Almost all quartic half-arc-transitive weak metacirculants of Class II are of Class IV

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A B S T R A C T

A half-arc-transitive graph is a vertex- and edge- but not arc-transitive graph. A weak metacirculant is a graph admitting a transitive metacyclic group that is a group generated by two automorphisms \( \rho \) and \( \sigma \), where \( \rho \) is \((m, n)\)-semiregular for some integers \( m \geq 1 \) and \( n \geq 2 \), and where \( \sigma \) normalizes \( \rho \). It was shown in [D. Marušič, P. Šparl, On quartic half-arc-transitive metacirculants, J. Algebr. Comb. 28 (2008) 365–395] that each connected quartic half-arc-transitive weak metacirculant \( X \) belongs to one (or possibly more) of four classes of such graphs, reflecting the structure of the quotient graph \( X_\rho \) relative to the semiregular automorphism \( \rho \). The first of these classes, called Class I, coincides with the class of so-called tightly attached graphs. Class II consists of the quartic half-arc-transitive weak metacirculants for which the quotient graph \( X_\rho \) is a cycle with a loop at each vertex. Class III consists of those graphs for which each vertex of the quotient graph \( X_\rho \) is connected to three other vertices, to one with a double edge. Finally, Class IV consists of those graphs for which \( X_\rho \) is a simple quartic graph.

This paper consists of two results concerning graphs of Class II. It is shown that, with the exception of the Doyle–Holt graph and its canonical double cover, each quartic half-arc-transitive weak metacirculant of Class II is also of Class IV. It is also shown that although quartic half-arc-transitive weak metacirculants of Class II which are not tightly attached exist they are “almost tightly attached”. More precisely, their radius is at most four times their attachment number.

1. Introductory remarks

Throughout this paper graphs are assumed to be finite and, unless stated otherwise, simple, connected and undirected (but with an implicit orientation of the edges when appropriate). For group- and graph-theoretic concepts not defined here we refer the reader to [4, 18], and [7], respectively.

Given a graph \( X \), we let \( V(X) \), \( E(X) \), \( A(X) \) and \( \text{Aut}X \) be the vertex set, the edge set, the arc set and the automorphism group of \( X \), respectively. A graph \( X \) is said to be vertex-transitive, edge-transitive and arc-transitive if its automorphism group \( \text{Aut}X \) acts transitively on \( V(X) \), \( E(X) \) and \( A(X) \), respectively. We say that \( X \) is half-arc-transitive if it is vertex- and edge-transitive but not arc-transitive.

The study of half-arc-transitive graphs, which was initiated more than forty years ago by Tutte [17], has received a lot of attention over the past two decades and is still a very active topic of research. Half-arc-transitive graphs have been considered from various different aspects and many interesting results have been obtained thus far (the interested reader...
is referred to [6,9,11,12,15] and the references therein). However, despite all the efforts there are still numerous questions about these graphs that need to be answered. Indeed, even the family of quartic half-arc-transitive graphs, which was studied the most, seems to be too rich to be completely classified. We are thus forced to restrict ourselves to some of its subfamilies.

One of the possible directions to take was proposed in [12] where the investigation of quartic half-arc-transitive weak metacirculants was initiated. (A weak metacirculant is a graph admitting a transitive metacyclic group \( \langle \rho, \sigma \rangle \) where \( \rho \) is a semiregular automorphism normalized by \( \sigma \).) It was shown that every quartic half-arc-transitive weak metacirculant belongs to one (or possibly more) of four classes of such graphs, reflecting the structure of the quotient graph relative to the semiregular automorphism \( \rho \). (The four classes are described in Section 2.) In [12] two of these classes (Class I and Class II) were thoroughly investigated. In particular, it was shown that Class I coincides with the class of so-called tightly attached graphs, which have already been completely classified [10,14]. (Quartic tightly attached graphs were introduced in [10] as quartic half-arc-transitive graphs in which two non-disjoint alternating cycles meet in precisely half of their vertices; see Section 2 for a precise definition.) The investigation of quartic half-arc-transitive weak metacirculants was continued in [15] where Class II was completely classified. Nevertheless, the question of how the four classes of quartic half-arc-transitive weak metacirculants relate to each other has not received much attention until now. It is this question that this paper is concerned with. In particular we focus on Class II and its connection to Classes I and IV.

The following are the main results of this paper.

**Theorem 1.1.** Let \( X \) be a connected quartic half-arc-transitive weak metacirculant of Class II. If \( X \) is not tightly attached then either \( R = 2A \) or \( R = 4A \) holds, where \( R \) and \( A \) are the radius and the attachment number of \( X \), respectively. Moreover, letting \( X \cong Y(m, n; r, t) \), where the parameters \( m, n, r \) and \( t \) satisfy the conditions of Proposition 2.1 and letting \( \alpha = m + \frac{m(m-1)}{2}(r-1) \), we have that

(i) \( X \) is tightly attached if and only if \( r-1 \in \langle t-\alpha \rangle \),

(ii) \( R = 2A \) if and only if \( 2(r-1) \in \langle t-\alpha \rangle \) and \( r-1 \notin \langle t-\alpha \rangle \) and

(iii) \( R = 4A \) if and only if \( 2(r-1) \notin \langle t-\alpha \rangle \).

**Theorem 1.2.** Let \( X \) be a connected quartic half-arc-transitive weak metacirculant of Class II. Then \( X \) is also of Class IV if and only if it is not isomorphic to the Doyle–Holt graph \( \overline{y}(3, 9; 4, 6) \) or its canonical double cover \( \overline{y}(3, 18; 7, 15) \).

The graphs \( \overline{y}(m, n; r, t) \) are defined in the following section.

The paper is organized as follows. In Section 2 notation is fixed, some results from the literature that will be needed in the subsequent sections are stated, and the four classes of quartic half-arc-transitive weak metacirculants and the graphs \( \overline{y}(m, n; r, t) \) are introduced. Then Sections 3 and 4 are devoted to the proofs of Theorems 1.1 and 1.2, respectively.

### 2. Preliminaries

We start by introducing basic notation that will be used throughout the rest of this paper. Let \( X \) be a graph. The fact that \( u \) and \( v \) are adjacent vertices of \( X \) will be denoted by \( u \sim v \); the corresponding edge will be denoted by \( uv \). In an oriented graph the fact that the edge \( uv \) is oriented from \( u \) to \( v \) will be denoted by \( u \rightarrow v \). In this case the vertex \( u \) is referred to as the tail and \( v \) is referred to as the head of the edge \( uv \).

Let \( m \geq 1 \) and \( n \geq 2 \) be integers. An automorphism of a graph is called \((m, n)\)-semiregular if it has \( m \) orbits of length \( n \) and no other orbit. We say that a graph \( X \) is an \((m, n)\)-weak metacirculant graph (in short an \((m, n)\)-weak metacirculant) if there exists an \((m, n)\)-semiregular automorphism \( \rho \) of \( X \), together with an additional automorphism \( \sigma \) of \( X \) normalizing \( \rho \) such that \((\rho, \sigma)\) is transitive. Note that this implies that \( \sigma \) cyclically permutes the orbits of \( \rho \). To stress the role of these two automorphisms in the definition of \( X \) we shall say that \( X \) is an \((m, n)\)-weak metacirculant relative to the ordered pair \((\rho, \sigma)\).

A graph \( X \) is a weak metacirculant if it is an \((m, n)\)-weak metacirculant for some \( m \) and \( n \). Note that a weak metacirculant is a generalization of a metacirculant, introduced by Alspach and Parsons [2], which is a weak metacirculant in which \( \sigma^n \) fixes at least one vertex. We remark that an \((m, n)\)-weak metacirculant need not be an \((m, n)\)-metacirculant. For instance, it can be seen that the graph \( \overline{y}(10; 100; 11, 90) \) (the \( \overline{y}(m, n; r, t) \) graphs are defined in the next paragraph), which is in fact a half-arc-transitive (10, 100)-weak metacirculant, is not a (10, 100)-metacirculant. Let us also point out that half-arc-transitivity of some special families of quartic metacirculant graphs has been considered before (see for instance [9,13,19]). In particular, the papers [9,19] consider graphs of prime power order, allowing a much more group-theoretic approach.

We now present a family of quartic weak metacirculants that will play a central role in this paper. For each \( m \geq 3, n \geq 3, r \in \mathbb{Z}_n^* \) and \( t \in \mathbb{Z}_n \), where \( r^m = 1 \) and \( t(r - 1) = 0 \), let \( \overline{y}(m, n; r, t) \) be the graph with vertex set \( V = \{u_i^j \mid i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\} \) and edges defined by the following adjacencies:

\[
\begin{align*}
\{u_i^j + r, u_i^j + 1, \; i \in \mathbb{Z}_m \setminus \{m - 1\}, \; j \in \mathbb{Z}_n \\
\{u_i^{j+m-1}, u_i^0 + r, \; i = m - 1, \; j \in \mathbb{Z}_n. \}
\end{align*}
\]

(Here \( \mathbb{Z}_n \) denotes the ring of residue classes modulo \( n \) and \( \mathbb{Z}_n^* \) denotes the multiplicative group of the invertible elements of \( \mathbb{Z}_n \).) Observe that the permutations \( \rho \) and \( \sigma \), defined by the rules
are automorphisms of $\mathcal{Y}(m, n; r, t)$. Clearly $\rho$ is $(m, n)$-semiregular and $(\rho, \sigma)$ is transitive. Moreover,

$$\sigma^{-1} \rho \sigma = \rho^m \quad \text{and} \quad \sigma^m = \rho^t,$$

and so $\mathcal{Y}(m, n; r, t)$ is a weak $(m, n)$-metacirculant relative to the ordered pair $(\rho, \sigma)$. We remark that the Doyle–Holt graph, the smallest half-arc-transitive graph (see [1, 5, 8]), is isomorphic to $\mathcal{Y}(3, 9; 4, 6)$ whereas its canonical double cover, which is also the smallest graph in the family of graphs constructed by Bouwer [3], is isomorphic to $\mathcal{Y}(3, 18; 7, 15)$. Note that each graph $X = \mathcal{Y}(m, n; r, t)$ admits the following orientation consistent with the action of $\rho$ and $\sigma$. We let $u_0^i \to u_0^{i+1}$ and $u_0 \to u_1$ and then distribute this orientation throughout $X$ according to the action of $\rho$ and $\sigma$. Therefore,

$$u_i^j \to \begin{cases} u_i^{j+1}, & i \in \mathbb{Z}_m, \ j \in \mathbb{Z}_n \ \text{or} \\ u_i^j, & i = m - 1, \ j \in \mathbb{Z}_n, \end{cases}$$

Let us now introduce the four classes of quartic half-arc-transitive weak metacirculants. Let $X$ be any connected quartic half-arc-transitive $(m, n)$-metacirculant relative to the ordered pair $(\rho, \sigma)$. Denote the orbits of $\rho$ by $X_i$, where $i \in \mathbb{Z}_m$, in such a way that $X_i \sigma = X_{i+1}$ and let $d_{im}$ denote the valency of the subgraphs induced on $X_i$. Furthermore, let $X_0$ denote the corresponding quotient (multi)graph relative to $\rho$, whose vertex set is the set of orbits of $\rho$ with orbits $X_i$ and $X_j$ connected by $d$ edges whenever each vertex of $X_i$ is adjacent to $d$ vertices of $X_j$. In [12] it was shown that $X$ belongs to one (or possibly more) of the following four classes of graphs.

- **Class I.** The graph $X$ belongs to Class I if $d_{im}(X) = 0$ and each orbit $X_i$ is connected (with a double edge) to two other orbits. In view of connectedness of $X$, we have that $X_0$ is a “double-edge” cycle.

- **Class II.** The graph $X$ belongs to Class II if $d_{im}(X) = 2$ and each orbit $X_i$ is connected (with a single edge) to two other orbits. In view of connectedness of $X$, we have that $X_0$ is a cycle (with a loop at each vertex).

- **Class III.** The graph $X$ belongs to Class III if $d_{im}(X) = 0$ and each orbit $X_i$ is connected to three other orbits, to one with a double edge and to two with a single edge. Clearly, $m$ must be even in this case and an orbit $X_i$ is connected to the orbit $X_{i+\frac{m}{2}}$ with a double edge. In short, $X_0$ is a connected circulant with double edges connecting antipodal vertices.

- **Class IV.** The graph $X$ belongs to Class IV if $d_{im}(X) = 0$ and each orbit $X_i$ is connected (with a single edge) to four other orbits. In short, $X_0$ is a connected circulant of valency 4 and is a simple graph.

As mentioned in the introduction Class II was completely classified in [15]. The classification is given in the following proposition.

**Proposition 2.1 ([15, Theorem 1.1]).** Let $m \geq 3$ and $n \geq 3$ be integers. A connected quartic graph $X$ is a half-arc-transitive weak $(m, n)$-metacirculant of Class II if and only if $X \cong \mathcal{Y}(m, n; r, t)$, where $r \in \mathbb{Z}_n^*$ and $t \in \mathbb{Z}_n$ satisfy the following conditions:

(i) $n = md_m$ with $d_m > 2$,

(ii) $r^2 \neq \pm 1$,

(iii) $r^m = 1$,

(iv) $m(r - 1) = t(r - 1) = (r - 1)^2 = 0$,

(v) $(m) = (t)$ in $\mathbb{Z}_n$,

(vi) there exists a unique $c \in \{0, 1, \ldots, d_m - 1\}$ such that $t = cm$ and $m = ct$,

(vii) there exists a unique $k \in \{0, 1, \ldots, d_m - 1\}$ such that $kt = -km = r - 1$, and

(viii) either $m \neq 4$ or $t \neq 2r$.

We conclude this section by introducing the radius and attachment number of a quartic half-arc-transitive graph. Let $X$ be a quartic half-arc-transitive graph and let $D(X)$ be one of the two (paired) oriented graphs associated with $X$ in a natural way via the half-arc-transitive action of Aut$X$. It was shown in [10] that the length of every alternating cycle, that is a cycle whose every two consecutive edges have opposite orientations, is the same. Half of this length is called the radius of $X$. Next, any two alternating cycles with nontrivial intersection meet in the same number of vertices. This number is called the attachment number of $X$. If the attachment number of $X$ coincides with the radius of $X$ then we say that $X$ is tightly attached.

### 3. The attachment number and the radius of Class II graphs

As already mentioned it was shown in [12] that the class of connected quartic tightly attached half-arc-transitive graphs coincides with Class I of connected quartic half-arc-transitive metacirculants. Moreover, not every graph of Class II is of Class I (although many are). In fact, in [12] an infinite family of Class II graphs that are not tightly attached (and thus not in Class I) was constructed. Nevertheless, by Theorem 1.1, which we now prove, even if a Class II graph is not tightly attached it is “very close” to being tightly attached.

Before proving Theorem 1.1 let us recall the following result from [10].
Proposition 3.1 ([10, Proposition 2.6.]). Let $X$ be a connected quartic half-arc-transitive graph of radius $\mathcal{R}$ and let $C = v_0v_1v_2 \ldots v_{2\mathcal{R} - 1}$ be an alternating cycle of $X$. Then there exists a divisor $d$ of $2\mathcal{R}$, such that $V(C) \cap V(C') = \{v_j : 0 \leq j < \frac{2\mathcal{R}}{d}\}$, where $C'$ is the other alternating cycle of $X$ containing $v_0$.

Proof of Theorem 1.1. By Proposition 2.1 $X$ is isomorphic to some $\mathcal{Y}(m, n; r, t)$, where $m, n, r$ and $t$ satisfy the conditions (i)–(vii) of that proposition. Since $(r - 1)^2 = 0$, we thus have that

$$1 + r + \cdots + r^{m-1} = 1 + (r - 1 + 1) + \cdots + (r - 1 + 1)^{m-1} = m + \frac{m(m - 1)}{2}(r - 1) = \alpha.$$  

(6)

By [15, Proposition 2.3.] it follows that in one of the two possible natural orientations of the edges of $X$ given by the half-arc-transitive action of $\text{Aut}X$, the orientation of the edges is as given in (5). It is thus clear that the alternating cycle $C_1$ of $X$ containing the edge $u_0^0u_1^0$ contains the vertices $u_0^0, u_1^0, u_2^0, u_3^0, u_4^{r-2}$, etc. Note that the first of these vertices (after $u_0^0$) which is contained in $X_0 = (u_0^j : j \in \mathbb{Z}_n)$ and is the tail of the corresponding edge of $C_1$ is $u_0^0$. The other alternating cycle $C_2$ containing $u_0^0$ contains the vertices $u_0^0, u_0^1, u_1^{1-r}, u_2^{-1}$, etc. Note that the first of these vertices (after $u_0^0$) which is contained in $X_0$ and is the head of the corresponding edge of $C_2$ is again $u_0^{r-\alpha}$.

Let us first show that $\mathcal{R}$ is a divisor of $4\mathcal{R}$ which proves the first part of Theorem 1.1. Since, by [14, Theorem 5.1.], $X$ is tightly attached if $m$ is odd, we can assume that $m$ is even. In view of Proposition 3.1 it suffices to show that the vertex $u = u_0^0 - r^{2} - r^{3}$ (which in the case of $m = 4$ is actually $u_0^0 - r^{2} - r^{3}$) of $C_2$ is contained in $C_1$. By the preceding paragraph this clearly holds in the case when $m = 4$. We can thus assume that $m \geq 6$. Then $u$ is contained in $C_1$ if and only if there exists some $s$ such that $-1 - r - r^2 - r^3 = -r - r^2 - r^3 + s(t - \alpha)$. In view of $(r - 1)^2 = 0$ this is equivalent to $4(r - 1) = s(t - \alpha)$. Thus, taking $s = 2k$, where $k$ is as in item (vii) of Proposition 2.1, we get the desired equality (recall that $m(r - 1) = 0$).

To prove the second part of Theorem 1.1 note first that, by Proposition 2.1, $X$ is tightly attached if and only if the vertex $u_0^{-1}$ of $C_2$ is contained in $C_1$, which occurs if and only if $1 = -r + s(t - \alpha)$ for some $s$. This proves claim (i) of Theorem 1.1. Finally, the vertex $u_0^{-1}$ of $C_2$ is contained in $C_1$ if and only if $-1 - r = -r - r^2 + s(t - \alpha)$ for some $s$ which is equivalent to $2(r - 1) \in \langle t - \alpha \rangle$. This completes the proof. $\blacksquare$

4. When is a Class II graph of Class IV?

In this section we prove Theorem 1.2. Let $X = \mathcal{Y}(m, n; r, t)$ be a connected quartic half-arc-transitive weak metacirculant of Class II, where $m, n, r$ and $t$ satisfy the conditions of Proposition 2.1. Throughout this section let $\rho$ and $\sigma$ be as in (2) and (3), let $d_m = \frac{n}{m}$ and let $\alpha = m + \frac{m(m - 1)}{2}(r - 1)$. Note that, as in the previous section, (6) holds. Moreover, as $m(r - 1) = 0$ we have that either $\alpha = m$ or $\alpha = m + \frac{n}{m}$, in which case $n$ is, of course, even.

In what follows we will prove that, except when $X$ is isomorphic to one of the graphs $\mathcal{Y}(3, 9; 4, 6)$ and $\mathcal{Y}(3, 18; 7, 15)$, there exists some $a \in \{0, 1, \ldots, d_m - 1\}$ such that $X$ is a weak metacirculant of Class IV relative to the ordered pair $\langle \rho^a, \sigma^a \rangle = (\rho^a, \sigma^a)$. The proof is split over a series of lemmas.

Lemma 4.1. Let $X = \mathcal{Y}(m, n; r, t)$ be a connected quartic half-arc-transitive weak metacirculant of Class II, where $m, n, r$ and $t$ satisfy the conditions of Proposition 2.1. If there exists $a \in \{0, 1, \ldots, d_m - 1\}$ such that $(r - 1) \sim (t + a\alpha)$ and $2a, a + 1 \not\sim (r - 1)$, then $X$ is of Class IV.

Proof. Suppose such an $a \in \{0, 1, \ldots, d_m - 1\}$ exists and let $\rho' = \sigma^a\rho$ and $\sigma' = \rho$. Clearly $\langle \rho, \sigma \rangle$ is regular, $\rho'$ is semi-regular. Let $n'$ be the order of $\rho'$ and let $m' = \frac{n}{m}$ so that $\rho'$ is $(m', n')$-semi-regular. Let us denote the $m'$ orbits of $\rho'$ by $X_{0}', X_{1}', \ldots, X_{m'-1}'$ in such a way that $X_i'\sigma' = X_{i+1}'$, holds for all $i$ and that $u_0^0 \in X_0'$. We now show that $\sigma'$ normalizes $\rho'$ which implies that $X$ is a weak metacirculant relative to the ordered pair $\langle \rho', \sigma' \rangle$. Observe first that (4) implies

$$(\sigma')^{-1}\rho'\sigma' = \rho^{-1}\sigma\rho^a\rho = \sigma\rho^{a+1-r}.$$  

Combining together items (i) and (vii) of Proposition 2.1 we have that $d_m(r - 1) = 0$ in $\mathbb{Z}_n$, and so $(r - 1) = (t + a\alpha)$ implies that there exists some $b_0 \in \{1, \ldots, d_m - 1\}$ for which $b_0(t + a\alpha) = 1 - r$. Let now $b = b_0m + 1$. Since $r^m = 1$, (6) implies that

$$1 + r + \cdots + r^{b-1} = (1 + r + \cdots + r^{m-1})(1 + r + r^{2m} + \cdots + r^{(b_0-1)m}) + r^{b_0m} = \alpha b_0 + 1.$$  

Thus (4) implies that

$$(\rho')^b = \sigma^b \rho^{a(1 + r + r^2 + \cdots + r^{b-1})} = \sigma \rho^{b_0 + (a + (b_0\alpha))} = \sigma \rho^{a + b_0(t + a\alpha)} = \sigma \rho^{a + 1 - r}.$$  

Hence $(\sigma')^{-1}\rho'\sigma' = (\rho')^b$, that is, $\sigma'$ normalizes $\rho'$, and so $X$ is a weak metacirculant relative to the ordered pair $\langle \rho', \sigma' \rangle$. We now only have to prove that the ordered pair $\langle \rho', \sigma' \rangle$ gives rise to a Class IV representation of $X$, that is, we need to show that the orbit $X_0'$ is adjacent to four different $\rho'$-orbits. Let $D(X)$ be the oriented graph corresponding to the half-arc-transitive action of $\text{Aut}X$ in which the orientation of edges is as in (5).
We first show that the ordered pair \((\rho', \sigma')\) does not give a Class I representation of \(X\). Suppose on the contrary it does. Since we have \(u_0^t \rightarrow u_0^{t'}\) and \(u_0^t \rightarrow u_0^{t''}\), and since \(u_0^{t'} = u_0^0\sigma'\) and \(u_0^{t''} = u_0^0\sigma\) holds, we must have that \(u_0^0\sigma' = u_0^0\sigma' (\rho')^b\) for some \(b\). Since \((\rho, \sigma)\) is regular, this implies \(\sigma = (\rho (\rho')^b)\). It follows that \(b = b_0m + 1\) for some \(b_0\), and so

\[
\sigma (\rho')^{-t'} = \rho^{-1} = (\sigma (\rho')^b)^{-1} = \sigma (\rho (\rho')^b)^{-1} = \sigma (\rho (\rho')^{-b_0 + a}) = \sigma (\rho (\rho')^{-b_0 + a}).
\]

Hence \(-r = a + b_0(t + a\alpha),\) and so \(a + r = -b_0(t + a\alpha)\). But as \((r - 1) = (t + a\alpha)\), we thus have that \(a + 1 + r - 1 = a + r \in (r - 1),\) so that \(a + 1 \in (r - 1)\), a contradiction.

Let us now show that the ordered pair \((\rho', \sigma')\) does not give a Class II representation of \(X\). If it does then as the outneighbors of \(u_0^t\) are \(u_0^t\) and \(u_0^{t'}\) and \(u_0^{t''}\), we must have that \(u_0^0\sigma' = u_0^1 = u_0^0(\rho')^b\) holds for some \(b\). Hence \(\sigma = (\sigma (\rho')^b)\), which, by the arguments of the above paragraph, implies that \(a + b_0(t + a\alpha) = 0\) holds for some \(b_0\). But then \(2a = -2b_0(t + a\alpha) \in (r - 1),\) which, by assumption, does not hold.

We finally show that the ordered pair \((\rho', \sigma')\) does not give a Class III representation of \(X\), which then proves that it gives a Class IV representation of \(X\). Suppose then that the ordered pair \((\rho', \sigma')\) gives a Class III representation of \(X\). Since in this case \(\rho'\) has at least four orbits and since the neighbors \(u_0^t\) and \(u_0^{t'}\) and \(u_0^{t''}\), depending on whether \(m \equiv 0 \pmod{2}\) or not.

It follows that \(a(2+r-1) = a(1+r) = -b_0(t+a\alpha) \in (r-1)\) and consequently \(2a \in (r-1)\), which, by assumption, does not hold. This completes the proof.

**Lemma 4.2.** Let \(X = \mathcal{Y}(m, n; r, t)\) be a connected quartic half-arc-transitive weak metacirculant of Class II, where \(m, n, r, t\) satisfy the conditions of Proposition 2.1. Then for each generator \(x\) of \(\langle r - 1 \rangle\) there exists some \(a \in \{0, 1, \ldots, d_m - 1\}\) such that \(x = t + a\alpha\).

**Proof.** Recall that \(\alpha \in \{m, m + \frac{n}{2}\}\). By Proposition 2.1 we have that \(r - 1 \in \langle m \rangle = \langle t \rangle\), and so if \(\alpha = m\), there is nothing to prove. Suppose then that \(\alpha = m + \frac{n}{2}\) in which case \(n\) must be even. Then \(m(r-1) = 0\) implies that \(m\) is even and \(\frac{m(r-1)}{2} = \frac{n}{2}\). It follows that \(d_m\) is also even while the number \(c\) from item (vi) of Proposition 2.1 is odd. Now, let \(s \in \{0, 1, \ldots, d_m - 1\}\) be such that \(x = t + sm = (c + s)m\). If \(s\) is even, then we can simply take \(a = s\), since then \(t + a\alpha = t + sm = x\). We can thus assume that \(s\) is odd. Then \(c + s\) is even, and so the fact that \(d_m\) is even forces \((r-1) \leq \langle m \rangle\) and \(r-1 \equiv 0 \pmod{4}\). Moreover, as \(\frac{m(r-1)}{2} = \frac{n}{2}\), we have that \(n \equiv 0 \pmod{4}\) and \(d_m \equiv 0 \pmod{4}\), and so \(\frac{d_m}{2} \equiv 0\). Hence \(\frac{d_m}{2} \alpha = \frac{n}{2}\) and so \(t + (s \pm \frac{d_m}{2})\alpha = t + sm + \frac{n}{2} \pm \frac{n}{2} = x\), so that we can take \(a = s + \frac{d_m}{2}\) or \(a = s - \frac{d_m}{2}\) depending on whether \(s < \frac{d_m}{2}\) or not. ■

Since \((r-1)a = 0\), the following lemma is self-explanatory.

**Lemma 4.3.** Let \(X = \mathcal{Y}(m, n; r, t)\) be a connected quartic half-arc-transitive weak metacirculant of Class II, where \(m, n, r, t\) satisfy the conditions of Proposition 2.1. If \(a, a' \in \{0, 1, \ldots, d_m - 1\}\) are such that \((t + a\alpha) = (t + a'\alpha) = (r - 1)\) and \(a, a' \in \langle r - 1 \rangle\), then \(a\alpha = a'\alpha\). Moreover, if \(a \in \{0, 1, \ldots, d_m - 1\}\) such that \((t + a\alpha) = (r - 1)\) and \(2a \in (r - 1)\), then \(\alpha \neq a\alpha\).

The proof of Theorem 1.2 is now at hand.

**Proof of Theorem 1.2.** That \(\mathcal{Y}(3, 9; 4, 6)\) and \(\mathcal{Y}(3, 18; 7, 15)\) are not of Class IV is easily verified using a computer. Let us now prove that all other connected quartic half-arc-transitive weak metacirculants of Class II are of Class IV.

Suppose first there exist \(a, a' \in \{0, 1, \ldots, d_m - 1\}\) such that \((t + a\alpha) = (t + a'\alpha) = (r - 1)\) with \(2a, 2a' \in (r - 1)\) and \(a\alpha = a'\alpha\). In view of Lemma 4.3 we can thus assume that \(a\alpha = 0\) and \(a\alpha = \frac{n}{2}\). Then \(n\) is even and \(\langle m \rangle = \langle t \rangle = \langle t + \frac{n}{2} \rangle = \langle r - 1 \rangle\). If \((a - a')\frac{m(r-1)}{2} = \frac{n}{2}\), then \((a - a')\alpha = \frac{n}{2}\) implies that \((a - a')m = 0\). But then \(\frac{n}{2} = (a - a')(m + \frac{m(r-1)}{2}) = (a - a')m = 0\), a contradiction. We thus have \((a - a')\frac{m(r-1)}{2} = 0\) and \((a - a')m = \frac{n}{2}\). It follows that \(d_m\) is even and either \(a - a' = \frac{d_m}{2}\) or \(a - a' = -\frac{d_m}{2}\). As \(2a, 2a' \in (r - 1)\), we have that \(2(a - a') \in (r - 1)\), and so \(d_m \in (r - 1)\), which, in view of \(m(r-1) = 0\), implies that \((d_m) = (r - 1)\). It follows that \(d_m = m\), that is \(n = m^2\), and \(\frac{m(r-1)}{2} = \frac{n}{2}\). Thus \((a - a')\frac{m(r-1)}{2} = 0\) implies that \(a - a'\) is even, and so \(a - a')m = \frac{n}{2}\) forces \(m \equiv 0 \pmod{4}\). If \(m = 4\), then \(X = \mathcal{Y}(4, 16; 5, 12)\) and \(\mathcal{Y}(4, 16; 13, 12)\) neither of which is half-arc-transitive by item (viii) of Proposition 2.1. Hence \(m \geq 5\), and so the fact that \(4\) divides \(m\) and that \(m\) is the order of \((r-1)\) implies that \((r-1)\) has at least four generators. Combining together Lemmas 4.1–4.3 we thus find that \(X\) is of Class IV.

Suppose now that at most one generator of \((r-1)\) of the form \(t + a\alpha\), where \(a \in \{0, 1, \ldots, d_m - 1\}\) and \(2a \in (r - 1)\), exists. If \((r - 1)\) has at least three generators, then Lemmas 4.1–4.3 imply that \(X\) is of Class IV. We can thus assume that \((r-1)\) has two generators (observe that \(2(r-1) \neq 0\) since \((r-1)^2 \equiv 0 \pmod{2}\), so that \((r-1)\) has at least two generators), say \(t + a\alpha\) and \(t + a'\alpha\), and that \(2a, 2a' \in (r - 1)\). Then \((r-1)\) is isomorphic to one of the cyclic groups \(Z_3, Z_4\) and \(Z_6\). We distinguish two cases depending on whether \(a\) is contained in \((r - 1)\) or not.
Case 1: $a \in (r - 1)$.

In this case $a = 0$, so $(m) = (t) = (r - 1)$. Since $t + a \alpha = -t$, that is $2t + a \alpha = 0$. Then $a' + 1 \in (r - 1)$ implies that $a \alpha = (a' + 1) \alpha = -\alpha$, and so $2t = \alpha$. Since $4t = 2\alpha = 2m$ we cannot have $(r - 1) \equiv \mathbb{Z}_3$. If $(r - 1) \equiv \mathbb{Z}_3$, then $d_m = 3$ and $a = m$, so $a \alpha = -a$ implies that $a' = 2$. Hence $3 = a' + 1 \in (r - 1)$ implies that $m = 3$ and so $X \equiv y(3, 9, 4, 6)$ is the Doyle–Holt graph. If however $(r - 1) \not\equiv \mathbb{Z}_3$, then $d_m = 6$. Thus $2t = \alpha$ and $a \alpha = -\alpha$ implies that $a'$ is one of 2 and 5, so $a' + 1 \in (r - 1)$ forces $6 \in (r - 1)$. If $m = 3$ then $n = 18$, but then $(r - 1) = (m)$ implies that $r \in [4, 16]$, contradicting $r \in \mathbb{Z}_m$. Hence $m = 6, n = 36, t = 30$ and $a = 24$. But in this case $(t + 2a) = (6) = (r - 1)$ and neither $X = 2 + 1$ nor $X = 2 \cdot 2$ is in $(r - 1)$. Taking $a = 2$ in Lemma 4.1 we thus find that $X$ is of Class IV, as claimed.

Case 2: $a \not\in (r - 1)$.

Note that we can assume that $(t) \neq (r - 1)$ since otherwise we can replace $a$ by 0 and proceed as in Case 1. This implies that $a = 0$, and so $n$ is even and $a = \frac{n}{2}$ (recall that $2a \in (r - 1)$, forcing $2a \alpha = 0$). We therefore have $(t + \frac{n}{2}) = (r - 1) \neq (m)$. Note that $a = \frac{n}{2}$ implies that $am = \frac{n}{2}$ and $a^{m(m-1)}(r - 1) = 0$. It follows that $a = \frac{d_m}{2}$, and so $2a = d_m \in (r - 1)$ implies that $(d_m) = (r - 1)$. Since $r - 1 \in (t)$ by Proposition 2.1 and $(t) \neq (r - 1)$, the fact that $(t + \frac{n}{2}) = (r - 1)$ implies that $\frac{n}{2} = (r - 1)$. But as $m$ is the order of $(d_m) = (r - 1)$, this implies that $m$ is odd, and so $(r - 1) \not\equiv \mathbb{Z}_3$ and $m = 3$. Since $n$ is even and $r - 1 \in (m)$ by Proposition 2.1, $(d_m) = (r - 1)$ implies that $d_m \equiv 0$ (mod 6), and so $n \equiv 0$ (mod 18). Moreover, as $r - 1$ is even and $t$ is odd, $(r - 1) = (t + \frac{n}{2})$ implies that $\frac{n}{2}$ is odd, so $n \equiv 18$ (mod 36). Since $m(r - 1) = 3(r - 1) = 0$, we have that $0 = 3(t + \frac{n}{2}) = 3t + \frac{3n}{2}$, and so $6t = 0$. Thus $6m = 0$, which finally implies that $n = 18$ and $X \equiv y(3, 18; 7, 15)$. This completes the proof. \hfill $\blacksquare$

5. Conclusions

The results of this paper have the following interesting corollary. By Theorem 1.1 the Doyle–Holt graph $y(3, 9, 4, 6)$ and its canonical double cover $y(3, 18; 7, 15)$ are both tightly attached and are thus of Class I. Combining together Theorem 1.2 and the results from [16], where it is shown that each connected quartic half-arc-transitive weak metacirculant of Class III belongs to Class I or Class II, we thus have that each connected quartic half-arc-transitive weak metacirculant belongs to Class I or Class IV. As was mentioned in the introduction Class I has already been completely classified (see [10, 14]). What remains to complete the project of classifying all connected quartic half-arc-transitive weak metacirculants is thus to classify Class IV graphs.

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