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Analytical Tableaux for da Costa's Hierarchy of Paraconsistent Logics

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Abstract

In this paper we present a new hierarchy of analytical tableaux systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, for da Costa's hierarchy of propositional paraconsistent logics $\mathbf{C}_n, 1 \leq n < \omega$. In our tableaux formulation, we introduce da Costa's "ball" operator "$\circ$", the generalized operators "k" and "(k)", and the negations "$\sim_k$", for $k \geq 1$, as primitive operators, differently to what has been done in the literature, where these operators are usually defined operators. We prove a version of Cut Rule for the $\mathbf{TNDC}_n, 1 \leq n < \omega$, and also prove that these systems are logically equivalent to the corresponding systems $\mathbf{C}_n, 1 \leq n < \omega$. The systems $\mathbf{TNDC}_n$ constitute completely automated theorem proving systems for the systems of da Costa's hierarchy $\mathbf{C}_n, 1 \leq n < \omega$.[3]

Keywords: paraconsistent logic, da Costa's hierarchy systems $\mathbf{C}_n, 1 \leq n < \omega$, decidability, hierarchy of analytical tableaux systems, cut rule, logical equivalency

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1 Introduction

A theory $\mathbf{T}$ is said to be inconsistent (contradictory) if it has as theorems a formula and its negation; otherwise, $\mathbf{T}$ is consistent (non-contradictory). A theory $\mathbf{T}$ is said to be trivial if every formula of its language is a theorem; otherwise, $\mathbf{T}$ is non-trivial.

A logic is paraconsistent if it can be used as the underlying logic to inconsistent but non-trivial theories, which we call paraconsistent theories.

D'Ottaviano [15] discusses that in paraconsistent logic the role of the Principle of Non-Contradiction is, in a certain sense, restricted. Although in those logics the Principle of Non-Contradiction is not necessarily invalid, from a formula and its negation it is not possible, in general, to deduce any formula.

In 1963, da Costa (see [9,10,11,12]) introduces his hierarchies of logical calculi for the study of inconsistent but non-trivial theories: the hierarchy of propositional calculi $\mathbf{C}_n, 1 \leq n \leq \omega$, the hierarchy of predicate calculi $\mathbf{C}_n^*, 1 \leq n \leq \omega$, the hierarchy of predicate calculi with equality $\mathbf{C}_n^=, 1 \leq n \leq \omega$, and the hierarchy of calculi of descriptions $\mathbf{D}_n, 1 \leq n \leq \omega$.

In 1976, da Costa and Alves (see [1,13]) introduce a semantics of valuations for the calculi $\mathbf{C}_n, 1 \leq n \leq \omega$, which generalizes the classical valuation semantics. By defining the quasi-matrices, they prove the completeness and the decidability of da Costa's propositional system $\mathbf{C}_1$, and indicate how to obtain these results for the calculi $\mathbf{C}_n, 2 \leq n < \omega$. [19], based on da Costa and Alves' work, proves the completeness and the decidability of the system $\mathbf{C}_\omega$. Loparic and Alves [20], solves a problem concerning Alves' quasi-matrices by modifying the conditions of Alves' definition of valuation, what allows them to prove the completeness and the decidability of the systems $\mathbf{C}_n, 1 \leq n < \omega$.

Marconi [21] introduces a variant of semantical tableaux systems, *à la* Beth [4], in order to prove the completeness and decidability of da Costa's propositional system $\mathbf{C}_1$. He also claims that his method can be expanded for the systems $\mathbf{C}_n, 2 \leq n < \omega$. The system introduced by Marconi is based on the same intuitions underlying the definition of quasi-matrices introduced by da Costa and Alves. In Marconi's tableaux system the rules for the connectives $\&, \vee$ and $\supset$ are the standard ones, and two special rules are added to operate with the paraconsistent negation.

Alves [1] also introduced the propositional paraconsistent system $\mathbf{C}_1^1$, by replacing the schema of axioms $\neg\neg A \supset A$ of $\mathbf{C}_1$ by the schema $\neg\neg A \equiv A$, in order to obtain a system stronger than da Costa's $\mathbf{C}_1$. Carnielli and Lima-Marques [6] introduce a semantical tableaux type approach, *à la* Smullyan [23], for Alves's paraconsistent propositional logic $\mathbf{C}_1^1$ and for the paraconsistent quantificational logic with equality $\mathbf{C}_1^{1=}$, namely the systems $\mathbf{TC}_1$ and $\mathbf{TC}_1^=$

respectively, and show that these systems are complete and decidable.

Buchsbaum and Pequeno [5] introduce a syntactical tableaux type approach, also *à la* Smullyan, for da Costa's $\mathbf{C}_1^*$, the system $S\mathbf{C}1^*$, showing that $S\mathbf{C}1^*$ is complete.

In a recent work [7,8] we introduce, through Fitch's [16] method of subordinate proofs, the hierarchy of natural deduction systems $\mathbf{NDC}_n, 1 \leq n \leq \omega$, and show that it is logically equivalent to da Costa's corresponding hierarchy $\mathbf{C}_n, 1 \leq n \leq \omega$. We prove a Normalization Theorem and a Subformula Property for these systems.

In this paper, based on the systems $\mathbf{NDC}_n, 1 \leq n < \omega$, and by using the method of analytical tableaux (see [23,17]), we introduce a hierarchy of syntactical tableaux systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, in which every system $\mathbf{TNDC}_n$ is equivalent to da Costa's corresponding system $\mathbf{C}_n, 1 \leq n < \omega$. In particular, our $\mathbf{TNDC}_1$ is distinct of Marconi's formulation, of Carnielli and Lima-Marques's tableaux system $\mathbf{TC}_1$ and of Buchsbaum and Pequeno's tableaux formulation $S\mathbf{C}1$.

In the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, we introduce da Costa's defined "ball" operator "$\circ$", the generalized operators "k" and "(k)", and the negations "$\sim_k$", for $k \geq 1$, as primitive operators, differently to what has been done in the literature, where these operators are usually defined operators. We prove a Cut Rule for the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$. Then, we prove that each system of this hierarchy is logically equivalent to the corresponding paraconsistent system $\mathbf{C}_n, 1 \leq n < \omega$.

Our systems $\mathbf{TNDC}_n$, as the other mentioned tableaux systems for da Costa's calculi, constitute automated theorem proving systems.

In the system $S\mathbf{C}1^*$ of Buchsbaum and Pequeno we do not have an explicit rule that determines *a priori* when the definition of the operator "$\circ$" must be used or must not be used during the derivations; on account of this it is possible to occur open branches which must be rebuilt, in a distinct way, from the mentioned occurrence of the operator "$\circ$". Also in Carnielli and Lima-Marques's systems $\mathbf{TC}_1$ and $\mathbf{TC}_1^=$ there are not specific rules that determine *a priori* when to use the definition of the operator "$\circ$", what can make necessary to rebuild branches; particularly, in these systems infinite loops may occur, 'postponing indefinitely', according to the own authors, the analysis of formulae that involve the operator of primitive negation and, as a natural consequence, the operator "$\circ$". Carnielli and Lima-Marques prove the decidability of $\mathbf{TC}_1$ and $\mathbf{TC}_1^=$, showing how to deal with the! infinite loops.

In our system $\mathbf{TNDC}_1$, as well as in every $\mathbf{TNDC}_n, 1 \leq n < \omega$, the branches of the tableaux are univocally and automatically generated and infinite loops do not occur. In fact, in the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, we do not apply definitions in generating the branches of the tableaux, for we have specific rules to directly deal with all the operators: the primitive classical connectives for conjunction, disjunction, implication and strong negations, the primitive non-classical connectives "ball" and paraconsistent negation; and the generalized paraconsistent operators "k" and "(k)", of all degree $k, k \geq 1$.

Another peculiarity of our tableaux systems is that, differently to what is in the literature, we define two conditions for the closure of the branches of the tableaux of $\mathbf{TNDC}_n$, for every $n, 1 \leq n < \omega$: either they are closed by the strong negation "$\sim_n$", as usual, or they are closed by the paraconsistent negation "$\neg$" and some additional conditions.

In the systems $\mathbf{TNDC}_n$ it was also necessary to deal with specific problems, concerning relationships between the generalized distinct operators for negation "$\sim_k$" and for the connectives "k" and "(k)", for any $k \geq 1$; and relationships between different systems of the hierarchy $\mathbf{TNDC}_n, 1 \leq n < \omega$.

Finally, we observe that every one of our systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, is introduced from a denumerable (infinite) set of primitive operators, what finally allows us to capture da Costa's systems $\mathbf{C}_n, 1 \leq n < \omega$, as paraconsistent extensions of classical logic.

2 Da Costa's propositional paraconsistent logics C_n

The language L of da Costa's [9,12] paraconsistent systems $\mathbf{C}_n, 1 \leq n \leq \omega$, has as primitive symbols propositional variables, the connectives $\neg, \vee, \&$ and $\supset$, and the parentheses.

The notions of formula and theorem, as well as the general conventions and notations, are the standard ones, as in [18].

Let A and B be formulae. The following operators are added, by definition, to the language L.

Definition 2.1 $A^\circ =_{df} \neg(A \& \neg A)$.

Definition 2.2 $A^k =_{df} A^{\circ\circ\ldots\circ}$ ("$\circ$" k times, for $k \geq 1$).

Definition 2.3 $A^{(k)} =_{df} A^1 \& A^2 \& \ldots \& A^k$, for $k \geq 1$.

Definition 2.4 $\sim_k A =_{df} \neg A \& A^{(k)}$, for $k \geq 1$.

Definition 2.5 $(A \equiv B) =_{df} (A \supset B) \& (B \supset A)$.

For each $\mathbf{C}_n, 1 \leq n < \omega$, the schemata of *axioms* and the *deduction rule* are the following.

AXIOM 1: $A \supset (B \supset A)$

AXIOM 2: $(A \supset B) \supset ((A \supset (B \supset C)) \supset (A \supset C)$

AXIOM 3: $A\&B \supset A$

AXIOM 4: $A\&B \supset B$

AXIOM 5: $A \supset (B \supset A\&B)$

AXIOM 6: $A \supset A \vee B$

AXIOM 7: $A \supset B \vee A$

AXIOM 8: $(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$

AXIOM 9: $\neg\neg A \supset A$

AXIOM 10: $A \vee \neg A$

AXIOM 11^n : $B^{(n)} \supset ((A \supset B) \supset ((A \supset \neg B) \supset \neg A))$

AXIOM 12^n : $A^{(n)}\&B^{(n)} \supset (A\&B)^{(n)}$

AXIOM 13^n : $A^{(n)}\&B^{(n)} \supset (A \vee B)^{(n)}$

AXIOM 14^n : $A^{(n)}\&B^{(n)} \supset (A \supset B)^{(n)}$

RULE OF MODUS PONENS (MP)

$$\frac{A, A \supset B}{B}$$

As in every system $\mathbf{C}_n, 1 \leq n < \omega$, the formulae $A \supset (\neg A \supset B)$ and $\neg A \supset (A \supset B)$ are not valid, da Costa's systems are paraconsistent systems *latu sensu*, that is, from a contradiction it is not possible in general to deduce any formula.

The following result was proved by Arruda (see [ALV 76]).

Theorem 2.6 (Arruda) *The systems* $\mathbf{C}_n, 1 \leq n < \omega$ *are not decidable by finite matrices.*

We indicate [2,3,14,22,15,8] for surveys on da Costa's paraconsistent systems and related results and topics.

3 Tableaux systems for $\mathbf{C}_n, 1 \leq n < \omega$

In this section, we introduce analytical tableaux versions, *à la* Smullyan [23], for the systems $\mathbf{C}_n, 1 \leq n < \omega$, named $\mathbf{TNDC}_n$. We adapt the notion of tableau sequence presented by van Fraassen [17].

The language L of the systems $\mathbf{TNDC}_n$ is the language of the logics $\mathbf{C}_n, 1 \leq n < \omega$, excepting that we consider the symbol "$\circ$" (the ball operator), the symbols "k" and "(k)" for $k \geq 1$, and the negations "$\sim_k$" for $k \geq 1$, as primitive symbols. So, L contains a (infinite) denumerable set of primitive connectives.

The tableaux method is based on expansion rules, which allow us to analyze the formulae of L. Essentially, the expansion rules allow us to expand a sequence of formulae into another sequence of formulae.

Definition 3.1 For every tableaux system $\mathbf{TNDC}_n, 1 \leq n < \omega$, a tableau sequence for a given formula S, or simply a tableau, is a sequence of expressions $A_1, A_2, \ldots, A_k$, such that the formula S is put at the origin of the tableau, as the initial expression A_1; and every expression $A_i, 1 < i \leq k$, corresponds to a finite disjunction A_i^1 or $\ldots$ or $A_i^m, m \geq 1$, where every $A_i^j, 1 \leq j \leq m$, is generated from the preceding expression(s) A_p^j, by applying one of the expansion rules of the system. We call each A_i^j a disjunct of the expression A_i.

Definition 3.2 For every system $\mathbf{TNDC}_n, 1 \leq n < \omega$, a branch j of a tableau sequence, $1 \leq j \leq m$, corresponds to a sequence of expressions $A_i^s, 1 \leq i \leq k$, with A_1^1 the first expression and A_k^j the last one. The superior index s is equal to $1 (s = 1)$, for $1 \leq i \leq i'$, for some $i' \leq k; s = j$, for $i'' \leq i \leq k$, for some $i'' > i'$; and for $i' < i < i''$, s assumes values between 1 and j.

NOTE. - We observe that the tableau sequence has the structure of a *tree*, if we leave out the disjunction, and write the results of applying any rule under the disjunct to which the rule was applied. Thus, by thinking the disjunction as indicating a branching, the tableau sequence has the structure of an ordered dyadic tree *à la* Smullyan [23].

For simplicity, the expressions of a given tableau branch j will be identified as of type A_i^j, with $1 \leq i \leq k$ and fixed $j, 1 \leq j \leq m$.

Definition 3.3 A *node* corresponds to every expression A_i^j of every branch of a tableau, with $1 \leq i \leq k$ and $1 \leq j \leq m$.

Let the letters $\alpha, \beta, \gamma, \ldots, \psi$, if necessary also with indexes, stand for formulae of L.

EXPANSION RULES 10. - *The* expansion rules *of the tableaux systems* $\mathbf{TNDC}_n, 1 \leq n < \omega$, *are the following.*

a) *Rules of Conjunctive Type C*:

$$\frac{\alpha}{\begin{array}{c}\delta_i^j\\ \delta_{i+1}^j\end{array}}$$

α	δ_i^j	δ_{i+1}^j	Name of the Rule
$A\&B$	A	B	$E\&$
$A^{(k)}$	A^k	$A^{(k-1)}$	$E(k), k > 1$
$\neg(A^k)$	A^{k-1}	$\neg(A^{k-1})$	$Ek\neg, k \geq 1$, where A° is A
$\neg(A^{(k)})$	A	$\neg A$	$E(k)\neg, k \geq 1$
$\sim_n \neg A$	$\neg\neg A$	$A^{(n)}$	$E \sim_n \neg$
$\sim_n (A^k)$	$\neg(A^k)$	$(A^k)^{(n)}$	$Ek \sim_n, k \geq 1$
$\sim_n (A \vee B)$	$\sim_n A$	$\sim_n B$	$DND \sim_n$
$\sim_n (A \supset B)$	A	$\sim_n B$	$DNI \sim_n$
$\sim_n (A^{(k)})$	A	$\neg A$	$E(k) \sim_n, k \geq 1$
$\sim_k A$	$\neg A$	$A^{(k)}$	$E \sim_k, k < n$ (i)

b) *Rules of Disjunctive Type D*:

$$\frac{\beta}{\delta_i^j \quad | \quad \delta_i^{j+1}}$$

β	δ_i^j	δ_i^{j+1}	Name of the rule
$A \vee B$	A	B	$E\vee$
$A \supset B$	$\sim_n A$	B	$E \supset$
$\neg(A\&B)$	$\neg A$	$\neg B$	$DNC\neg$, where B is distinct of $\neg A$ (ii)
$\neg(A \vee B)$	$\neg(A^{(n)}\&B^{(n)})$	$\neg A\&\neg B$	$DND\neg$
$\neg(A \supset B)$	$\neg(A^{(n)}\&B^{(n)})$	$A\&\neg B$	$DNI\neg$
$\sim_n (A\&B)$	$\sim_n A$	$\sim_n B$	$DNC \sim_n$

c) *Rules of Special Type* S_1:

$$\frac{\gamma}{\delta_i^j}$$

γ	δ_i^j	Name of the rule
$\neg\neg A$	A	$E\neg\neg$
$\neg \sim_k A$	A	$E\neg \sim_k, k \geq 1$
$\sim_n\sim_k A$	A	$E \sim_n\sim_k, k \geq 1$
$\sim_k A$	$\sim_{k-1} A$	$R \sim_k, k > n$
A^k	$\neg(A^{k-1}\&\neg A^{k-1})$	$Rk, k \geq 1$, where A° is A (iii)
$A^{(1)}$	A^1	$E(1)$

d) *Rules of Special Type* S_2:

$$\frac{\begin{array}{c}\varphi_1\\ \vdots\\ \varphi_m\end{array}}{\delta_i^j}$$

$\varphi_1, \ldots, \varphi_m$	δ_i^j	Name of the Rule
$\{\neg A, A^1, \ldots, A^k\}$	$\sim_k A$	$I \sim_k, k < n$
$\{A^1, A^2, \ldots, A^k\}$	$A^{(k)}$	$I(k), k < n$ (i)

e) *Rules of Special Type* S_3 (iv):

$$\frac{\epsilon}{\varsigma_i^j}$$

ϵ	ς_i^j	Name of the Rule
$A^{\circ\circ\ldots\circ}$	A^k	$E_\circ$ (with "$\circ$" k-times)
$\neg(A^{k-1}\&\neg A^{k-1})$	A^k	$Ik, k \geq 1$, where A° is A
$(A^s)^k$	A^{s+k}	$Is + k$, for $s, k \geq 1$
$A^1\&A^2\&\ldots\&A^k$	$A^{(k)}$	$I'(k), k \geq 1$ (v)
$\neg A\&A^{(k)}$	$\sim_k A$	$I' \sim_k, k \geq 1$ (v)

(i) *This rule must be applied only once, on every branch and for every formula.*

(ii) *If A is of type $(C^{k-1}\&\neg(C^{k-1}))$, then B must be distinct of C^k.*

(iii) *This rule must be applied only when there is no possibility of applying any* *other Rule; it can be applied in subformulas of formulas that occur in the nodes and, in these cases, it must be applied "from outside to inside", that is, from the connective of largest scope to the connective of smallest scope.*

(iv) *The Rules of Special Type S_3 must be immediately applied, in every case, after applying the first Rule in the initial node of the tableau; they can be applied to subformulas of formulas that occur in the nodes and, in these cases, they must be applied "from outside to inside".*

(v) *These rules, under conditions (iv), can only be applied to proper subformulas of formulas that occur in the nodes and, in these cases, they must be applied "from outside to inside".*

Note 1

- *In the application of the Expansion Rules, it is more efficient to give priority to the Rules of Type C and to the Rules of Special Type.*
- *We observe that A°, which corresponds to the formula A with superior index "0" (numeral 0), coincides with the formula A. This formula is distinct of the formula A° ("A-ball").*

In the Rules of Special Type S_2 we use the notation of set in order to indicate that it is not important the order in which the formulas occur in the nodes of a branch.

Also, the only rules that can be applied to subformulas, are the Rules of Special Type S_3 and the Rule Rk.

Definition 3.4 For every system $\mathbf{TNDC}_n, 1 \leq n < \omega$, a branch $A_1^j, \ldots, A_s^j$ of a tableau is called a closed branch if there exist nodes $A_r^j, 1 \leq r \leq s$, that correspond either to formulae B and $\sim_n B$, or to formulae $B, \neg B$ and $B^1, B^2, \ldots, B^n$.

Definition 3.5 Given a formula S, a tableau for S is closed if all its branches are closed; otherwise, it is said to be open.

Definition 3.6 A set of formulae Γ is said to be closed if, and only if, there exists a finite subset Γ_0 of Γ, such that there exists a closed tableau for the conjunction of the formulae of Γ_0; otherwise, it is said to be open.

In what follows, we use Γ, A as an abbreviation for $\Gamma \cup \{A\}$.

Definition 3.7 For every tableaux system $\mathbf{TNDC}_n, 1 \leq n < \omega$, a formula S is said to be an analytical consequence of a set Γ of formulae if, and only if, $\Gamma, \sim_n S$ is closed. We also say that Γ, by the Expansion Rules, generates S.

This is denoted by: $\Gamma \vdash_{TNDC_n} S$.

Definition 3.8 For every tableaux system $\mathbf{TNDC}_n, 1 \leq n < \omega$, a formula S is said to be provable if, and only if, there is a closed tableau for $\sim_n S$, that is, if $\{\sim_n S\}$ is closed.

This is denoted by: $\vdash_{TNDC_n} S$.

Example 3.9 Next, we present some examples of proofs in the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$. The rules used are indicated to the right of each step of the proof; the numbers on the left side are added only to facilitate mentioning the tableau.

a) $\vdash_{TNDC_1} ((A^{(2)})^\circ)$

$$
\begin{array}{clc}
1 & \sim_1 ((A^{(2)})^\circ) & \\
 & \downarrow & \\
2 & \sim_1 ((A^{(2)})^1) & 1,\ \mathrm{E}\circ \\
 & \downarrow & \\
3 & \neg((A^{(2)})^1) & 2,\ \mathrm{E}1{\sim_1} \\
 & \downarrow & \\
4 & (((A^{(2)})^1))^{(1)} & 2,\ \mathrm{E}1{\sim_1} \\
 & \downarrow & \\
5 & A^{(2)} & 3,\ \mathrm{E}1\neg \\
 & \downarrow & \\
6 & \neg(A^{(2)}) & 3,\ \mathrm{E}1\neg \\
 & \downarrow & \\
7 & A^2 & 5,\ \mathrm{E}(2) \\
 & \downarrow & \\
8 & A^{(1)} & 5,\ \mathrm{E}(2) \\
 & \downarrow & \\
9 & A^1 & 8,\ \mathrm{E}(1) \\
 & \downarrow & \\
10 & A & 6,\ \mathrm{E}(2)\neg \\
 & \downarrow & \\
11 & \neg A & 6,\ \mathrm{E}(2)\neg
\end{array}
$$

The tableau is closed by the formulas that occur in the nodes 9, 10 and 11.

b) $\vdash_{TNDC_{15}} ((A \supset B) \supset A) \supset A$

$$
\begin{array}{llcl}
1 & & \sim_{15} (((A \supset B) \supset A) \supset A) & \\
 & & \downarrow & \\
2 & & ((A \supset B) \supset A) & 1, \text{DNI}\sim_{15} \\
 & & \downarrow & \\
3 & & \sim_{15} A & 1, \text{DNI}\sim_{15} \\
 & & \swarrow \qquad\qquad \searrow & \\
4 & \sim_{15} (A \supset B) \;\; 2, \text{DNI}\sim_{15} & & 5 \; A \; 2, \text{DNI}\sim_{15} \\
 & \downarrow & & \\
6 & A & 4, \text{DNI}\sim_{15} & \\
 & \downarrow & & \\
7 & \sim_{15} B & 4, \text{DNI}\sim_{15} &
\end{array}
$$

The tableau is closed by the formulas $\sim_{15} A$ and A, that occur in the nodes 3 and 6 of the first branch, and in the nodes 3 and 5 of the second branch.

c) $\vdash_{TNDC_n} (\neg(\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))\&\neg\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))))^s \supset (A^{\circ\circ\circ})^s$

$$
\begin{array}{lll}
1 & \sim_n ((\neg(\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))\&\neg\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))))^s \supset (A^{\circ\circ\circ})^s) & \\
 & \downarrow & \\
2 & (\neg(\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))\&\neg\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))))^s & 1, \text{DNI}\sim_n \\
 & \downarrow & \\
3 & \sim_n ((A^{\circ\circ\circ})^s) & 1, \text{DNI}\sim_n \\
 & \downarrow & \\
4 & ((\neg(\neg(A\&\neg A)\&\neg\neg(A\&\neg A))^1)^s & 2, \text{I1} \\
 & \downarrow & \\
5 & (((\neg(A\&\neg A))^1)^1)^s & 4, \text{I1} \\
 & \downarrow & \\
6 & (((A^1)^1)^1)^s & 5, \text{I1} \\
 & \downarrow & \\
7 & ((A^1)^1)^{1+s} & 6, \text{I1}+s \\
 & \downarrow & \\
8 & (A^1)^{2+s} & 7, \text{I1}+(1+s) \\
 & \downarrow & \\
9 & A^{3+s} & 8, \text{I1}+(2+s) \\
 & \downarrow & \\
10 & \sim_n ((A^3)^s) & 3, \text{E}\circ \\
 & \downarrow & \\
11 & \sim_n (A^{3+s}) & 10, \text{I3}+s
\end{array}
$$

The tableau is closed by the formulas that occur in the nodes 9 and 11, that is, A^{3+s} and $\sim_n (A^{3+s})$.

4 The Cut Rule for the systems $\mathbf{TNDC}_n$

Next, we present a special version of the Cut Rule for the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$.

Theorem 4.1 (CUT RULE [8]) *For every system* $\mathbf{TNDC}_n, 1 \leq n < \omega$, *there exists a closed tableau for a set* Γ *of formulae if, and only if, for a given formula* S *there exist closed tableaux either for* $\Gamma \cup \{S\}$ *and* $\Gamma \cup \{\sim_n S\}$, *or for* $\Gamma \cup \{S\}$ *and* $\Gamma \cup \{\neg S, S^1, S^2, \ldots, S^n\}$.

Proof. If there exists a closed tableau for Γ, it is immediate that there are closed tableaux either for Γ, S and $\Gamma, \sim_n S$, or for Γ, S and $\Gamma, \neg S, S^1, S^2, \ldots, S^n$.

Now, suppose that either there exist closed tableaux for Γ, S and $\Gamma, \sim_n S$, or there exist closed tableaux for Γ, S and $\Gamma, \neg S, S^1, S^2, \ldots, S^n$. The proof that there exists a closed tableau for Γ is done by induction on the complexity of the formula S.

1) Let S be an atomic formula A.

Suppose that there are closed tableaux for Γ, A and $\Gamma, \sim_n A$. In the cases when either $A \in \Gamma$ or $\sim_n A \in \Gamma$, it is immediate that Γ is closed. Hence, we have only to analyze the case when $A \notin \Gamma$ and $\sim_n A \notin \Gamma$. If either Γ, A or $\Gamma, \sim_n A$ is closed only on account of formulae of Γ, then Γ is closed and we have nothing to prove; the same reasoning is applicable to the case when we have that Γ, A and $\Gamma, \neg A, A^1, A^2, \ldots, A^n$ are closed.

1.1) Suppose that there are closed tableaux for Γ, A and for $\Gamma, \sim_n A$. Observe that from A atomic we can not generate any formula, and from $\sim_n A$ we also can not generate any formula.

If Γ, A is closed then there is a tableau **T** such that its branches are closed either by $\sim_n A$, or by $\neg A, A^1, A^2, \ldots, A^n$. As $\Gamma, \sim_n A$ is also closed, then there is a closed tableau **T**' such that its branches are closed by A, or by $\sim_n \sim_n A$, or by $\neg \sim_n A$ and $(\sim_n A)^1, (\sim_n A)^2, \ldots, (\sim_n A)^n$; that is, by Rules E$\sim_n\sim_k$ and E$\neg \sim_k$, the formula A appears in all the branches of **T**'.

Therefore, in the tableaux **T** and **T**' the formulae $\sim_n A, \neg A, A^1, A^2, A^n$ (in **T**), and A (in **T**'), respectively, are directly generated, by the Expansion Rules from Γ, because, neither $\sim_n A, \neg A, A^1, A^2, A^n$ could be generated from A, nor A could be generated from $\sim_n A$.

Hence, there is a closed tableau for Γ and Γ is closed.

1.2) Suppose that there are closed tableaux for Γ, A and for $\Gamma, \neg A, A^1, A^2, \ldots, A^n$. Observe that it is not possible to generate any formula from A and from $\neg A, A^1, A^2, \ldots, A^n$ it is only possible to generate $\sim_k A$ and $A^{(k)}, k < n$ (by Rules I$\sim_k$ and I(k)).

If Γ, A is closed, then there is a tableau **T** such that its branches are closed either by $\sim_n A$, or by $\neg A$ and $A^1, A^2, \ldots, A^n$. As $\Gamma, \neg A, A^1, A^2, \ldots, A^n$ is also closed, then there is a closed tableau **T**' such that its branches are closed by A; or by $\sim_n \neg A$; or by $\neg\neg A, (\neg A)^1, (\neg A)^2, \ldots, (\neg A)^n$; or by $\sim_n (A^i)$, for every $i, 1 \leq i \leq n$; or by $\neg(A^i), (A^i)^1, (A^i)^2, \ldots, (A^i)^n$, for every $i, 1 \leq i \leq n$. So, by Rules $E \sim_n \neg, E\neg\neg, Ek \sim_n, E\&$ and $Ek\neg$, the formula A appears in all the branches of **T**'.

Therefore, in the tableaux **T** and **T**' the formulae $\sim_n A$ or $\neg A, A^1, A^2, \ldots, A^n$ (in **T**) and A (in **T**'), respectively, are directly generated, by the Expansion Rules, from Γ.

Hence, there exists a closed tableau for Γ and so, Γ is closed.

2) Suppose that the result holds for formulae S of complexity $p, p > 0$.

3) Let S be a formula of complexity $p + 1$.

3.1) Let S be of type $\neg B$, with B of complexity p.

3.1.1) Suppose that Γ, $\neg B$ and $\Gamma, \sim_n \neg B$ are closed, considering that $\neg B$ and $\sim_n \neg B$ are not formulae of Γ.

3.1.2) Suppose that $\Gamma, \neg B$ and $\Gamma, \neg\neg B, (\neg B)^1, (\neg B)^2, \ldots, (\neg B)^n$ are closed, also considering that $\neg B, \neg\neg B$ and $(\neg B)^i$, for every $i, 1 \leq i \leq n$, are not formulae of Γ.

3.2) Let S be of type $B^k, k \geq 1$.

3.2.1) Suppose that Γ, B^k and $\Gamma, \sim_n (B^k)$ are closed.

3.2.2) Suppose that there are closed tableaux for Γ, B^k and for $\Gamma, \neg(B^k), (B^k)^1, (B^k)^2, \ldots, (B^k)^n$, observing that from $\neg(B^k)$, by Rule $Ek\neg$, it is only possible to generate B and $\neg B$.

3.3) Let S be of type $B^{(k)}$, with $k \geq 1$.

3.3.1) Suppose that $\Gamma, B^{(k)}$ and $\Gamma, \sim_n (B^{(k)})$ are closed. Observe that, by Rule $E(k)$, from $B^{(k)}$ it is only possible to generate $B^{(k-1)}, B^{(k-2)}, \ldots, B^{(1)}, B^k, B^{k-1}, \ldots, B^1$; and by Rule $E(k) \sim_n$, from $\sim_n (B^{(k)})$ it is only possible to generate $\neg(B^{(k)}), B$ and $\neg B$, for every k.

3.3.2) Suppose that $\Gamma, B^{(k)}$ and $\Gamma, \neg(B^{(k)}), (B^{(k)})^1, (B^{(k)})^2, \ldots, (B^{(k)})^n$ are closed. Observe that, from $\neg(B^{(k)})$ by Rule $E(k)\neg$, it is only possible to generate B and $\neg B$; and that, from $(B^{(k)})^i$, for $1 \leq i \leq k$, in fact it is only possible to generate the formula $\neg((B^{(k)})^{i-1}\&\neg((B^{(k)})^i))$.

3.4) Let S be of type $\sim_k B$, with $k \geq 1$.

3.5) Let S be of type $(B\&C)$.

3.6) Let S be of type $(B \vee C)$.

3.7) Let S be of type $(B \supset C)$.

Hence, by *Cases* 1-3, we have proved the theorem. $\square$

5 The logical equivalence between the systems of the hierarchy $\mathbf{TNDC}_n$ and the corresponding da Costa's systems $\mathbf{C}_n, 1 \leq n < \omega$

Now, based on the Cut Rule for $\mathbf{TNDC}_n, 1 \leq n < \omega$, we can prove the equivalence between the systems $\mathbf{TNDC}_n$ and the corresponding da Costa's paraconsistent systems $\mathbf{C}_n, 1 \leq n < \omega$.

Theorem 5.1 ([8]) *The systems* $\mathbf{TNDC}_n, 1 \leq n < \omega$, *constitute a hierarchy of tableaux systems, such that every system* $\mathbf{TNDC}_n$ *is equivalent to da Costa's corresponding paraconsistent system* $\mathbf{C}_n, 1 \leq n < \omega$.

Proof.
1) *If* $\Gamma \vdash_{C_n} S$, then $\Gamma \vdash_{TNDC_n} S$, *for every* $n, 1 \leq n < \omega$.

Suppose that $\Gamma \vdash_{C_n} S$. If $S \in \Gamma$ then, for every $n, 1 \leq n < \omega$, it is immediate that $\Gamma \vdash_{TNDC_n} S$. So, let us suppose that S is not in Γ.

1.1) Let S be an axiom schema of $\mathbf{C}_n, 1 \leq n < \omega$.

Let us prove that $\Gamma \vdash_{TNDC_n} S$, that is, we have to prove that $\Gamma, \sim_n S$ is closed in $\mathbf{TNDC}_n, 1 \leq n < \omega$.

Here, we only present the proof for Axiom schemata 11^n.

Let S be Axiom 11^n, that is, S is $A^{(n)} \supset ((B \supset A) \supset ((B \supset \neg A) \supset \neg B))$. We shall generate a closed tableau, whose initial node, $\Gamma, \sim_n S$, constitutes the step 1 below.

1	$\Gamma, \sim_n (A^{(n)} \supset ((B \supset A) \supset ((B \supset \neg A) \supset \neg B))$	
2	$\Gamma, A^{(n)}$	1, DNI$\sim_n$
3	$\Gamma, \sim_n (((B \supset A) \supset ((B \supset \neg A) \supset \neg B)))$	1, DNI$\sim_n$
4	$\Gamma, (B \supset A)$	3, DNI$\sim_n$
5	$\Gamma, \sim_n ((B \supset \neg A) \supset \neg B)$	3, DNI$\sim_n$
6	$\Gamma, (B \supset \neg A)$	5, DNI$\sim_n$
7	$\Gamma, \sim_n \neg B$	5, DNI$\sim_n$
8	$\Gamma, \neg\neg B$	7, E$\sim_n \neg$
9	$\Gamma, B^{(n)}$	7, E$\sim_n \neg$
10	Γ, B	8, E$\neg\neg$
11	Γ, A^n	2, E(k)
12	$\Gamma, A^{(n-1)}$	2, E(k)
13	Γ, A^{n-1}	12, E(k)
14	$\Gamma, A^{(n-2)}$	12, E(k)
15	Γ, A^{n-2}	14, E(k)
$\vdots$	$\vdots$	
$i-1$	$\Gamma, A^{(1)}$	$i-3$ E(k)
i	Γ, A^1	$i-1$ E(1)

$i+1$ $\Gamma, \sim_n B$ 4, E$\supset$ $i+2$ $\Gamma,\ A$ 4, E$\supset$

$i+3$ $\Gamma, \sim_n B$ 6, E$\supset$ $i+4$ $\Gamma, \neg A$ 6, E$\supset$

In this case, the branches of the tableau close by the two distinct closure conditions: the tableau closes in the first and the second branches by the formulae $\sim_n B$ and B, occurring on the nodes 10 e $(i+1)$, 10 and $(i+3)$, respectively; in the third branch closes by the formulae $A^n, A^{n-1}, \ldots, A^1, A$ and $\neg A$, that occur on the nodes $11, 13, 15, \ldots, i, (i+2)$ and $(i+4)$.

1.2) Now, let us consider that the formula S is a consequence of preceding formulae in the proof, in $\mathbf{C}_n, 1 \leq n < \omega$, by *Modus Ponens*; that is, we have that $\Gamma \vdash_{C_n} S$ is a consequence of $\Gamma \vdash_{C_n} A$ and $\Gamma \vdash_{C_n} A \supset S$. Then, as we have that $\Gamma \vdash_{TNDC_n} A$ and $\Gamma \vdash_{TNDC_n} A \supset S$, the sets $\Gamma \cup \{\sim_n A\}$ and $\Gamma \cup \{\sim_n (A \supset S)\}$ are closed in $\mathbf{TNDC}_n$, and so, by Rule **DNI**$\sim_n$ $\Gamma \cup \{\sim_n A\}$ and $\Gamma \cup \{A, \sim_n S\}$ are closed. So, $\Gamma, \sim_n S, A$ and $\Gamma, \sim_n S, \sim_n A$ are closed and, by the Cut Rule, $\Gamma, \sim_n S$ is closed. Therefore, Γ generates S in $\mathbf{TNDC}_n, 1 \leq n < \omega$, that is, $\Gamma \vdash_{TNDC_n} S$.

2) *If* $\Gamma \vdash_{TNDC_n} S$, *then* $\Gamma \vdash_{C_n} S$.

Suppose that $\Gamma \vdash_{TNDC_n} S$ and S is not in Γ.

In order to prove the theorem, let us consider S as a formula generated from Γ by the expansion rules of $\mathbf{TNDC}_n, 1 \leq n < \omega$.

We shall transform every Expansion Rule of $\mathbf{TNDC}_n, 1 \leq n < \omega$, into a correspondent valid proof in $\mathbf{C}_n, 1 \leq n < \omega$. That is, the rules of conjunctive, disjunctive and special types will be transformed into the proofs of $\alpha \vdash_{C_n} (\delta_i^j)\&(\delta_{i+1}^j); \beta \vdash_{C_n} (\delta_i^j) \vee (\delta_{i+1}^j); \gamma \vdash_{C_n} \delta_i^j; \varphi_1, \ldots, \varphi_n \vdash_{C_n} \delta_i^j$ and $\epsilon \vdash_{C_n} \delta_i^j$, respectively.

We shall only present the complete proofs relative to some of the Expansion Rules involving the strong and weak negations, and the operator "k".

2.1) Let S be of type $\neg\neg A\&(A)^{(n)}$, generated, in $\mathbf{TNDC}_n, 1 \leq n < \omega$, from $\sim_n \neg A$, by Rule E$\sim_n \neg$. We have to prove that $\sim_n \neg A \vdash_{C_n} \neg\neg A\&(A)^{(n)}$.

1.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \sim_n (\neg\neg A\&(A)^{(n)})$	property of $\vdash_{C_n}$
2.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \sim_n \neg\neg A \vee \sim_n ((A)^{(n)})$	1, property of $\vdash_{C_n}$
3.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \sim_n \neg A$	property of $\vdash_{C_n}$
4.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg\neg A\&(\neg A)^{(n)}$	3, Definition 2.4
5.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg\neg A$	4, Axiom 3, MP
6.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \sim_n ((A)^{(n)})$	2, 5, property of $\vdash_{C_n}$
7.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg((A)^{(n)})\&(((A)^{(n)})^{(n)})$	6, Definition 2.4
8.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg((A)^{(n)}))$	7, Axiom 3, MP
9.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg A$	8, (1), Axiom 4, MP
10.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg A\& \sim_n \neg A$	9, 3, Axiom 5, MP
11.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} \neg A\&(\neg\neg A\&(\neg A)^{(n)})$	10, Definition 2.4
12.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} (\neg A\&(\neg\neg A\&(\neg A)^{(n)})) \supset (\neg\neg A\&(A)^{(n)})$	property of $\vdash_{C_n}$
13.	$\sim_n \neg A, \sim_n (\neg\neg A\&(A)^{(n)}) \vdash_{C_n} (\neg\neg A\&(A)^{(n)})$	11, 12, MP
14.	$\sim_n \neg A \vdash_{C_n} (\sim_n (\neg\neg A\&(A)^{(n)})) \supset (\neg\neg A\&(A)^{(n)}))$	13, Deduction Theorem
15.	$\sim_n \neg A \vdash_{C_n} (\sim_n\sim_n (\neg\neg A\&(A)^{(n)})) \vee (\neg\neg A\&(A)^{(n)}))$	14, property of $\vdash_{C_n}$
16.	$\sim_n \neg A \vdash_{C_n} (\sim_n\sim_n (\neg\neg A\&(A)^{(n)})) \supset (\neg\neg A\&(A)^{(n)}))$	property of $\vdash_{C_n}$
17.	$\sim_n \neg A \vdash_{C_n} (\neg\neg A\&(A)^{(n)}) \supset (\neg\neg A\&(A)^{(n)}))$	property of $\vdash_{C_n}$
18.	$\sim_n \neg A \vdash_{C_n} \neg\neg A\&(A)^{(n)}$	16, 17, 15, Axiom 8, MP

2.2) Let S be of type $((A^{k-1})\&\neg(A^{k-1}))$, generated, in $\mathbf{TNDC}_n, 1 \leq n < \omega$, from $\neg(A^k)$, by Rule E$k\neg, k > 1$. We have to prove that $\neg(A^k) \vdash_{C_n} (A^{k-1})\&\neg(A^{k-1})$.

1.	$\neg(A^k) \vdash_{C_n} \neg(A^k)$	property of $\vdash_{C_n}$
2.	$\neg(A^k) \vdash_{C_n} \neg(\neg((A^{k-1})\&\neg(A^{k-1})))$	1, Definition 2.2
3.	$\neg(A^k) \vdash_{C_n} (A^{k-1})\&\neg(A^{k-1})$	2, Axiom 9, MP

2.3) Let S be of type $\neg(A^k)$, generated, from $\sim_n (A^k)$, by Rule E$k \sim_n (k \geq 1)$. We have to prove that $\sim_n (A^n) \vdash_{C_n} \neg(A^k)$.

By Definition 2.4 and Axiom 3, the proof is immediate.

2.4) Let S be of type $(A^k\&A^{(k-1)})$, generated, in $\mathbf{TNDC}_n, 1 \leq n < \omega$, from $A^{(k)}$, by Rule E(k), with $k > 1$. We have to prove that $A^{(k)} \vdash_{C_n}$

$A^k \& A^{(k-1)}$.

The proof is immediate, by Definition 2.3.

2.5) Let S be (A^k), generated, in $\mathbf{TNDC}_n, 1 \leq n < \omega$, from $\neg(A^{k-1} \& \neg A^{k-1})$, by Rule E$\neg$. We have to prove that $\neg(A^{k-1} \& \neg A^{k-1}) \vdash_{C_n} A^k$.

The result is immediate, by Definition 2.2.

□

As every system $\mathbf{TNDC}_n, 1 \leq n < \omega$, is equivalent to the corresponding $\mathbf{C}_n, 1 \leq n < \omega$, the syntactical and semantical results concerning the $\mathbf{TNDC}_n$ are immediate. So, the soundness and completeness of our tableaux systems are immediate.

Besides, the decidability of the systems $\mathbf{TNDC}_n, 1 \leq n < \omega$, can also be proved, from the characteristics of the Expansion Rules of the systems. For every formula S we have to check, in a finite number of steps, either if $\sim_n S$ is closed, or if $\sim_n S$ is not closed; for every tableau for $\sim_n S$, in the case when $\sim_n S$ is not closed, we have to generate at least a finite, open and complete branch.

We intend to develop this proof in a future paper.

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