{E_i, F_i, L_{jk}, L_{ij}, L_{jm}, L_{jn}\}\).
For $g \in G$ and $\sigma \in \Sigma_6$, $\sigma gM$ is of the same type as $gM$ in the above list. When $\sigma$ runs over $\Sigma_6$ the set $(\sigma gM)$ covers the whole of the type of $gM$. Therefore it suffices to show that, for each type in the above list, there is a $g \in G$ such that $gM$ is of the type. For the type (1), there is nothing to say. $rM$ is of type (2) and the relations of the generators (12), (23), (34), (45), and (56), $G$ is generated by these reflections and $r$.

For a point $P = ((x_1 : x_2 : x_3), (y_1 : y_2 : y_3)) \in U$, if $gP = ((X_1 : X_2 : X_3), (Y_1 : Y_2 : Y_3))$ we express $g$ by $((X_1 : X_2 : X_3), (Y_1 : Y_2 : Y_3))$.

**Lemma 4.3.** Under the above notation, the generators (12), (23), (34), (45), (56), and $r$ are expressed as the following:

- $(12) = ((x_2 : x_1 : x_3), (y_2 : y_1 : y_3));$
- $(23) = ((x_1 : x_3 : x_2), (y_1 : y_3 : y_2));$
- $(34) = ((x_3 - x_1 : x_3 - x_2 : x_3), (y_3 - y_1 : y_3 - y_2 : y_3));$
- $(45) = ((x_2 x_3 : x_1 x_3 : x_1 x_2), (y_1 x_2 x_3 : y_2 x_1 x_3 : y_3 x_1 x_2));$
- $(56) = ((y_1 : y_2 : y_3), (x_1 : x_2 : x_3));$
- $r = ((x_2 x_3 : x_1 x_3 : x_1 x_2), (y_2 y_3 : y_1 y_3 : y_1 y_2)).$

**Proof:** The proof is straightforward by definitions.

It is easily seen that the relations of the generators (12), (23), (34), (45), (56), and $r$ are expressed by the Dynkin diagram:

$$\begin{array}{c}
(12) \\
\downarrow \\
(23) \\
\downarrow \\
(34) \\
\downarrow \\
(45) \\
\downarrow \\
(56)
\end{array}$$

That is to say, the reflections (12), (23), (45), and (56) are commutative with $r$ and $(34)r$ is of order 3.
5. AUTOMORPHISM GROUPS

For an automorphism \( g \in G \), put \( U_g = \{ P \in U \mid g(P) = P \} \). For a point \( P \in U \), put \( G_P = \{ g \in G \mid g(P) = P \} \). Then for \( g \in G \) and \( P \in U \), \( g \in G_P \) if and only if \( P \in U_g \). And \( G_{g(P)} = gG_P g^{-1} \). For \( g, h \in G \), \( hU_g = U_{hgh^{-1}} \).

There are 25 conjugacy classes including unity \([C]\). We follow the notation from \([C]\) to denote conjugacy classes of \( G \). For a conjugacy class \( C \) of \( G \), denote by \( \langle C \rangle \) the set of all elements in \( G \) which belong to the class \( C \). With a conjugacy class \( C \) of \( G \), if the invariant locus \( U \neq \emptyset \) for \( g \in (C) \), we can associate the isomorphic class of the stabilizer group \( G_P \) for a general point \( P \in U \). Thus we could obtain all the isomorphic classes of stabilizer groups, in principle. But it is hard to calculate by hand. We use Mathematica on a UNIX machine. In this section, we merely mention the conclusions. We gather data for Mathematica in the final section.

**Theorem 5.1** (cf. \([S, pp. 151, 152]\)). *If a stabilizer group is nontrivial it contains an element of \((4A_1)\). The isomorphic class of the stabilizer group \( G_P \) for a point \( P \in U \) is determined by the order \( n(G_P) = \#G_P \) and the number \( e(G_P) = \#(G_P \cap (4A_1)) \).*

**Proof.** The proof follows from the next proposition.

Denote by \( G_P^* \) the isomorphic class of a stabilizer group \( G_P \) with \( n(G_P) = n \) and \( e(G_P) = e \). We say a conjugacy class \( C \) is effective if

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<th>TABLE 1</th>
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<tr>
<td>2A_1</td>
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<tr>
<td>4A_1</td>
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<tr>
<td>A_2</td>
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<td>2A_2</td>
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<td>3A_2</td>
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<tr>
<td>A_3 + A_1</td>
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<tr>
<td>D_4 (a_1)</td>
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<tr>
<td>A_4</td>
</tr>
<tr>
<td>A_2 + 2A_1</td>
</tr>
<tr>
<td>D_4</td>
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<tr>
<td>A_1 + A_1</td>
</tr>
<tr>
<td>E_6 (a_1)</td>
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<tr>
<td>D_5</td>
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<td>E_6 (a_2)</td>
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<td>E_6</td>
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For an effective conjugacy class $C$, denote by $d(C)$ the dimension of the invariant locus $U_\Sigma$ for $g \in (C)$. For a conjugacy class $C$, denote by $o(C)$ the order of $g \in (C)$.

**Proposition 5.2.** There are 16 effective conjugacy classes including unity, and there are 12 isomorphic classes of stabilizer groups including the trivial one. The associations of effective conjugacy classes and isomorphic classes of stabilizer groups are gathered in Table 1. A numeral after the vertical line indicates the number $(G^g_n \cap (C))$, and a blank means 0.

The proof is postponed until the final section.

Here we summarize the structure of each group. We also give a point $P \in U$ such that $G^g_n \equiv G_p$.

**Theorem 5.3.** The automorphism group of a cubic surface is isomorphic to one of the following:

1. $G^g_3 \equiv (e)$. For a general point $P \in U$, $G^0_1 \equiv G_p$, e.g., $P = ((1:2:5),(1:3:8))$.

2. $G^g_3 \equiv \mathbb{Z}/2\mathbb{Z}$. Let $g \in (4A_1)$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:5:4),(1:3:12))$.

3. $G^g_3 \equiv \mathbb{Z}/4\mathbb{Z}$. Let $g \in (D_4(a_1))$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:1/2:1 + \sqrt{-1}/2),(1:-\sqrt{-1}:2))$.

4. $G^g_3 \equiv \mathbb{Z}/8\mathbb{Z}$. Let $g \in (D_4)$. For any point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g.,

$$P = ((1:1 - \sqrt{-1} - \sqrt{-1}),(1:-\sqrt{-1} - (1 + \sqrt{-1})/\sqrt{2}:

(1 - \sqrt{-1})(1 + 1/\sqrt{2})).$$

5. $G^g_3 \equiv (\mathbb{Z}/2\mathbb{Z})^2$. Let $g \in (2A_1)$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:9:8),(1:-2:-3))$.

6. $G^g_3 \equiv \Sigma_3$. Let $g \in (2A_2)$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:3:5),(1:3/5:3))$.

7. $G^g_3 \equiv \Sigma_3 \times (\mathbb{Z}/2\mathbb{Z})$. Let $g \in (A_5 + A_1)$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:4:3),(1:3:2))$.

8. $G^g_3 \equiv \Sigma_4$. Let $g \in (A_3 + A_2)$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g., $P = ((1:-2/3:2/3),(1:-4:2))$.

9. $G^g_3 \equiv (G^g_3 \cap (3A_2)) \equiv (\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})$, where $C(G^g_3) = (G^g_3 \cap (3A_2)) \equiv \mathbb{Z}/2\mathbb{Z}$ is the center of $G^g_3$. Let $g \in (3A_2) \cup (E_6(a_2))$. For a general point $P \in U_\Sigma$, $G^3_2 \equiv G_p$, e.g.,

$$P = ((1:\sqrt{3}:(3 + \sqrt{3})/2),(1:(3 + \sqrt{3}/2:2))).$$
(10) $G_{108}^9$ has a normal subgroup of index 2 which is isomorphic to $G_{54}^9$. Let $g \in (E_6)$. For any point $P \in U_k$, $G_{108}^9 \cong G_p$, e.g.,

$$P = \left((1:1 + (\sqrt{-1} - \sqrt{3})/2, (1 - \sqrt{-3})/2), \right)$$

(11) $G_{120}^{10} = \Sigma_5$. Let $g \in (A_4)$. For any point $P \in U_k$, $G_{120}^{10} \cong G_p$, e.g.,

$$P = \left((1:(-1 + \sqrt{5})/2:(3 - \sqrt{5})/2), (1:(3 - \sqrt{5})/2:(-1 + \sqrt{5})/2)\right).$$

(12) $G_{48}^{18} = (\mathbb{Z}/3\mathbb{Z}) \times \Sigma_4$. Let $g \in (A_4) \cup (A_2 + 2A_1) \cup (D_2) \cup (E_6(a_1))$. For any point $P \in U_k$, $G_{48}^{18} \cong G_p$, e.g.,

$$P = \left((1:(1 + \sqrt{-3})/2:(3 + \sqrt{-3})/2), \right)$$

Except for the groups $G_{4}, G_{9},$ and $G_{108}$, the group $G_5$ is generated by the set $G_5^c \cap (4A_1)$. Only the groups $G_{4}^1, G_{5}^1,$ and $G_{108}^9$ contain an element of $(D_2a_1)$. The cubic surfaces $S$ with $\text{Aut} S \cong G_4^1, G_5^1,$ or $G_{108}^9$ form a one-dimensional family (cf. [N, Theorem 2]).

Remark 5.4. B. Segre gave the list of nontrivial automorphism groups of cubic surfaces [S, pp. 147–152]. But in his list, the group $G_{8}^1$ is lacking. He also describes the structure of each group. His description of the groups $G_{54}^9$ and $G_{108}$ are misleading. His description of the group $G_{54}^9$ says that it is the direct product of the cyclic group of order 3 and the group of order 18 which is generated by the set $G_{54}^9 \cap (4A_1)$. But, by our computations, the center of $G_{54}^9$ is of order 3 and the set $G_{54}^9 \cap (4A_1)$ generates the whole group. Since $G_{108}^9$ has a unique normal subgroup of index 2 which is isomorphic to $G_{54}^9$, his description of $G_{108}$ is also misleading.

6. DATA FOR THE PROOF OF PROPOSITION 5.2

To get the table of Proposition 5.2, we used Mathematica on a Unix machine. For the convenience of the reader, we disclose all the data and describe a sketch of the procedure to get the table.

6.1. Representatives of Right Cosets Modulo $\Sigma_6$

First of all, we should express each element of $G$ in a unique way.

Set $N = \{0, 1, 2, 3, 4, 5\} \times \{0, 1, 2, 3, 4\} \times \{0, 1, 2, 3\} \times \{0, 1, 2\} \times \{0, 1\}$ and define the map $s: N \rightarrow \Sigma_6$ by

$$s(i, j, k, l, m) = (123456)^i(12345)^j(1234)^k(123)^l(12)^m.$$
It is easily seen that the map $s$ is a bijection. Therefore elements of the group $\Sigma_6$ are uniquely indexed by the set $N$. Now we give a complete set of representatives of the right cosets of $G$ modulo $\Sigma_6$ in the following:

\[
\begin{align*}
    r[1] &= e; \\
    r[2] &= r; \\
    r[3] &= (14)r[2](14); \\
    r[4] &= (15)r[2](15); \\
    r[5] &= (16)r[2](16); \\
    r[6] &= (24)r[2](24); \\
    r[7] &= (25)r[2](25); \\
    r[8] &= (26)r[2](26); \\
    r[9] &= (34)r[2](34); \\
    r[10] &= (35)r[2](35); \\
    r[11] &= (36)r[2](36); \\
    r[12] &= (25)(14)r[2](14)(25); \\
    r[13] &= (26)(14)r[2](14)(26); \\
    r[14] &= (35)(14)r[2](14)(35); \\
    r[15] &= (36)(14)r[2](14)(36); \\
    r[16] &= (26)(15)r[2](15)(26);
\end{align*}
\]
\[ r[17] = (36)(15) r[2](15)(36); \]
\[ r[18] = (35)(24) r[2](24)(35); \]
\[ r[19] = (36)(24) r[2](24)(36); \]
\[ r[20] = (36)(25) r[2](25)(36); \]
\[ r[21] = (36)(25)(14) r[2](14)(25)(36); \]
\[ r[22] = (24) r[2](25)(36) r[2](24); \]
\[ r[23] = (34) r[2](25)(36) r[2](34); \]
\[ r[24] = r[2](25)(36) r[2]; \]
\[ r[25] = (45) r[24](45); \]
\[ r[26] = (46) r[24](46); \]
\[ r[27] = (12)(24) r[24](24)(12); \]
\[ r[28] = (12)(34) r[24](34)(12); \]
\[ r[29] = (12) r[24](12); \]
\[ r[30] = (12)(45) r[24](45)(12); \]
\[ r[31] = (12)(46) r[24](46)(12); \]
\[ r[32] = (13)(34) r[24](34)(13); \]
\[ r[33] = (13)(24) r[24](24)(13); \]
\[ r[34] = (13)r[24](13); \]
\[ r[35] = (13)(45)r[24](45)(13); \]
\[ r[36] = (13)(46)r[24](46)(13); \]
\[ r[37] = (14)r[24](14); \]
\[ r[38] = (14)(24)r[24](24)(14); \]
\[ r[39] = (14)(34)r[24](34)(14); \]
\[ r[40] = (14)(45)r[24](45)(14); \]
\[ r[41] = (14)(46)r[24](46)(14); \]
\[ r[42] = (15)(45)r[24](45)(15); \]
\[ r[43] = (15)(24)r[24](24)(15); \]
\[ r[44] = (15)(34)r[24](34)(15); \]
\[ r[45] = (15)r[24](15); \]
\[ r[46] = (15)(46)r[24](46)(15); \]
\[ r[47] = (16)(46)r[24](46)(16); \]
\[ r[48] = (16)(24)r[24](24)(16); \]
\[ r[49] = (16)(34)r[24](34)(16); \]
\[ r[50] = (16)r[24](16); \]
\[ r[51] = (16)(45)r[24](45)(16); \]

\[ r[52] = r[2](14)(25)(36)r[2]; \]

\[ r[53] = (14)r[52](14); \]

\[ r[54] = (15)r[52](15); \]

\[ r[55] = (16)r[52](16); \]

\[ r[56] = (24)r[52](24); \]

\[ r[57] = (25)r[52](25); \]

\[ r[58] = (26)r[52](26); \]

\[ r[59] = (34)r[52](34); \]

\[ r[60] = (35)r[52](35); \]

\[ r[61] = (36)r[52](36); \]

\[ r[62] = (25)(14)r[52](14)(25); \]

\[ r[63] = (26)(14)r[52](14)(26); \]

\[ r[64] = (35)(14)r[52](14)(35); \]

\[ r[65] = (36)(14)r[52](14)(35); \]

\[ r[66] = (26)(15)r[52](15)(26); \]
From now on, we represent an element of $G$ by $s(\alpha)r[t]$ ($\alpha \in \mathbb{N}$, $1 \leq t \leq 72$). By Lemma 4.3, for every $g = s(\alpha)r[t] \in G$, we have an expression $g = ((X_1 : X_2 : X_3), (Y_1 : Y_2 : Y_3))$ so that we can solve the invariant locus $U_g$. For our purpose, it is we only need the invariant locus of a $g \in (C)$ for each conjugacy class $C$.

6.2. Conjugacy Classes $A_1$ and $4A_1$

Among the 25 conjugacy classes, the most important classes are $A_1$ and $4A_1$. By virtue of $[C]$, we can express an element of $(C)$ for every conjugacy class $C$ as a product of elements of $(A_1)$.

The set $(A_1)$ consists of 36 elements. They are 15 reflections in $\Sigma_6$ and $r[2], \ldots, r[21]$, and $r[72]$. Among the relations in $(A_1)$, we need the following ones expressed as Dynkin diagrams:

```
       r[72]  \\
       |  \\
       |  \\
(12) — (23) — (34) — (45) — (56)  \\
       |  \\
(12) — (23) — (34) — (45)  \\
(23) — (34) — (45).
```
The set \((4 \mathcal{A}_1)\) consists of 45 elements, which are listed in the following:

\[
\begin{align*}
& s(4, 0, 2, 0, 0) \in [22], \ s(2, 2, 1, 0, 0) \in [23], \ s(2, 1, 1, 1, 1) \in [24], \ s(2, 3, 3, 1, 1) \in [25], \\
& s(0, 3, 0, 2, 0) \in [26], \ s(4, 0, 2, 0, 0) \in [27], \ s(1, 3, 1, 0, 0) \in [28], \ s(1, 2, 1, 1, 0) \in [29], \\
& s(1, 4, 3, 1, 0) \in [30], \ s(0, 3, 3, 0, 1) \in [31], \ s(2, 2, 1, 0, 1) \in [32], \ s(4, 3, 0, 2, 0) \in [33], \\
& s(2, 0, 2, 0, 0) \in [34], \ s(2, 3, 2, 1, 0) \in [35], \ s(0, 1, 1, 1, 1) \in [36], \ s(2, 1, 1, 1, 1) \in [37], \\
& s(1, 2, 1, 1, 0) \in [38], \ s(2, 0, 2, 0, 0) \in [39], \ s(2, 3, 0, 2, 0) \in [40], \ s(0, 3, 1, 1, 1) \in [41], \\
& s(2, 3, 3, 1, 1) \in [42], \ s(4, 1, 1, 1, 1) \in [43], \ s(2, 3, 2, 1, 0) \in [44], \ s(2, 3, 0, 2, 0) \in [45], \\
& s(0, 2, 0, 0, 0) \in [46], \ s(0, 3, 2, 0, 0) \in [47], \ s(0, 3, 3, 0, 1) \in [48], \ s(0, 4, 1, 0, 1) \in [49], \\
& s(0, 3, 1, 1, 1) \in [50], \ s(0, 0, 3, 1, 1) \in [51], \ s(1, 1, 1, 1, 1) \in [72], \ s(1, 2, 0, 2, 0) \in [72], \\
& s(1, 3, 1, 2, 1) \in [72], \ s(2, 0, 1, 1, 0) \in [72], \ s(2, 1, 0, 2, 1) \in [72], \ s(2, 2, 1, 2, 0) \in [72], \\
& s(3, 0, 0, 0, 0) \in [72], \ s(3, 1, 1, 1, 1) \in [72], \ s(3, 4, 1, 0, 1) \in [72], \ s(4, 0, 2, 0, 1) \in [72], \\
& s(4, 3, 0, 1, 1) \in [72], \ s(4, 4, 3, 1, 0) \in [72], \ s(5, 1, 1, 1, 1) \in [72], \ s(5, 1, 2, 0, 0) \in [72], \\
& s(5, 1, 3, 1, 1) \in [72].
\end{align*}
\]

6.3. **Non–Effective Conjugacy Classes**

Here we exclude the non–effective conjugacy classes. We should give an example \(g \in (C)\) for each conjugacy class \(C\) not appearing in Table 1 (Proposition 5.2):

1. \(g = s(0, 0, 0, 0, 1) = (12)\) is in \((4 \mathcal{A}_1)\) and
   \[
   g = (x_2 : x_1 : x_3), (y_2 : y_1 : y_3).
   \]
2. \(g = s(0, 0, 1, 0, 0) = (1234)\) is in \((4 \mathcal{A}_3)\) and
   \[
   g = ((−x_3 : x_1 − x_3 : x_2 − x_3), (−y_3 : y_1 − y_3 : y_2 − y_3)).
   \]
3. \(g = s(0, 3, 2, 0, 1) = (12)(345)\) is in \((4 \mathcal{A}_2 + 4 \mathcal{A}_1)\) and
   \[
   g = (x_1(x_2 − x_3) : x_2(x_1 − x_3) : x_1x_2),
   \]
   \[
   (x_1(x_3y_2 − x_2y_3) : x_2(x_3y_1 − x_1y_3) : −x_1x_2y_3)).
   \]
4. \(g = s(5, 1, 3, 1, 1) = (12)(34)(56)\) is in \((3 \mathcal{A}_1)\) and
   \[
   g = ((−y_2 + y_3 : −y_1 + y_3 : y_2), (x_2 − x_3 : x_1 − x_3 : −x_3)).
   \]
5. \(g = s(1, 0, 0, 0, 0, 0) = (123456)\) is in \((4 \mathcal{A}_9)\) and
   \[
   g = ((−y_1y_2 : y_2(−y_1 + y_3) : y_1(−y_2 + y_3)),
   \]
   \[
   (−x_3y_1y_2 : y_2(−x_3y_1 + x_1y_3) : y_1(−x_3y_2 + x_2y_3))).
   \]
(6) \( g = s(3, 3, 0, 0, 1) r [2] = (12)(3456)r[2] \) is in \( (A_4 + A_3) \) and
\[
g = \left( (y_2 - y_3 : y_1 - y_3), \right.
\left. (x_1(-x_3y_2 + x_2y_3) : x_2(-x_3y_1 + x_1y_3) : x_4x_2y_3) \right).
\]

(7) \( g = s(4, 3, 1, 2, 1)r[52] = (12)(34)(56)r[2]r[72] \) is in \( (A_3 + 2A_2) \) and
\[
g = \left( (x_1(x_2 - x_3)(-x_2y_1 + x_3y_2 - x_3y_1 + x_2y_3), \right.
\left. x_2(-x_1 + x_3)(x_2y_1 - x_3y_2 + x_3y_2 + x_1y_3 - x_2y_3), \right.
\left. (x_1 - x_3)(x_2 - x_3)(-x_2y_1 + x_1y_2), \right.
\left. (y_1(y_2 - y_3)(x_2y_1 - x_3y_2 + x_3y_2 + x_1y_3 - x_2y_3), \right.
\left. y_2(y_1 - y_3)(x_2y_1 - x_3y_2 - x_3y_1 + x_2y_3), \right.
\left. (x_2y_1 - x_1y_2)(-y_1 + y_3)(-y_2 + y_3) \right).
\]

(8) \( g = s(4, 2, 0, 1, 0)r[2] = (123)(456)r[2] \) is in \( (2A_2 + A_1) \) and
\[
g = \left( (y_3 : y_1 : y_2), (x_1x_2y_3 : x_2x_3y_1 : x_1x_3y_2) \right).
\]

(9) \( g = s(4, 4, 2, 1, 0)r[11] = (2345)r[11]r[2] \) is in \( (D_3(a_1)) \) and
\[
g = \left( ((-x_1 + x_3)(y_2 - y_3)y_3 : x_3(y_1 - y_3)(-y_2 + y_3) : (-x_2 + x_3) \times (y_1 - y_3)y_3, (x_1y_2y_3 : x_3y_3y_2 : x_2y_1y_3) \right).
\]

In every case, the invariant locus \( U_g \) is empty. Be careful to seek for the solutions in \( U \).

6.4. Stabilizer Groups

Take a \( g \in C \) for a conjugacy class \( C \) appearing in Table 1 (Proposition 5.2). If a point \( P \in U_g \) satisfies the condition
\[
U_g \subset U_h \quad \text{for all} \quad h \in G_p,
\]
the point \( P \) is a general point in \( U_g \) and the isomorphic class of \( G_p \) is the associated isomorphic class with the conjugacy class \( C \). Here we take points given in Theorem 5.3 and check the above conditions:

(1) Let \( P = ((1: 2: 5), (1: 3: 8)) \). Then \( G_p = \{e\} \).
(2) Let $P = ((1 : 5 : 4), (1 : 3 : 12))$. Then $G_p = \{ e, g = s(1, 1, 1, 1, 1, 1) r[72] \}$ and $g \in (4A_1)$. We see that

$$g = s(1, 1, 1, 1, 1) r[72] = ((x_3 y_2 : x_2 y_3 : x_3 y_3), (x_3 y_1 : x_1 y_3 : x_3 y_3)),$$

$$U_g = \{ ((1 : x_2 : x_3), (1 : y_2 : x_3 y_2)) \in U \}$$

and $\dim U_g = 3$. $G_p \cong \mathbb{Z}/2\mathbb{Z}$.

(3) Let $P = ((1 : 1/2 : 1 + \sqrt{-1} /2), (1 : - \sqrt{-1} : 2))$. Then $G_p = \{ e, g = s(5, 1, 1, 0, 0) r[46], g^2 = s(5, 1, 2, 0, 0) r[72], g^3 = s(5, 1, 0, 1, 1) r[51] \}$. $G_p \cap (D_4(a_1)) = \{ g, g^3 \}$ and $G_p \cap (4A_1) = \{ g^2 \}$. We see that

$$g = s(5, 1, 1, 0, 0) r[46]
= \left(\left( (x_2 y_1 - x_1 y_2) y_3 (x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3)
\right)
\right)
= \left(\left( y_2 (x_2 y_1 - x_1 y_3) (x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3)
\right)
\right)
= \left(\left( x_2 y_1 - x_1 y_2)(-y_2 + y_3)(-x_3 y_1 + x_1 y_3)
\right)
\right)
= \left(\left( x_2 - x_2 : x_1 - x_3 : x_1 \right)
\right).

U_g = \{ (1 : 1 - 1/y_3 : 1 + \sqrt{-1} (1 - 1/y_3)), (1 : - \sqrt{-1} (1 - y_3) : y_3) \in U \}$ and $\dim U_g = 1$. $G_p \cong \mathbb{Z}/4\mathbb{Z}$.

(4) Let

$$P = ((1 : 1 - \sqrt{-1} : -\sqrt{-1}), (1 : -\sqrt{-1} - (1 + \sqrt{-1}) / \sqrt{2} : (1 - \sqrt{-1})(1 + 1/\sqrt{2})))$$

Then

$$G_p = \{ e, g = s(0, 1, 0, 0, 0) r[2],
\}
= \{ s(0, 0, 1, 1, 1, 1) r[10],
\}
= \{ s(0, 2, 0, 0, 1) r[26],
\}
= \{ s(0, 0, 2, 0, 0) r[46],
\}
= \{ s(0, 4, 2, 0, 1) r[41],
\}
= \{ s(0, 0, 2, 0, 1) r[12],
\}
= \{ s(0, 4, 0, 0, 0) r[3] \}.$$

$G_p \cap (D_4) = \{ g, g^3, g^5, g^7 \}$, $G_p \cap (D_4(a_1)) = \{ g^2, g^6 \}$, and $G_p \cap (4A_1) = \{ g^4 \}$. We see that $g = s(0, 1, 0, 0, 0) r[2] = ((-x_3 : x_1 - x_3 : x_2 - x_3),
(-x_3 y_1 y_2 : y_2 (-x_3 y_1 + x_1 y_3) : y_3 (-x_3 y_2 + x_2 y_3))). \#U_g = 4$. $G_p \cong \mathbb{Z}/8\mathbb{Z}$. 

(5) Let \( P = ((1:9:8), (1:-2:-3)) \). Then

\[
G_p = \{ e, g_1 = s(5,1,1,1) r[72], \\
g_2 = s(5,1,3,1,1) r[72], \\
g = g_1 g_2 = s(0,0,2,0,0) \}.
\]

\( G_p \cap (2A_1) = \{ g \} \) and \( G_p \cap (4A_1) = \{ g_1, g_2 \} \). We see that

\[
g_1 = s(5,1,1,1,1) r[72]
\]

\[
= \left( \left( x_2 + x_3 \right) \left( -x_2 y_1 + x_1 y_2 \right) \left( -x_3 y_1 + x_1 y_3 \right) : x_2 \left( x_3 y_1 - x_1 y_3 \right) \right)
\times \left( -x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3 \right) ;
\]

\[
x_3 \left( -x_2 y_1 + x_1 y_2 \right) \left( x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3 \right) \),
\]

\[
\left( -x_2 y_1 + x_1 y_2 \right) \left( y_2 - y_3 \right) \left( x_3 y_1 - x_1 y_3 \right) y_2 \left( x_3 y_1 - x_1 y_3 \right) ;
\]

\[
x_3 \left( -x_2 y_1 + x_1 y_2 \right) \left( x_2 y_1 - x_3 y_1 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3 \right) \),
\]

\[
g_2 = s(5,1,3,1,1) r[72]
\]

\[
= \left( \left( x_1 y_2 - x_3 y_2 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3 \right) \left( -x_3 y_2 + x_2 y_3 \right) \right) ;
\]

\[
\left( -x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3 \right) ;
\]

\[
\left( x_1 y_2 - x_3 y_2 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3 \right) ;
\]

\[
\left( x_1 y_2 - x_3 y_2 - x_1 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3 \right) ;
\]

\[
\left( -y_1 + y_2 \right) \left( x_3 y_1 - x_1 y_3 \right) \left( x_3 y_2 - x_2 y_3 \right) \),
\]

and

\[
g = s(0,0,2,0,0)
\]

\[
= \left( \left( -x_2 + x_3 : -x_2 : x_1 - x_2 \right), \left( -y_2 + y_3 : -y_2 : y_1 - y_2 \right) \right) .
\]

\( U^s = \{(1:1 + x_3 : x_3), (1:1 + y_3 : y_3) \} \in U \) = \( U_{s_1} \cap U_{s_2} \) and \( \dim U_s = 2 \).

\( G_p \equiv \mathbb{Z}/2^2 \).

(6) Let \( P = ((1:3:5), (1:3/5:3)) \). Then \( G_p = \{ e, g = s(4,2,0,1,0), g^{-1} = s(5,0,1,2,0), g_1 = s(1,1,1,1) r[72], g_2 = s(2,1,0,2,1) r[72], \)
\( g_3 = s(3, 4, 1, 0, 1)r[72] \). \( G_p \cap (2A_2) = \{ g, g^{-1} \} \) and \( G_p \cap (4A_1) = \{ g_1, g_2, g_3 \} \). We see that

\[
g_1 = s(1, 1, 1, 1, 1)r[72]
= ((x_3y_2 : x_2y_3 : x_3y_3), (x_3y_1 : x_1y_3 : x_3y_3)),
\]
\[
g_2 = s(2, 1, 0, 2, 1)r[72]
= ((x_1y_1 : x_1y_3 : x_3y_1), (x_1y_2 : x_2y_1 : x_2y_1)),
\]
\[
g_3 = s(3, 4, 1, 0, 1)r[72]
= ((x_1y_2 : x_2y_2 : x_2y_1), (x_3y_2 : x_2y_2 : x_2y_3))
\]

and

\[
g = s(4, 2, 0, 1, 0) = ((y_1y_2 : y_2y_3 : y_1y_3), (x_3y_1y_2 : x_1y_2y_3 : x_2y_1y_3)).
\]

\( U_3 = ((1 : y_3 / x_3), (1 : y_3 / x_3) \in U) = U_{g_1} \cap U_{g_2} \) and \( \dim U_g = 2 \).

\( G_p \cong \Sigma_3 \).

(7) Let \( P = ((1 : 4 : 3), (1 : 3 : 2)) \). Then \( G_p \cap (A_5 + A_1) = \{ g = s(1, 3, 0, 2, 0)r[72], g^{-1} = s(3, 0, 2, 0, 0)r[72] \} \) and \( G_p \cap (4A_1) = \{ g_1 = s(1, 3, 1, 2, 1)r[72], g_2 = s(3, 0, 0, 0, 0)r[72], g_3 = s(5, 1, 3, 1, 1)r[72], g_4 = s(5, 1, 1, 1, 1)r[72] \} \). We see that

\[
g_1 = s(1, 3, 1, 2, 1)r[72]
= ((x_3y_2 - x_3y_2 : x_2(y_2 - y_3)) :
\]
\[
x_3(y_2 - y_3), (x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3):
\]
\[
(-x_1 + x_3)(y_2 - y_3)) = (-x_1 + x_3)((y_2 - y_3)),
\]
\[
g_2 = s(3, 0, 0, 0, 0)r[72]
= ((x_1y_1 : x_2y_1 : x_1y_2), (x_3y_1 : x_3y_1 : x_1y_3)),
\]
\[
g_3 = (5, 1, 3, 1, 1)r[72]
= ((x_1y_2 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3):
\]
\[
(x_3y_2 - x_3y_2) : x_2(x_3y_1 - x_1y_3)
\]
\[
\times (-x_3y_2 + x_3y_3) : x_2(x_3y_1 - x_1y_3)
\]
\[
\times (-x_3y_2 + x_3y_3) : x_2(x_3y_1 - x_1y_3)
\]
\[
\times (x_2y_1 - x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3):
\]
\[
(x_1 - x_2)(x_3y_1 - x_1y_3)
\]
\[
\times (x_2y_2 - x_3y_2), (y_1(x_3y_2 - x_2y_3)
\]
\[
\times (x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3):
\]
\[
y_2(x_3y_1 - x_1y_3)(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3):
\]
\[
(-y_1 + y_2)(x_3y_1 - x_1y_3)(x_3y_2 - x_2y_3)),
\]
\[ g_4 = s(5, 1, 1, 1)r[72] \]
\[ = \left( (-(x_2 + x_3)(-x_2y_1 + x_1y_2)(-x_3y_1 + x_3y_2) : \right. \]
\[ \left. \begin{array}{l}
x_2(x_3y_1 - x_1y_3)(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) : \\
x_3(-x_2y_1 + x_1y_2)(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3) : \\
( (-(x_2y_1 + x_1y_2)(y_2 - y_3)(x_3y_1 - x_1y_3) \\
\times (-x_2y_1 + x_1y_2 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) : \\
( -x_2y_1 + x_1y_2) y_3 \\
\times (x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3)), \right) \]
\[ \text{and} \]
\[ g = s(1, 3, 0, 2, 0)r[72] \]
\[ = \left( (x_2y_1 - x_3y_2 : x_2y_1 : (x_2 + x_3)y_1), \right. \]
\[ \left. \begin{array}{l}
(x_3y_1 - x_1y_3 : x_3y_1 : (x_1 + x_3)y_1)) . \right) \]
\[ U_3 = \{(1 : 1 + x_3 : x_3), (1 : x_3 : -1 + x_3) \in U) = U_{g_{1}} \cap U_{g_{2}} \cap U_{g_{3}} \text{ and} \]
\[ \dim U_{g} = 1. \text{ If } G_{p} = \Sigma_3 \times (Z/2Z). \]
\[ (8) \text{ Let } P = ((1 : -2/3 : 2/3), (1 : -4 : 2)). \text{ Then} \]
\[ \#(G_{p} \cap (A_3 + A_1)) = 6 \text{ and} \]
\[ G_{p} \cap (4A_1) \]
\[ = \{s(2, 3, 2, 1, 0)r[35], s(0, 3, 1, 1, 1)r[41], s(2, 3, 2, 1, 0)r[44], \]
\[ s(0, 3, 1, 1, 1)r[50], s(3, 1, 1, 1, 1)r[72], s(5, 1, 3, 1, 1)r[72]) . \]
\[ \text{Let} \]
\[ g_1 = s(2, 3, 2, 1, 0)r[35], \]
\[ g_2 = s(0, 3, 1, 1, 1)r[41], \]
\[ \text{and} \]
\[ g_3 = s(2, 3, 2, 1, 0)r[44]. \]
\[ \text{Then } g_1g_2g_3 = s(3, 3, 0, 0, 1) \text{ is in } G_{p} \cap (A_3 + A_1) \text{ and } (g_1, g_2, g_3) \]
\[ \text{generate } G_{p}. \text{ We see that} \]
\[ g_1 = s(2, 3, 2, 1, 0)r[35] \]
\[ = \left( (x_1(-x_1x_3y_1 + x_2x_3y_1y_2 + x_1x_2y_1y_3) \right. \]
\[ \left. -x_1x_3y_1y_3 - x_1x_2y_2y_3 + x_1x_3y_2y_3) : \\
x_2(-x_1x_3y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3) \right. \]
\[ \left. -x_2x_3y_1y_3 - x_1x_2y_2y_3 + x_1x_3y_2y_3) : \\
x_1x_2y_3(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3) : \\
( y_1 : y_2 : y_3)), \right) \]
\[ g_2 = s(0, 3, 1, 1, 1)r[41] \]
\[ = ((-x_1 : -x_2 : -x_3), (x_1(y_1 - y_2)(-y_1 + y_3)(-x_3y_2 + x_2y_3) : \]
\[ x_2(y_1 - y_2)(-y_2 + y_3)(-x_3y_1 + x_1y_3) : \]
\[ x_3(x_2y_1 - x_1y_2)(-y_1 + y_3)(-y_2 + y_3)) \]\\
\[ g_3 = s(2, 3, 2, 1, 0)r[44] \]
\[ = ((y_2 - y_3)(-x_1x_3y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 - x_2x_3y_1y_3 \]
\[ -x_1x_2y_2y_3 + x_1x_3y_2y_3) : y_2(y_1 - y_3) \]
\[ \times (-x_1x_3y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 - x_2x_3y_1y_3 \]
\[ -x_1x_2y_2y_3 + x_1x_3y_2y_3) : (-x_1 + x_2) \]
\[ \times x_3y_1y_2(y_1 - y_3)(y_2 - y_3), (y_2(y_1 - x_1y_2) : \]
\[ y_2(x_2y_1 - x_1y_2) : (-x_1 + x_2) : y_1y_2) \),
and
\[ g_1g_2g_3 = s(3, 3, 0, 0, 1) \]
\[ = ((y_2 + y_3) : y_2(-y_1 + y_3) : -y_1y_2), (y_2(-x_3y_2 + x_2y_3) : \]
\[ y_2(-x_3y_1 + x_1y_3) : -x_3y_1y_2)) \).

For any 
\[ g \in G_p \cap (A_3 + A_1), \]
\[ U_g = U_{g_1} \cap U_{g_2} \cap U_{g_3} \]
\[ = \{(1 : (y_3 - y_3^2)/(1 + y_3) : y_3/(1 + y_3)), (1 : -y_3^2 : y_3) \} \in U \}
and dim \( U_g \) = 1. \( G_p \equiv \Sigma_4 \).

(9) Let 
\[ P = ((1 : \sqrt{3} : (3 + \sqrt{3})/2), (1 : (3 + \sqrt{3})/2 : 2)) \].
Then \(#(G_p \cap (3A_2)) = 2, #(G_p \cap (E_4(a_2))) = 18 and \)
\[ G_p \cap (4A_1) \]
\[ = \{s(0, 3, 0, 2, 0)r[26], s(4, 0, 2, 0, 0)r[27], s(2, 0, 2, 0, 0)r[34], \]
\[ s(2, 3, 0, 2, 0)r[40], s(2, 3, 2, 1, 0)r[44], s(0, 3, 3, 0, 1)r[48], \]
\[ s(3, 0, 0, 0, 0)r[72], s(4, 3, 0, 1, 1)r[72], s(5, 1, 2, 0, 0)r[72] \} \).

Let 
\[ g_1 = s(0, 3, 0, 2, 0)r[26], \]
\[ g_2 = s(4, 0, 2, 0, 0)r[27] \]
and
\[ g_3 = s(2, 0, 2, 0, 0)r[34]. \]

Then
\[ g_1 g_2 g_3 = s(1, 2, 3, 2, 0)r[44] \in (E_6(a_2)), \]
\[ (g_1 g_2 g_3)^2 = s(4, 2, 2, 0, 1)r[61] \in (3A_2), \]

and \( g_1, g_2, g_3 \) generate \( G_P \). We see that
\[ g_1 = s(0, 3, 0, 2, 0)r[26] \]
\[ = \left( (x_1 : x_2 : x_3) \left( x_1 y_2 y_3 (-x_2 y_1 + x_3 y_1 + x_4 y_2 - x_3 y_3 - x_1 y_3 + x_2 y_3) : y_2 (x_1 x_3 y_1 y_2 - x_2 x_3 y_2 y_1 - x_1 x_2 y_1 y_3 + x_2 x_3 y_1 y_3) + x_1 x_2 y_2 y_3 - x_1 x_3 y_2 y_3) : y_3 (-x_1 x_3 y_1 y_2 - x_2 x_3 y_1 y_2 - x_1 x_2 y_1 y_3 + x_2 x_3 y_1 y_3) \right) \],
\[ g_2 = s(4, 0, 2, 0, 0)r[27] \]
\[ = \left( (x_1 y_1 - y_2) (-x_3 y_2 + x_2 y_3) : x_3 (x_2 y_1 - x_1 y_2) \times (-y_2 + y_3) : x_3 (y_1 - y_2) (-x_3 y_2 + x_2 y_3) \right) \]
\[ \times (x_1 - x_2) y_1 (-x_3 y_2 + x_2 y_3) : (x_2 - x_3) \times (-x_2 y_1 + x_1 y_2) y_3 : (-x_2 + x_1) y_3 (x_3 y_2 - x_2 y_3) \],

and
\[ g_3 = s(2, 0, 2, 0, 0)r[34] \]
\[ = \left( (x_1 y_3 (-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3) : x_2 y_3 \times (-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3) : x_1 x_3 y_1 y_2 - x_2 x_3 y_1 y_2 - x_1 x_2 y_1 y_3 + x_2 x_3 y_1 y_3 + x_1 x_2 y_2 y_3 - x_1 x_3 y_2 y_3) \right) \]
\[ \times (x_1 y_1 (-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3) : x_3 y_2 \times (-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3) : x_1 x_3 y_1 y_2 - x_2 x_3 y_1 y_2 - x_1 x_2 y_1 y_3 + x_2 x_3 y_1 y_3 + x_1 x_2 y_2 y_3 - x_1 x_3 y_2 y_3) \].
and that
\[ g_1g_2g_3 = s(1, 2, 3, 2, 0)r[44] \]
\[ = \left( (x_1y_3(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 + x_1y_3 + x_2y_3) : \
\quad (x_1 - x_3)(-x_2y_1 + x_1y_2)y_3 : \
\quad x_1x_3y_1y_2 - x_2x_3y_3y_1 - x_1x_2y_1y_3 \
\quad + x_2x_3y_1y_3 + x_1x_2y_2y_3 - x_1x_3y_2y_3), \
\quad (x_3(-y_1 + y_3) : \right) \]
\[ x_3(x_2y_1 - x_1y_2)(-y_1 + y_3) : x_1x_3y_1y_2 \
\quad - x_2x_3y_1y_3 - x_1x_3y_2y_3 + x_1x_2y_2y_3 - x_1x_3y_2y_3) \]
\[ \right) \]
\[ = s(4, 2, 2, 0, 1)r[61] \]
\[ = \left( (x_2y_3(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3) : \
\quad (x_2 - x_3)(-x_2y_1 + x_1y_2)(-y_1 + y_3) : \
\quad x_2(-x_3y_1 + x_1y_2)(-y_1 + y_3), (x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 \
\quad + x_1y_3 - x_2y_3)(-x_2 + x_3)(-y_1 + y_3) : (x_1 - x_2)(-y_1 + y_3)) \right) \]

For any \( g \in G_p \cap (3A_2) \cup (E_8(a_2)) \),
\[ U_g = U_u \cap U_v \cap U_s \]
\[ = \left\{ ((1 : 1 + (-1 ± \sqrt{-3})y_3)/2 : (1 + y_3 ± \sqrt{-3}(-1 + y_3))/2, \
\quad (1 : 1 + y_3 ± \sqrt{-3}(-1 + y_3))/2 : y_3) \in U \right\}, \]
and \( \dim U_g = 1 \). Let \( C(G_p) \) be the center of \( G_p \). Then \( C(G_p) = \langle G_p \cap (3A_2) \rangle \cong \mathbb{Z}/3\mathbb{Z} \) and \( G_p / C(G_p) \cong (\mathbb{Z}/3\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z}) \).

(10) Let \( P = ((1 : 1 + (\sqrt{-1} - \sqrt{-3})/2 : \sqrt{-1} + (1 - \sqrt{-3})/2), \
\quad (1 : \sqrt{-1} + (1 - \sqrt{-3})/2 : (\sqrt{-1} + \sqrt{-3})/2) \). Then \( n(G_p) = \#(G_p) = 
\quad 108, \ #(G_p \cap (E_8)) = 36 \), and \( G_p \cap (4A_1) \) is the same as above. Let \( g_1, g_2, \) and \( g_3 \) be as above. Let \( h = s(0, 1, 2, 1, 2, 0)r[2] \). Then \( h \) is in \( G_p \cap (D_4(a_1)) \) and \( h^2 = g_1 \). We see that \( h^2g_2 = s(4, 0, 1, 1, 0)r[70] \) is in \( G_p \cap (E_8) \) and \( g_3 = h^2g_2h^2g_1h^2 \). Moreover, \( \langle h, g_2 \rangle \) generates \( G_p \). We see that \( h = s(0, 2, 1, 2, 0)r[2] \)
\[ = \left( ((x_1 - x_2)x_3 : x_2(x_1 - x_3) : (x_1 - x_3)x_3), \
\quad (-x_1x_2y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 - x_2x_3y_1y_3 - x_1x_2y_2y_3 \
\quad + x_1x_3y_2y_3 : x_2(-x_1 + x_3)y_1(y_2 - y_3) : (-x_1 + x_3)y_1 \
\quad \times (x_3y_2 - x_2y_3)) \right) \]
and

\[ hg_2 g_3 = s(4, 0, 1, 1, 0) r[70] \]

\[ = \left( \frac{y_3(x_1 x_3 y_1 y_2 - x_2 x_3 y_1 y_2 - x_1 x_2 y_1 y_3 + x_2 x_3 y_1 y_3)}{x_1 x_2 y_2 y_3 - x_1 x_3 y_2 y_3} \right) \]

\[ (x_1 - x_3)(-x_2 y_1 + x_1 y_2) y_3(-y_2 + y_3) : \]

\[ (y_2 - y_3)(-x_1 x_3 y_1 y_2 + x_2 x_3 y_1 y_2 + x_1 x_2 y_1 y_3 \]

\[ - x_2 x_3 y_1 y_3 - x_1 x_2 y_2 y_3 + x_3 x_2 y_2 y_3) \}

\[ ((y_1 - y_2) y_3(-x_1 x_3 y_1 y_2 + x_2 x_3 y_1 y_2 + x_1 x_2 y_1 y_3 - x_2 x_3 y_1 y_3 \]

\[ - x_1 x_2 y_2 y_3 + x_1 x_3 y_2 y_3) : \]

\[ x_2(-x_1 + x_3) y_1(-y_1 + y_2) y_3(-y_2 + y_3) : \]

\[ y_1(y_2 - y_3)(x_1 x_3 y_1 y_2 - x_2 x_3 y_1 y_2 - x_1 x_2 y_1 y_3 \]

\[ + x_2 x_3 y_1 y_3 + x_1 x_2 y_2 y_3 - x_1 x_3 y_2 y_3) \}) . \]

For any \( g \in G_p \cap (E_g), U_g = U_n \cap U_{g_2} \) and \( \#U_g = 4. \)

(11) Let

\[ P = \left( (1 : (-1 + \sqrt{5})/2 : (3 - \sqrt{5})/2), (1 : (3 - \sqrt{5})/2 : (-1 + \sqrt{5})/2) \right). \]

Then \( \#(G_p \cap (A_p)) = 24 \) and

\[ G_p \cap (4A_1) \]

\[ = \{ s(1, 2, 0, 2, 0) r[72], s(1, 3, 1, 2, 1) r[72], s(2, 0, 1, 1, 0) r[72], s(2, 1, 0, 2, 1) r[72], s(3, 0, 0, 0, 0) r[72], s(3, 1, 1, 1, 1) r[72], s(4, 3, 0, 1, 1) r[72], s(4, 4, 3, 1, 0) r[72], s(5, 1, 2, 0, 0) r[72], s(5, 1, 3, 1, 1) r[72] \}. \]

Let

\[ g_1 = s(1, 2, 0, 2, 0) r[72], \]

\[ g_2 = s(1, 3, 1, 2, 1) r[72], \]

\[ g_3 = s(2, 0, 1, 1, 0) r[72], \]

and

\[ g_4 = s(2, 1, 0, 2, 1) r[72]. \]
Then \( g_1 g_2 g_3 g_4 = s(0, 3, 0, 0, 0) \) is in \( G_p \cap (A_4) \) and \((g_1, g_2, g_3, g_4)\) generates \( G_p \). We see that

\[
g_1 = s(1, 2, 0, 2, 0) r[72]
\]

\[
= ((x_2 y_3 : x_2 y_2 : x_3 y_2), (x_2 y_1 : x_2 y_2 : x_1 y_2))
\]

\[
g_2 = s(1, 3, 1, 2, 1) r[72]
\]

\[
= ((x_3 y_2 - x_2 y_3 : x_2 (y_2 - y_3) : x_3 (y_2 - y_3)),

(x_2 y_1 - x_3 y_2 + x_3 y_2 + x_1 y_3 - x_2 y_3):

(-x_1 + x_2)(y_2 - y_3) : (x_1 + x_3)(y_2 - y_3)))
\]

\[
g_3 = s(2, 0, 1, 1, 0) r[72]
\]

\[
= ((x_1 y_3 : x_3 y_2 : x_3 y_3), (x_2 y_3 : x_3 y_2 : x_3 y_3))
\]

\[
g_4 = s(2, 1, 0, 2, 1) r[72]
\]

\[
= ((x_1 y_1 : x_1 y_3 : x_3 y_1), (x_1 y_1 : x_1 y_2 : x_2 y_1))
\]

and

\[
g_1 g_2 g_3 g_4 = s(0, 3, 0, 0, 0)
\]

\[
= ((x_2 (x_1 - x_3) : x_1 (x_2 - x_3) : x_2 (x_2 - x_3)),

(x_2 (-x_2 y_1 + x_3 y_1 + x_1 y_2 - x_3 y_2 - x_1 y_3 + x_2 y_3) :

(x_2 - x_3) (-x_2 y_1 + x_1 y_2) : x_2 (x_2 - x_3) (-y_1 + y_2))).
\]

For any \( g \in G_p \cap (A_4) \), \( U_g = U_{g_1} \cap U_{g_2} \cap U_{g_3} \cap U_{g_4} \) and \( \#U_g = 2 \).

Let

\[
P = ((1 : (1 + \sqrt{3}) / 2 : (3 + \sqrt{3}) / 2),

(1 : (3 + \sqrt{3}) / 2 : (3 - \sqrt{3}) / 2)).
\]

Then

\[
\#(G_p \cap (A_2)) = 6,
\]

\[
\#(G_p \cap (A_2 + 2A_1)) = 54,
\]

\[
\#(G_p \cap (D_4)) = 36,
\]

\[
\#(G_p \cap (E_6(a_1))) = 144,
\]
and
\(G_p \cap (4A_1)\)

\[= \{ s(4, 0, 2, 0, 0)r[22], s(2, 3, 3, 1, 1)r[25], s(1, 3, 1, 0, 0)r[28],
\ s(1, 2, 1, 1, 0)r[29], s(2, 2, 1, 0, 1)r[32], s(0, 1, 1, 1, 1)r[36],
\ s(2, 1, 1, 1, 1)r[37], s(0, 3, 1, 1, 1)r[41], s(2, 3, 2, 1, 0)r[44],
\ s(2, 3, 0, 2, 0)r[45], s(0, 3, 3, 0, 1)r[48], s(0, 0, 3, 1, 1)r[51],
\ s(1, 2, 0, 2, 0)r[72], s(1, 3, 1, 2, 1)r[72], s(2, 0, 1, 1, 0)r[72],
\ s(3, 0, 0, 0, 0)r[72], s(4, 4, 3, 1, 0)r[72], (5, 1, 3, 1, 1)r[72]\}.
\]

Let
\[g_1 = s(4, 0, 2, 0, 0)r[22],
\ g_2 = s(2, 3, 3, 1, 1)r[25],
\ g_3 = s(1, 3, 1, 0, 0)r[28],
\]

and
\[g_4 = s(1, 2, 1, 1, 0)r[29].\]

Then
\[g_1g_2g_3g_4 = s(0, 3, 0, 0, 0)r[31] \in (A_2 + 2A_1),
\ (g_1g_2g_3g_4)^2 = s(0, 2, 2, 0, 1)r[12] \in (A_2),
\ g_1g_3g_2g_4 = s(3, 4, 1, 2, 1)r[19] \in (E_8(a_1)),
\]

and \(g_1g_3g_2g_4s_1g_3g_2 = s(3, 2, 0, 2, 0)r[16] \in (D_4).\) We see that
\[g_1 = s(4, 0, 2, 0, 0)r[22]
\]

\[= ((x_3(x_2y_1 - x_1y_2))(-y_1 + y_3) : x_2(y_1 - y_2)(-x_3y_1 + x_1y_3) :
\ x_3(y_1 - y_2)(-x_3y_1 + x_1y_3)) : ((x_1 - x_3)(-x_2y_1 + x_1y_2)y_3 :
\ (x_1 - x_2)y_3(-x_3y_1 + x_1y_3) :
\ (-x_1 + x_2)y_3(x_3y_1 - x_1y_3)),
\]

\[g_2 = s(2, 3, 3, 1, 1)r[25]
\]

\[= ((x_2x_3y_1(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3) :
\ x_2(-x_1x_3y_1y_2 + x_2x_3y_1y_2
\ + x_1x_2y_1y_3 - x_2x_3y_1y_3 - x_1x_2y_2y_3 + x_1x_3y_2y_3) :
\ x_3(-x_1x_3y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 - x_2x_3y_1y_3
\ - x_1x_2y_2y_3 + x_1x_3y_2y_3)) : (y_1 : y_2 : y_3)).
\]
$g_3 = s(1, 3, 1, 0, 0)r[28]$

$$= ((x_1(x_2y_1 - x_1y_2)(-y_2 + y_3): x_3(y_1 - y_2))$$

$$\times (-x_3y_2 + x_2y_3): x_3(x_2y_1 - x_1y_2)(-y_2 + y_3)),$$

$$(( -x_2 + x_3)y_1(-x_2y_1 + x_1y_2): (x_1 - x_2)y_1$$

$$\times (x_3y_2 - x_2y_3): (-x_2 + x_3)(-x_2y_1 + x_1y_2)y_3)),$$

and

$g_4 = s(1, 2, 1, 1, 0)r[29]$

$$= ((x_1y_2(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3):$$

$$-x_1x_3y_1y_2 + x_2x_3y_1y_2 + x_1x_2y_1y_3 - x_2x_3y_1y_3$$

$$-x_1x_2y_2y_3 + x_1x_3y_2y_3: x_3y_2$$

$$\times (x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3),$$

$$(x_2y_3(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3):$$

$$x_1x_3y_1y_2$$

$$-x_2x_3y_1y_2 - x_1x_2y_1y_3 + x_2x_3y_1y_3 + x_1x_2y_2y_3 - x_1x_3y_2y_3: x_2y_3$$

$$\times (-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3))$$

and that

$g_1g_2g_3g_4 = s(0, 3, 0, 0, 0)r[31]$

$$= ((( -x_1 + x_2)(x_2 - x_3): x_1(-x_2 + x_3): (x_1 - x_2)x_3),$$

$$y_1(-y_2 + y_3): (-x_3y_1 + x_1y_3):$$

$$y_1(-y_2)y_3(-x_3y_1 + x_1y_3)),$$

$(g_1g_2g_3g_4)^2$

$$= s(0, 2, 2, 0, 1)r[12]$$

$$= ((x_2(-x_1 + x_3): (-x_1 + x_2)(-x_1 + x_3): (-x_1 + x_2)x_3),$$

$$(x_2y_1 - y_2): (-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2$$

$$-x_1y_3 + x_2y_3): (x_1 - x_2)y_2$$

$$\times (-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3):$$

$$( -x_1 + x_2)(y_1 - y_2)(-x_3y_2 + x_2y_3)).$$
\[
g_1g_3g_2g_4 = s(3, 4, 1, 2, 1)r[19] = ((y_2(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) : (x_2y_1 - x_1y_2)(-y_2 + y_3) : (-x_1 + x_2)y_2(-y_2 + y_3)),
\]
\[
(y_3(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) : (y_2 - y_3)(-x_3y_1 + x_1y_3) : (x_1 - x_3)(y_2 - y_3)y_3)\]

and
\[
g_1g_3g_2g_4g_2g_3g_4g_1g_4g_3g_2 = s(3, 2, 0, 2, 0)r[16] = ((y_2(-x_2y_1 + x_3y_1 + x_1y_2 - x_3y_2 - x_1y_3 + x_2y_3) : (x_2y_1 - x_1y_2) \times (-y_2 + y_3) : (-x_1 + x_2)y_2(-y_2 + y_3)),
\]
\[
(x_1y_3(x_2y_1 - x_3y_1 - x_1y_2 + x_3y_2 + x_1y_3 - x_2y_3) : x_3(y_2 - y_3) \times (x_3y_1 - x_1y_3) : (-x_1 + x_2)(-y_2 + y_3)(x_3y_1 - x_1y_3)).\]

Moreover \( U_g = U_{g_1} \cap U_{g_2} \cap U_{g_3} \cap U_{g_4} \) for any \( g \in G_p \cap (A_4) \cup (A_2 + 2A_1) \cup (D_9) \cup (E_6(a_1)) \) and \( \{g_1, g_2, g_3, g_4\} \) generate \( G_p \). \( #U_g = 2. \ G_p \cong (\mathbb{Z}/3\mathbb{Z})^3 \rtimes \Sigma_4. \)

REFERENCES


