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# Time–space integrable decompositions of nonlinear evolution equations<sup>☆</sup>

Wen-Xiu Ma<sup>a,\*</sup>, Hongyou Wu<sup>b</sup><sup>a</sup> *Department of Mathematics, University of South Florida, Tampa, FL 33620-5700, USA*<sup>b</sup> *Department of Mathematical Sciences, Northern Illinois University, DeKalb, IL 60115-2888, USA*

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We dedicate this paper to the memory of our great mentor Professor Shiing-Shen Chern

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## Abstract

Separation of the time and space variables of evolution equations is analyzed, without using any structure associated with evolution equations. The resulting theory provides techniques for constructing time–space integrable decompositions of evolution equations, which transform an evolution equation into two compatible Liouville integrable ordinary differential equations in the time and space variables. The techniques are applied to the KdV, MKdV and diffusion equations, thereby yielding several new time–space integrable decompositions of these equations.

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## 1. Introduction

Symmetry constraints [1] provide a way to decompose soliton equations into two ordinary differential equations (ODEs), one of which is spatial and the other is temporal [2,3]. The re-

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<sup>\*</sup> Corresponding author.

*E-mail addresses:* [mawx@math.usf.edu](mailto:mawx@math.usf.edu) (W.X. Ma), [wu@math.niu.edu](mailto:wu@math.niu.edu) (H. Wu).

sulting two ODEs possess the Hamiltonian structures inherited from the symmetry property, and thus they can be shown to be Liouville integrable [1,4]. Actually, symmetry constraints yield Bäcklund transformations from soliton equations to finite-dimensional integrable systems [5]. It has been shown that soliton solutions can be generated through the resulting Bäcklund transformations [6]. The basic idea of both symmetry constraints [1] and Bäcklund transformations [7] is to use nice structures of differential equations in analyzing properties of the equations under consideration and their solution manifolds [8].

In order to exhibit more solution submanifolds for soliton equations, we would like to explore possible transformations for evolution equations, which aim to turn the equations into simpler and integrable ones. Time–space integrable decompositions are such transformations, which separate the time variable and the space variable of evolution equations and pave a way to solve evolution equations through integrating related ODEs. Once an integrable decomposition is found, an associated solution submanifold, which can be determined by quadratures, can be obtained, and thus the integrability by quadratures can be shown for the evolution equations under investigation. Therefore, it is very important how to find integrable decompositions of evolution equations. The main topic of this paper is to analyze structures of time–space integrable decompositions of evolution equations.

There are two basic questions on time–space integrable decompositions. One is what conditions time–space integrable decompositions need to satisfy. The other is what kind of time–space integrable decompositions can exist. To answer these questions, we will characterize integrable decompositions of evolution equations, and analyze the key conditions that integrable decompositions should satisfy. The primary objective of our work is to establish solution connections between soliton equations and finite-dimensional integrable systems, and explore the integrability by quadratures for soliton equations. The resulting theory will also generalize the theory of symmetry constraints. One of the advantages of our results is that we do not require any structure associated with the equations under investigation, such as Lax pairs for soliton equations and the symmetry property in symmetry constraints.

This paper is structured as follows. In Section 2, we will consider general time–space integrable decompositions of evolution equations, and derive the conditions that time–space integrable decompositions need to satisfy. In particular, we will present concrete criteria for the existence of time–space integrable decompositions in the case where the ansätze equations are scalar evolution equations and one-dimensional Hamiltonian systems. In Section 3, the criteria will be applied to the KdV, MKdV and diffusion equations, thereby yielding several new examples of time–space integrable decompositions of these equations. Some concluding remarks will be given in Section 4.

## 2. Integrable decompositions

We consider the problem of finding time–space integrable decompositions of a given evolution equation

$$u_t = f(x, t, u, u_x, \dots, u^{(n)}), \quad u^{(k)} = \frac{\partial^k u}{\partial x^k}, \quad k \geq 1, \quad (2.1)$$

where  $f: \mathbb{R}^{n+3} \rightarrow \mathbb{R}$  is a function. Our aim is to separate the time variable  $t$  and the space variable  $x$ , and to decompose the evolution equation into two ODEs, which are integrable by quadratures [9]. We will divide our discussions into several cases according to the number of dependent variables in the ODEs.

### 2.1. Into two scalar ordinary differential equations

Let us first assume that the evolution equation (2.1) accepts a time–space decomposition

$$u = g(\phi), \quad (2.2)$$

where  $g: \mathbb{R} \rightarrow \mathbb{R}$  is a function and new dependent variable  $\phi$  satisfies two scalar ODEs

$$\begin{cases} \phi_x = \Gamma(\phi), \\ \phi_t = \Omega(\phi). \end{cases} \quad (2.3)$$

Solutions to each of these two ansätze equations can be determined by quadratures, but we require the compatibility condition for the two equations,  $\phi_{xt} = \phi_{tx}$ , which is equivalent to

$$\frac{d\Gamma}{d\phi} \Omega = \Gamma \frac{d\Omega}{d\phi}. \quad (2.4)$$

If one of the two functions  $\Gamma$  and  $\Omega$  is identically zero, the resulting solution to (2.1) will be special: either  $u_x = 0$  or  $u_t = 0$ , both of which transform the evolution equation (2.1) into an ODE. If none of them is identically zero, then there is a nonzero constant  $v$  such that  $\Omega = v\Gamma$  holds locally, because of (2.4). In what follows, we assume that

$$\Omega = v\Gamma \quad (2.5)$$

holds globally, to make our discussion simple; and thus, we have

$$v\phi_x - \phi_t = 0.$$

This implies that a solution  $\phi(x, t)$  to the system (2.3) is a traveling wave

$$\phi(x, t) = \chi(x + vt),$$

for some function  $\chi = \chi(\xi)$  which solves  $\chi_\xi = \Gamma(\chi)$ .

Since we can compute that

$$\begin{aligned} u_t &= \frac{dg}{d\phi} \phi_t = v \frac{dg}{d\phi} \Gamma, & u_x &= \frac{dg}{d\phi} \phi_x = \frac{dg}{d\phi} \Gamma, & \dots, \\ u^{(n)} &= \underbrace{\frac{\partial}{\partial \phi} \left( \dots \left( \frac{\partial}{\partial \phi} \left( \frac{\partial g}{\partial \phi} \Gamma \right) \Gamma \right) \dots \right)}_n \Gamma, \end{aligned}$$

the evolution equation (2.1) becomes

$$\begin{aligned} v \frac{dg}{d\phi} \Gamma &= f(g, \tilde{u}^{(1)}, \dots, \tilde{u}^{(n)}), \\ \tilde{u}^{(k)} &= \underbrace{\frac{\partial}{\partial \phi} \left( \dots \left( \frac{\partial}{\partial \phi} \left( \frac{\partial g}{\partial \phi} \Gamma \right) \Gamma \right) \dots \right)}_k, \quad 1 \leq k \leq n. \end{aligned} \quad (2.6)$$

This gives us the following result.

**Theorem 2.1.** *If two functions  $g = g(\phi)$  and  $\Gamma = \Gamma(\phi)$  satisfy Eq. (2.6) with some constant  $v$ , then*

$$u(x, t) = g(\chi(x + vt)), \quad \chi_\xi = \Gamma(\chi) \quad (2.7)$$

*gives a traveling wave solution to the evolution equation (2.1).*

## 2.2. Into two Hamiltonian systems

### 2.2.1. General case

Let us now assume that the evolution equation (2.1) accepts a time–space decomposition

$$u = g(\Phi, \Psi), \quad \Phi = (\phi_1, \dots, \phi_m)^T, \quad \Psi = (\psi_1, \dots, \psi_m)^T, \quad (2.8)$$

where  $g: \mathbb{R}^{2m} \rightarrow \mathbb{R}$  is a function, and the new dependent variables  $\Phi$  and  $\Psi$  satisfy a spatial integrable Hamiltonian system with a Hamiltonian function  $H = H(x, t, \Phi, \Psi)$ :

$$\phi_{i,x} = \{\phi_i, H\}, \quad \psi_{i,x} = \{\psi_i, H\}, \quad 1 \leq i \leq m, \quad (2.9)$$

and a temporal integrable Hamiltonian system with a Hamiltonian function  $G = G(x, t, \Phi, \Psi)$ :

$$\phi_{i,t} = \{\phi_i, G\}, \quad \psi_{i,t} = \{\psi_i, G\}, \quad 1 \leq i \leq m, \quad (2.10)$$

where the Poisson bracket is defined by

$$\{H_1, H_2\} = \sum_{i=1}^m \left( \frac{\partial H_1}{\partial \phi_i} \frac{\partial H_2}{\partial \psi_i} - \frac{\partial H_1}{\partial \psi_i} \frac{\partial H_2}{\partial \phi_i} \right). \quad (2.11)$$

Since these two ansätze Hamiltonian systems (2.9) and (2.10) hold simultaneously, they need to satisfy the compatibility conditions

$$\phi_{i,tx} = \phi_{i,x t}, \quad \psi_{i,tx} = \psi_{i,x t}, \quad 1 \leq i \leq m.$$

Equivalently, this leads to

$$H_t + G_x + \{H, G\} = c(x, t) \quad (2.12)$$

for some function  $c = c(x, t)$ , which does not explicitly depend on any of the variables:  $\phi_i, \psi_j$ ,  $1 \leq i, j \leq m$ . In order to construct such time–space integrable compositions, we are going to check on all conditions which the three functions  $g$ ,  $H$  and  $G$  need to satisfy.

Let us focus on the case of time–space implicit dependence for brevity:  $H = H(\Phi, \Psi)$  and  $G = G(\Phi, \Psi)$ . In this case, we first have

$$\{H, G\} = c, \quad (2.13)$$

where  $c$  is an arbitrary constant. Second, taking advantage of the Hamiltonian structure, we can compute that

$$u_t = \hat{G}g, \quad u_x = \hat{H}g, \quad u_{xx} = \hat{H}^2g, \quad \dots, \quad u^{(n)} = \hat{H}^n g,$$

where the hat denotes the adjoint operation of the associated Poisson algebra:

$$\hat{P}Q = \{Q, P\}. \quad (2.14)$$

Therefore, the evolution equation (2.1) becomes

$$\hat{G}g = f(x, t, g, \hat{H}g, \dots, \hat{H}^n g). \quad (2.15)$$

Equations (2.13) and (2.15) are only two conditions which guarantee the existence of a time–space integrable decomposition of the evolution equation (2.1). Thus, we have the following theorem.

**Theorem 2.2.** Let  $\Phi = (\phi_1, \dots, \phi_m)^T$  and  $\Psi = (\psi_1, \dots, \psi_m)^T$ . If three functions  $g = g(\Phi, \Psi)$ ,  $H = H(\Phi, \Psi)$  and  $G = G(\Phi, \Psi)$  satisfy two equations (2.13) and (2.15), then  $u = g(\Phi, \Psi)$  decompose the evolution equation (2.1) into a spatial integrable Hamiltonian system (2.9) and a temporal integrable Hamiltonian system (2.10), determined by two Hamiltonian functions  $H$  and  $G$ , respectively.

The two conditions (2.13) and (2.15) restrict three functions:  $g, H, G$ , and thus there should exist choices of these functions for determining time–space integrable decompositions of the given evolution equations.

### 2.2.2. One-dimensional case

Let us consider the one-dimensional case (i.e.,  $m = 1$ ):

$$\Phi = (\phi), \quad \Psi = (\psi), \quad H = H(\phi, \psi), \quad G = G(\phi, \psi), \quad (2.16)$$

where  $\phi$  and  $\psi$  are new scalar dependent variables. In this case, the conditions (2.13) and (2.15) read as

$$-\frac{\partial H}{\partial \psi} \frac{\partial G}{\partial \phi} + \frac{\partial H}{\partial \phi} \frac{\partial G}{\partial \psi} = c, \quad (2.17)$$

where  $c$  is an arbitrary constant, and

$$\alpha(\phi, \psi) \frac{\partial G}{\partial \phi} + \beta(\phi, \psi) \frac{\partial G}{\partial \psi} = \gamma(\phi, \psi), \quad (2.18)$$

where  $\alpha, \beta$  and  $\gamma$  are introduced for the sake of convenience:

$$\alpha = -\frac{\partial g}{\partial \psi}, \quad \beta = \frac{\partial g}{\partial \phi}, \quad \gamma = f(x, t, g, \hat{H}g, \dots, \hat{H}^n g). \quad (2.19)$$

Further, suppose that  $g$  and  $H$  have already been fixed, and then we think about how to get  $G$ . Let us first observe the linear system for  $\frac{\partial G}{\partial \phi}$  and  $\frac{\partial G}{\partial \psi}$ , i.e., (2.17) and (2.18). Obviously, the determinant of its coefficient matrix can be computed as

$$\delta := \begin{vmatrix} -\frac{\partial H}{\partial \psi} & \frac{\partial H}{\partial \phi} \\ \alpha & \beta \end{vmatrix} = -\alpha \frac{\partial H}{\partial \phi} - \beta \frac{\partial H}{\partial \psi} = \{H, g\}. \quad (2.20)$$

There are two subcases:  $\delta$  is identically zero and  $\delta$  is not identically zero. If  $\delta$  is identically zero, then  $u_x = \{g, H\} = 0$ . Thus, the evolution equation (2.1) becomes an ODE and any decomposition will only give a special solution to (2.1). We would like to discuss the other subcase:  $\delta$  is not identically zero. In this second subcase, (2.17) and (2.18) yield the unique solution

$$\frac{\partial G}{\partial \phi} = \frac{1}{\delta} \begin{vmatrix} c & \frac{\partial H}{\partial \phi} \\ \gamma & \beta \end{vmatrix} = \frac{1}{\delta} \left( c \frac{\partial g}{\partial \phi} - \gamma \frac{\partial H}{\partial \phi} \right), \quad (2.21)$$

$$\frac{\partial G}{\partial \psi} = \frac{1}{\delta} \begin{vmatrix} -\frac{\partial H}{\partial \psi} & c \\ \alpha & \gamma \end{vmatrix} = \frac{1}{\delta} \left( c \frac{\partial g}{\partial \psi} - \gamma \frac{\partial H}{\partial \psi} \right). \quad (2.22)$$

Hence we have

$$\frac{\partial^2 G}{\partial \psi \partial \phi} = \frac{1}{\delta^2} \left[ \left( c \frac{\partial^2 g}{\partial \psi \partial \phi} - \frac{\partial \gamma}{\partial \psi} \frac{\partial H}{\partial \phi} - \gamma \frac{\partial^2 H}{\partial \psi \partial \phi} \right) \delta - \left( c \frac{\partial g}{\partial \phi} - \gamma \frac{\partial H}{\partial \phi} \right) \frac{\partial \delta}{\partial \psi} \right],$$

$$\frac{\partial^2 G}{\partial \phi \partial \psi} = \frac{1}{\delta^2} \left[ \left( c \frac{\partial^2 g}{\partial \phi \partial \psi} - \frac{\partial \gamma}{\partial \phi} \frac{\partial H}{\partial \psi} - \gamma \frac{\partial^2 H}{\partial \phi \partial \psi} \right) \delta - \left( c \frac{\partial g}{\partial \psi} - \gamma \frac{\partial H}{\partial \psi} \right) \frac{\partial \delta}{\partial \phi} \right],$$

which implies that the compatibility condition of the two equations (2.21) and (2.22),  $\frac{\partial^2 G}{\partial \psi \partial \phi} = \frac{\partial^2 G}{\partial \phi \partial \psi}$ , is equivalent to

$$\{\gamma, H\}\delta - \gamma\{\delta, H\} - c\{g, \delta\} = 0. \quad (2.23)$$

In terms of  $g$  and  $H$ , the above equality can be written as

$$(\hat{H}f(g, \hat{H}g, \dots, \hat{H}^n g))\hat{H}g - f(g, \hat{H}g, \dots, \hat{H}^n g)(\hat{H}^2 g) + c\hat{g}\hat{H}g = 0, \quad (2.24)$$

where  $c$  is an arbitrary constant and  $\hat{P}$  is an adjoint operator of  $P$  defined by (2.14). This is the only condition that  $g$ ,  $H$  and  $f$  need to satisfy in order to guarantee the existence of  $G$  by (2.21) and (2.22). We sum up the above result in the following theorem.

**Theorem 2.3.** *If two functions  $g = g(\phi, \psi)$  and  $H = H(\phi, \psi)$  have nonzero Poisson bracket and satisfy Eq. (2.24) for some constant  $c$ , then Eqs. (2.21) and (2.22) can determine a function  $G = G(\phi, \psi)$ , and  $u = g(\phi, \psi)$  decomposes the evolution equation (2.1) into two one-dimensional integrable Hamiltonian systems:*

$$\phi_x = \{\phi, H\}, \quad \psi_x = \{\psi, H\}, \quad (2.25)$$

and

$$\phi_t = \{\phi, G\}, \quad \psi_t = \{\psi, G\}. \quad (2.26)$$

This theorem gives an approach for constructing one-dimensional time–space integrable decompositions of evolution equations. The starting point is to search for two functions  $g$  and  $H$  satisfying (2.24) such that  $\delta = \{H, g\}$  is not identically zero. Then, integrating (2.21) and (2.22) leads to a third function  $G$ . Note that the Hamiltonian systems (2.25) and (2.26) are one-dimensional and thus they are always integrable by quadratures. Therefore, the whole process of determining  $g$ ,  $H$  and  $G$  will finally result in integrable decomposition  $u = g(\phi, \psi)$  associated with  $H$  and  $G$ .

It is also not difficult to see that if  $G = G(H)$  is a function of  $H$  and  $\frac{dG}{dH}$  is an integral of motion of the Hamiltonian systems (2.25) and (2.26) (for example, this is true if  $G = G(H)$  is a constant coefficient polynomial in  $H$  and  $\{H, G\} = 0$ ), then we have

$$\phi_x = \frac{\partial H}{\partial \psi}, \quad \phi_t = \frac{\partial G}{\partial \psi} = \frac{dG}{dH} \frac{\partial H}{\partial \psi} = \frac{dG}{dH} \phi_x,$$

and

$$\psi_x = -\frac{\partial H}{\partial \phi}, \quad \psi_t = -\frac{\partial G}{\partial \phi} = -\frac{dG}{dH} \frac{\partial H}{\partial \phi} = \frac{dG}{dH} \psi_x,$$

and by a similar argument in Section 2.1, these two equations generate traveling wave solutions  $\phi$  and  $\psi$  with a traveling speed  $v = \frac{dG}{dH}$ . Therefore, the resulting decomposition will only provide a traveling wave solution  $u = g(\phi, \psi)$  with the traveling speed  $v = \frac{dG}{dH}$ .

### 3. Applications

We apply the idea of construction in the previous section to the KdV equation, the MKdV equation and three nonlinear diffusion equations. Many new time–space integrable decompositions of these equations can be presented.

### 3.1. KdV equation

We first consider the KdV equation

$$u_t = u_{xxx} + 6uu_x. \quad (3.1)$$

In this KdV case, we have

$$f = f(u, u_x, u_{xx}, u_{xxx}) = u_{xxx} + 6uu_x,$$

and thus

$$\gamma = \hat{H}^3 g + 6g\hat{H}g.$$

Further, Eqs. (2.6) and (2.24) under  $c = 0$  become

$$v \frac{dg}{d\phi} = 6g \frac{dg}{d\phi} + \frac{d^3 g}{d\phi^3} + 2 \frac{d^2 g}{d\phi^2} \frac{d\Gamma}{d\phi} + \frac{dg}{d\phi} \frac{d^2 \Gamma}{d\phi^2} \Gamma + \frac{dg}{d\phi} \left( \frac{d\Gamma}{d\phi} \right)^2, \quad (3.2)$$

and

$$(\hat{H}^4 g + 6\hat{H}(g\hat{H}g))\hat{H}g - (\hat{H}^3 g + 6g\hat{H}g)\hat{H}^2 g = 0, \quad (3.3)$$

respectively. More generally, according to the conditions (2.13) and (2.15), we can begin with

$$\hat{G}H = 0, \quad \hat{G}g = \hat{H}^3 g + 6g\hat{H}g. \quad (3.4)$$

We will look for polynomial type solutions  $g$ ,  $\Gamma$ ,  $H$  and  $G$  of (3.2), (3.3) and (3.4), and then in the one-dimensional case, determine a nontrivial associated third function  $G$ .

#### 3.1.1. Scalar case

Among the fifth order and the sixth order polynomials of  $\phi$  for  $g$  and  $\Gamma$ , respectively, we found the following four nontrivial examples.

The first example is

$$v = 4b_1^2 + 6a_0, \quad g = a_0 - 8b_3b_1\phi^2 - 8b_3^2\phi^4, \quad \Gamma = b_1\phi + b_3\phi^3,$$

where  $a_0$ ,  $b_1$  and  $b_3$  are arbitrary constants. The second example is

$$v = b_1^2 + 8b_0b_2 + 6a_0, \quad g = a_0 - 2b_2b_1\phi - 2b_2^2\phi^2, \quad \Gamma = b_0 + b_1\phi + b_2\phi^2,$$

where  $a_0$ ,  $b_0$ ,  $b_1$  and  $b_2$  are arbitrary constants. The third example is

$$v = \frac{2(2b_2^4 + 243a_0b_3^2)}{81b_3^2},$$

$$g = a_0 - \frac{16}{27} \frac{b_2^3}{b_3} \phi - \frac{40}{9} b_2^2 \phi^2 - \frac{32}{3} b_2 b_3 \phi^3 - 8b_3^2 \phi^4,$$

$$\Gamma = \frac{2}{9} \frac{b_2^2}{b_3} \phi + b_2 \phi^2 + b_3 \phi^3,$$

where  $a_0$ ,  $b_2$  and  $b_3 \neq 0$  are arbitrary constants. The fourth example is

$$v = \frac{2}{27} \frac{81a_0b_3^2 - 10b_2^4 + 36b_2^2b_3b_1 + 54b_3^2b_1^2}{b_3^2},$$

$$g = a_0 - \frac{16}{27} \frac{b_2(-b_2^2 + 9b_3b_1)}{b_3} \phi + \left(-\frac{8}{3}b_2^2 - 8b_3b_1\right)\phi^2 - \frac{32}{3}b_2b_3\phi^3 - 8b_3^2\phi^4,$$

$$\Gamma = \frac{1}{27} \frac{b_2(9b_3b_1 - 2b_2^2)}{b_3^2} + b_1\phi + b_2\phi^2 + b_3\phi^3,$$

where  $a_0$ ,  $b_1$ ,  $b_2$  and  $b_3 \neq 0$  are arbitrary constants.

### 3.1.2. One-dimensional case

Among the second order homogeneous polynomials of  $\phi$  and  $\psi$  for  $g$ , we found the following two nontrivial examples with a special polynomial for  $H$  shown below.

The first example is

$$g = 8b_1b_2\phi^2 \quad \text{or} \quad g = 8b_2b_3\psi^2, \quad H = b_1\phi^2 + b_2\phi^2\psi^2 + b_3\psi^2,$$

$$G = (8b_2^3\psi^4 + 16b_1b_2^2\psi^2 + 8b_1^2b_2)\phi^4 + (16b_2^2b_3\psi^4 - 16b_1^2b_3)\phi^2$$

$$+ 8b_2b_3^2\psi^4 - 16b_1b_3^2\psi^2$$

$$= 8b_2H^2 - 16b_1b_3H,$$

where  $b_1$ ,  $b_2$  and  $b_3$  are arbitrary constants. The second example is

$$g = 8b_1b_2\phi^2 + 8b_2b_3\psi^2, \quad H = b_1\phi^2 + b_2\phi^2\psi^2 + b_3\psi^2,$$

$$G = (8b_2^3\psi^4 + 16b_1b_2^2\psi^2 + 8b_1^2b_2)\phi^4 + (16b_2^2b_3\psi^4 - 48b_1b_2b_3\psi^2 - 64b_1^2b_3)\phi^2$$

$$+ 8b_2b_3^2\psi^4 - 64b_1b_3^2\psi^2$$

$$= 8b_2H^2 - 64b_1b_3H,$$

where  $b_1$ ,  $b_2$  and  $b_3$  are arbitrary constants.

### 3.1.3. Higher-dimensional case

Motivated by symmetry constraints, we found the following two higher-dimensional examples. The first example is

$$g = -8_1a_2(\phi_{11}^2 + \phi_{12}^2),$$

$$H = a_1(\phi_{21}^2 + \phi_{22}^2) - a_2(\phi_{11}^2 + \phi_{12}^2)^2 - a_3(\lambda_1\phi_{11}^2 + \lambda_2\phi_{12}^2),$$

$$G = 16a_1^2a_2(\phi_{11}\phi_{21} + \phi_{12}\phi_{22})^2 - 16a_1^2a_2(\phi_{11}^2 + \phi_{12}^2)(\phi_{21}^2 + \phi_{22}^2)$$

$$+ 16a_1^2a_3(\lambda_1\phi_{21}^2 + \lambda_2\phi_{22}^2) - 16a_1a_3^2(\lambda_1^2\phi_{11}^2 + \lambda_2^2\phi_{12}^2)$$

$$- 16a_1a_2a_3(\phi_{11}^2 + \phi_{12}^2)(\lambda_1\phi_{11}^2 + \lambda_2\phi_{12}^2),$$

where  $a_1$ ,  $a_2$ ,  $a_3$  and  $\lambda_1 \neq \lambda_2$  are arbitrary constants. The Liouville integrability of the associated two two-dimensional Hamiltonian systems

$$\phi_{1j,x} = \frac{\partial H}{\partial \phi_{2j}}, \quad \phi_{2j,x} = -\frac{\partial H}{\partial \phi_{1j}}, \quad 1 \leq j \leq 2,$$

and

$$\phi_{1j,t} = \frac{\partial G}{\partial \phi_{2j}}, \quad \phi_{2j,t} = -\frac{\partial G}{\partial \phi_{1j}}, \quad 1 \leq j \leq 2,$$



has been shown in the case of  $a_1 = a_2 = a_3$  in [10].

The second example is

$$\begin{aligned} g &= 2(\phi_{11}\psi_{21} + \phi_{12}\psi_{22}), \\ H &= \phi_{21}\psi_{11} + \phi_{22}\psi_{12} - (\phi_{11}\psi_{21} + \phi_{12}\psi_{22})^2 + \lambda_1\phi_{11}\psi_{21} + \lambda_2\phi_{12}\psi_{22}, \\ G &= (\phi_{11}\psi_{11} + \phi_{12}\psi_{12} - \phi_{21}\psi_{21} - \phi_{22}\psi_{22})^2 + 4(\phi_{11}\psi_{21} + \phi_{12}\psi_{22})(\phi_{21}\psi_{11} + \phi_{22}\psi_{12}) \\ &\quad + 4\lambda_1\phi_{21}\psi_{11} + 4\lambda_2\phi_{22}\psi_{12} + 4\lambda_1^2\phi_{11}\psi_{21} + 4\lambda_2^2\phi_{12}\psi_{22} \\ &\quad - 4(\phi_{11}\psi_{21} + \phi_{12}\psi_{22})(\lambda_1\phi_{11}\psi_{21} + \lambda_2\phi_{12}\psi_{22}), \end{aligned}$$

where  $\lambda_1 \neq \lambda_2$  are arbitrary constants. The Liouville integrability of the associated two two-dimensional Hamiltonian systems

$$\phi_{ij,x} = \frac{\partial H}{\partial \psi_{ij}}, \quad \psi_{ij,x} = -\frac{\partial H}{\partial \phi_{ij}}, \quad 1 \leq i, j \leq 2,$$

and

$$\phi_{ij,t} = \frac{\partial G}{\partial \psi_{ij}}, \quad \psi_{ij,t} = -\frac{\partial G}{\partial \phi_{ij}}, \quad 1 \leq i, j \leq 2,$$

has been shown in [11].

### 3.2. MKdV equation

We now consider the MKdV equation

$$u_t = u_{xxx} - 6u^2u_x. \quad (3.5)$$

In this MKdV case, we have

$$f = f(u, u_x, u_{xx}, u_{xxx}) = u_{xxx} - 6u^2u_x,$$

and thus

$$\gamma = \hat{H}^3 g - 6g^2 \hat{H} g.$$

Further, Eqs. (2.6) and (2.24) under  $c = 0$  become

$$v \frac{dg}{d\phi} = -6g^2 \frac{dg}{d\phi} + \frac{d^3 g}{d\phi^3} + 2 \frac{d^2 g}{d\phi^2} \frac{d\Gamma}{d\phi} + \frac{dg}{d\phi} \frac{d^2 \Gamma}{d\phi^2} \Gamma + \frac{dg}{d\phi} \left( \frac{d\Gamma}{d\phi} \right)^2, \quad (3.6)$$

and

$$(\hat{H}^4 g - 6\hat{H}(g^2 \hat{H} g)) \hat{H} g - (\hat{H}^3 g - 6g^2 \hat{H} g) \hat{H}^2 g = 0, \quad (3.7)$$

respectively. More generally, based on the conditions (2.13) and (2.15), we can begin with

$$\hat{G}H = 0, \quad \hat{G}g = \hat{H}^3 g - 6g^2 \hat{H} g. \quad (3.8)$$

Similarly, we will search for polynomial type solutions  $g$ ,  $\Gamma$ ,  $G$  and  $G$  of (3.6)–(3.8), and then in the one-dimensional case, try to determine a nontrivial third function  $G$ .

### 3.2.1. Scalar case

Among the third order polynomials of  $\phi$  for  $g$  and  $\Gamma$ , we found the following four nontrivial examples.

The first two examples are

$$v = -\frac{1}{2}b_1^2 \pm 2a_1b_0, \quad g = \pm\frac{1}{2}b_1 + a_1\phi, \quad \Gamma = b_0 + b_1\phi \pm a_1\phi^2,$$

where  $a_1$ ,  $b_0$  and  $b_1$  are arbitrary constants. The second two examples are

$$\begin{aligned} v &= \frac{\pm 48a_1^2a_2b_1 - 9a_1^4 - 64a_2^2b_1^2}{32a_2^2}, \\ g &= \frac{1}{8a_2}(\pm 8a_2b_1 - a_1^2) + a_1\phi + a_2\phi^2, \\ \Gamma &= \frac{a_1(4a_2b_1 \mp a_1^2)}{8a_2^2} + b_1\phi \pm \frac{3}{4}a_1\phi^2 \pm \frac{1}{2}a_2\phi^3, \end{aligned}$$

where  $a_1$ ,  $a_2 \neq 0$  and  $b_1$  are arbitrary constants.

### 3.2.2. One-dimensional case

Among the second order homogeneous polynomials of  $\phi$  and  $\psi$  for  $g$ , we found the following two nontrivial examples with special polynomials for  $H$  shown in the two examples. The first example is

$$\begin{aligned} g &= \pm 2b_2\phi\psi, \quad H = b_1\phi^2 + b_2\phi^2\psi^2 + b_3\psi^2, \\ G &= (-4b_2^3\psi^4 - 8b_1b_2^2\psi^2 - 4b_1^2b_2)\phi^4 + (-8b_2^2b_3\psi^4 - 24b_1b_2b_3\psi^2 - 16b_1^2b_3)\phi^2 \\ &\quad - 4b_2b_3^2\psi^4 - 16b_1b_3^2\psi^2 \\ &= -4b_2H^2 - 16b_1b_3H, \end{aligned}$$

where  $b_1$ ,  $b_2$  and  $b_3$  are arbitrary constants. The second example among constant coefficient polynomials  $H = b_1\phi^4 + b_2\phi^4\psi^4 + b_3\psi^4$  is

$$\begin{aligned} g &= -\frac{a^3}{32b^2}\phi^2 + a\psi^2, \quad H = \frac{a^4}{2^{10}b^3}\phi^4 + b\psi^4, \\ G &= -\frac{3a^{10}}{2^{21}b^7}\phi^8 - \frac{3a^6}{2^{10}b^3}\psi^4\phi^4 - \frac{3}{2}a^2b\psi^8 = -\frac{3a^2}{2b}H^2, \end{aligned}$$

where  $a$  and  $b$  are arbitrary constants.

### 3.2.3. Higher-dimensional case

Motivated by symmetry constraints, we found the following two higher-dimensional examples. The first example is

$$\begin{aligned} g &= \frac{1}{8}(\phi_{11}\phi_{21} + \phi_{12}\phi_{22}), \\ H &= -\frac{1}{16}(\phi_{11}\phi_{21} + \phi_{12}\phi_{22})^2 - a_2(\lambda_1\phi_{21}^2 + \lambda_2\phi_{22}^2) - a_3(\phi_{11}^2 + \phi_{12}^2), \\ G &= 2a_2a_3(\phi_{11}\phi_{21} + \phi_{12}\phi_{22})(\lambda_1\phi_{11}\phi_{21} + \lambda_2\phi_{12}\phi_{22}) \\ &\quad - a_2a_3(\phi_{11}^2 + \phi_{12}^2)(\lambda_1\phi_{21}^2 + \lambda_2\phi_{22}^2) + 16a_2^2a_3(\lambda_1^2\phi_{21}^2 + \lambda_2^2\phi_{22}^2) \end{aligned}$$

$$+ 16a_2a_3^2(\lambda_1\phi_{11}^2 + \lambda_2\phi_{12}^2) + \frac{1}{4}H^2,$$

where  $a_2, a_3$  and  $\lambda_1 \neq \lambda_2$  are arbitrary constants. The Liouville integrability of the associated two two-dimensional Hamiltonian systems

$$\phi_{1j,x} = \frac{\partial H}{\partial \phi_{2j}}, \quad \phi_{2j,x} = -\frac{\partial H}{\partial \phi_{1j}}, \quad 1 \leq j \leq 2,$$

and

$$\phi_{1j,t} = \frac{\partial G}{\partial \phi_{2j}}, \quad \phi_{2j,t} = -\frac{\partial G}{\partial \phi_{1j}}, \quad 1 \leq j \leq 2,$$

has been shown in the case of  $a_2 = a_3 = -\frac{1}{16}$  in [12].

The second example is

$$g = \frac{1}{2}(\phi_{11}\psi_{11} + \phi_{12}\psi_{12} - \phi_{21}\psi_{21} - \phi_{22}\psi_{22}),$$

$$H = \frac{1}{4}(\phi_{11}\psi_{11} + \phi_{12}\psi_{12} - \phi_{21}\psi_{21} - \phi_{22}\psi_{22})^2 \\ + \lambda_1\phi_{21}\psi_{11} + \lambda_2\phi_{22}\psi_{12} - \phi_{11}\psi_{21} - \phi_{12}\psi_{22},$$

$$G = -2(\phi_{11}\psi_{11} + \phi_{12}\psi_{12} - \phi_{21}\psi_{21} - \phi_{22}\psi_{22})(\lambda_1\phi_{11}\psi_{11} + \lambda_2\phi_{12}\psi_{12} \\ - \lambda_1\phi_{21}\psi_{21} - \lambda_2\phi_{22}\psi_{22}) - 4(\phi_{11}\psi_{21} + \phi_{12}\psi_{22})(\lambda_1\phi_{21}\psi_{11} + \lambda_2\phi_{22}\psi_{12}) \\ - 4\lambda_1^2\phi_{21}\psi_{11} - 4\lambda_2^2\phi_{22}\psi_{12} + 4\lambda_1\phi_{11}\psi_{21} + 4\lambda_2\phi_{12}\psi_{22} - H^2,$$

where  $\lambda_1 \neq \lambda_2$  are arbitrary constants. The Liouville integrability of the associated two two-dimensional Hamiltonian systems

$$\phi_{ij,x} = \frac{\partial H}{\partial \psi_{ij}}, \quad \psi_{ij,x} = -\frac{\partial H}{\partial \phi_{ij}}, \quad 1 \leq i, j \leq 2,$$

and

$$\phi_{ij,t} = \frac{\partial G}{\partial \psi_{ij}}, \quad \psi_{ij,t} = -\frac{\partial G}{\partial \phi_{ij}}, \quad 1 \leq i, j \leq 2,$$

has been shown in [13].

In addition, we know that the Miura transformation

$$u = w_x - w^2$$

presents a link between the KdV equation (3.1) and the MKdV equation (3.5), where  $u$  and  $w$  are solutions to the KdV and MKdV equations, respectively. Therefore, we can also obtain other integrable decompositions of the KdV equation from the obtained integrable decompositions of the MKdV equation by the Miura transformation.

### 3.3. Diffusion equations

We would finally like to exhibit three examples of nonlinear diffusion equations of the Kolmogorov–Petrovskii–Piskunov (KPP) type [14], which means that  $u = 0$  is always a solution to the equation under investigation, and show that our method of constructing time–space integrable decompositions work well for them. But we will only discuss the case of scalar ansätze equations, which generates traveling wave solutions.

### 3.3.1. KPP equation with special third-order nonlinearity

We consider a diffusion equation with a special third-order nonlinearity:

$$u_t = u_{xx} + A_1 u + A_2 u^2 - (A_1 + A_2) u^3, \quad (3.9)$$

where  $A_1$  and  $A_2$  are arbitrary constants. Note that the function

$$F(u) := A_1 u + A_2 u^2 - (A_1 + A_2) u^3 = u(1-u)[A_1 + (A_1 + A_2)u]$$

satisfies  $F(0) = F(1) = 0$ . In the scalar case of integrable decompositions, suppose that

$$g = a_0 + a_1 \phi + a_2 \phi^2, \quad \Gamma = b_1 \phi - b_1 \phi^3,$$

where  $a_i$ ,  $0 \leq i \leq 2$ , and  $b_1$  are constants to be determined. Then, we have

$$\begin{aligned} u_t &= -2va_2b_1\phi^4 - va_1b_1\phi^3 + 2va_2b_1\phi^2 + va_1b_1\phi, \\ u_{xx} &= 8a_2b_1^2\phi^6 + 3a_1b_1^2\phi^5 - 12a_2b_1^2\phi^4 - 4a_1b_1^2\phi^3 + 4a_2b_1^2\phi^2 + a_1b_1^2\phi. \end{aligned}$$

Substituting these into the diffusion equation (3.9) and comparing each powers of  $\phi$  in the resulting equation will lead to six algebraic equations on the constants  $a_i$ ,  $0 \leq i \leq 2$ ,  $b_1$  and  $v$ . By solving these algebraic equations, we found the following six nontrivial examples.

Two of the examples are

$$v = \frac{3A_1 + A_2}{2B_1}, \quad g = \phi^2, \quad \Gamma = \frac{B_1}{2}\phi - \frac{B_1}{2}\phi^3; \quad (3.10)$$

$$v = -\frac{3A_1 + A_2}{2B_1}, \quad g = 1 - \phi^2, \quad \Gamma = \frac{B_1}{2}\phi - \frac{B_1}{2}\phi^3; \quad (3.11)$$

where  $B_1$  is determined by

$$2B_1^2 - A_1 - A_2 = 0. \quad (3.12)$$

The other four examples are

$$\begin{cases} v = \frac{3A_1 + 2A_2}{4(A_1 + A_2)B_2}, & g = -\frac{A_1}{A_1 + A_2}\phi^2, \\ \Gamma = A_1 B_2 \phi - A_1 B_2 \phi^3; \end{cases} \quad (3.13)$$

$$\begin{cases} v = -\frac{A_2}{4(A_1 + A_2)B_2}, & g = 1 - \frac{2A_1 + A_2}{A_1 + A_2}\phi^2, \\ \Gamma = (2A_1 + A_2)B_2 \phi - (2A_1 + A_2)B_2 \phi^3; \end{cases} \quad (3.14)$$

$$\begin{cases} v = -\frac{3A_1 + 2A_2}{4(A_1 + A_2)B_2}, & g = -\frac{A_1}{A_1 + A_2} + \frac{A_1}{A_1 + A_2}\phi^2, \\ \Gamma = A_1 B_2 \phi - A_1 B_2 \phi^3; \end{cases} \quad (3.15)$$

$$\begin{cases} v = \frac{A_2}{4(A_1 + A_2)B_2}, & g = -\frac{A_1}{A_1 + A_2} + \frac{2A_1 + A_2}{A_1 + A_2}\phi^2, \\ \Gamma = (2A_1 + A_2)B_2 \phi - (2A_1 + A_2)B_2 \phi^3, \end{cases} \quad (3.16)$$

where  $B_2$  is determined by

$$(8A_1 + 8A_2)B_2^2 - 1 = 0. \quad (3.17)$$

Note that a solution to the ansätze equation

$$\phi_\xi = b_1 \phi - b_1 \phi^3 \quad (3.18)$$

is given by

$$\phi(\xi) = \left[ \tanh\left(\frac{1}{2}b_1\xi + C\right) \right]^2, \quad (3.19)$$

where  $C$  is an arbitrary constant. Therefore, when

$$A_1 + A_2 > 0, \quad (3.20)$$

there are solutions to the diffusion equation (3.9), determined by  $u(x, t) = a_0 + a_1\phi(x - vt) + a_2(\phi(x - vt))^2$ .

### 3.3.2. KPP equation with general third-order nonlinearity

Let us now consider a diffusion equation with a general third-order nonlinearity:

$$u_t = u_{xx} + A_1u + A_2u^2 + A_3u^3, \quad (3.21)$$

where  $A_1, A_2$  and  $A_3$  are arbitrary constants. We will use another ansätze equation

$$\phi_\xi = b_0 + b_1\phi + b_2\phi^2, \quad (3.22)$$

where  $b_0, b_1$  and  $b_2 \neq 0$  are constants. When  $\Delta := b_1^2 - 4b_0b_2 \leq 0$ , the ansätze equation has a solution

$$\phi(\xi) = \frac{-b_1 + \sqrt{-\Delta} \tan(\frac{1}{2}\sqrt{-\Delta}\xi + \frac{1}{2}\sqrt{-\Delta}C)}{2b_2}, \quad (3.23)$$

where  $C$  is an arbitrary constant. When  $\Delta := b_1^2 - 4b_0b_2 \geq 0$ , it has a solution

$$\phi(\xi) = \frac{-b_1 - \sqrt{\Delta} \tanh(\frac{1}{2}\sqrt{\Delta}\xi + \frac{1}{2}\sqrt{\Delta}C)}{2b_2}, \quad (3.24)$$

where  $C$  is an arbitrary constant.

Among the choices for  $g$  and  $\Gamma$ :

$$g = a_0 + a_1\phi, \quad \Gamma = b_0 + b_1\phi + b_2\phi^2,$$

where  $a_0, a_1, b_0, b_1$  and  $b_2$  are constants, the diffusion equation (3.21) is found to have the following two integrable decompositions

$$v = \frac{3B_1B_2 + A_2}{B_1}, \quad g = a_1\phi, \quad \Gamma = B_2\phi + a_1B_1\phi^2, \quad (3.25)$$

and

$$v = -\frac{A_2}{2B_1}, \quad g = a_1\phi, \quad \Gamma = -\frac{A_1}{2a_1B_1} - \frac{A_2}{2B_1}\phi + a_1B_1\phi^2, \quad (3.26)$$

where  $a_1 \neq 0$  is an arbitrary constant, and  $B_1$  and  $B_2$  are two constants determined by

$$2B_1^2 + A_3 = 0, \quad 2A_3B_2^2 - 2A_2B_1B_2 - A_1A_3 = 0.$$

To guarantee the existence of  $B_1$  and  $B_2$  and to have that  $B_1 \neq 0$  ensuring that  $\Gamma$  is nonlinear, we need the conditions

$$A_3 < 0, \quad A_2^2 - 4A_1A_3 \geq 0 \quad (3.27)$$

for the first decomposition (3.25). But the second decomposition (3.26) only needs  $A_3 < 0$ , since  $B_2$  is not involved.

Note that in the first decomposition (3.25), we have  $\Delta = b_1^2 - 4b_0b_2 = B_2^2 \geq 0$ , but in the second decomposition (3.26), we have  $\Delta = b_1^2 - 4b_0b_2 = \frac{A_2^2}{4B_1^2} + 2A_1 = -\frac{A_2^2 - 4A_1A_3}{2A_3}$ , which can

be positive as well as negative. Therefore, from the first decomposition, we can obtain the traveling wave solution  $u(x, t) = a_1\phi(x - vt)$  with  $\phi(\xi)$  being defined by (3.23). From the second decomposition, we can obtain the traveling wave solution  $u(x, t) = a_1\phi(x - vt)$ , where  $\phi(\xi)$  is given by (3.23) if  $A_2^2 - 4A_1A_3 \leq 0$ , and  $\phi(\xi)$ , by (3.24) if  $A_2^2 - 4A_1A_3 \geq 0$ . For the diffusion equation (3.21), there are also other integrable decompositions to generate traveling wave solutions, using different ansätze equations in [15].

### 3.3.3. KPP equation with fifth-order nonlinearity

We want to discuss the following diffusion equation with fifth-order nonlinearity:

$$u_t = u_{xx} + A_1u + A_3u^3 + A_5u^5, \quad (3.28)$$

where  $A_1, A_3$  and  $A_5$  are constants. Among the selection of  $g$  and  $\Gamma$ :

$$g = a_0 + a_1\phi + a_2\phi^2, \quad \Gamma = b_1\phi + b_3\phi^3,$$

the diffusion equation (3.28) is found to have the following integrable decomposition:

$$v = \frac{A_3 + 4B_1B_2}{B_1}, \quad g = a_1\phi, \quad \Gamma = B_2\phi + a_1^2B_1\phi^3, \quad (3.29)$$

where the constant  $a_1 \neq 0$  is arbitrary, and the other two constants  $B_1$  and  $B_2$  are determined by

$$3B_1^2 + A_5 = 0, \quad 3A_5B_2^2 - 3A_3B_1B_2 - A_1A_5 = 0.$$

Note that the above equations for  $B_1$  and  $B_2$  are all quadratic. Thus, in order to guarantee the existence of  $B_1$  and  $B_2$  and to have that  $B_1 \neq 0$ , which ensures that  $\Gamma$  is nonlinear, we need the conditions:

$$A_5 < 0, \quad A_3^2 - 4A_1A_5 \geq 0, \quad (3.30)$$

under which we have

$$B_1 = \sqrt{-\frac{A_5}{3}}, \quad B_2 = \frac{\sqrt{-3A_5}(A_3 \pm \sqrt{A_3^2 - 4A_1A_5})}{6A_5}. \quad (3.31)$$

On the other hand, a solution to the ansätze equation

$$\phi_\xi = b_1\phi + b_3\phi^3 \quad (3.32)$$

is give by

$$\phi(\xi) = \pm b_1[-b_1b_3 + Cb_1^2 \exp(-2b_1\xi)]^{-1/2}, \quad (3.33)$$

where  $C$  is an arbitrary constant. The condition  $b_1b_3 \leq 0$  guarantees that the solution (3.33) is defined over the whole line, together with a selection of a positive constant  $C$ . In our case,  $b_1 = B_2$  and  $b_3 = a_1^2B_1$ . Therefore, the condition  $b_1b_3 \leq 0$  requires

$$A_3 + \sqrt{A_3^2 - 4A_1A_5} \geq 0 \quad \text{or} \quad A_3 - \sqrt{A_3^2 - 4A_1A_5} \geq 0, \quad (3.34)$$

which contains two cases of  $A_1, A_3$  and  $A_5$  discussed in [16].

To conclude, under the conditions (3.30) and (3.34), the diffusion equation (3.28) has the analytic traveling wave solution determined by  $u(x, t) = a_1\phi(x - vt)$ . Note that  $B_2$  has two cases as shown in (3.31). The plus and minus cases correspond to the first and second conditions in (3.34), respectively.

#### 4. Concluding remarks

Time–space integrable decompositions have been introduced and analyzed for evolution equations, without using any structure associated with evolution equations. Concrete criteria guaranteeing the existence of time–space integrable decompositions have been established, especially in the case where the ansätze equations are scalar evolution equations and one-dimensional integrable Hamiltonian systems. The criteria have been applied to the celebrated KdV, MKdV and nonlinear diffusion equations, and the examples of time–space integrable decompositions, furnished in the previous section, have shown the integrability by quadratures for the equations investigated. In particular, the resulting integrable decompositions have exhibited many interesting solution relations with integrable ODEs, including those relations of traveling wave solutions with scalar differential equations and one-dimensional Hamiltonian systems.

We remark that the established theory also provides techniques for constructing evolution equations which possess given time–space integrable decompositions. This is an interesting problem itself. For example, in Eq. (2.24), conversely we can first fix two functions  $g$  and  $H$  and then look for another function  $f$ . Equations (2.21) and (2.22) will yield the other Hamiltonian function  $G$  to govern the evolution of  $\phi$  and  $\psi$ . The resulting evolution equation  $u_t = f$  will then possess a given integrable decomposition  $u = g(\phi, \psi)$ , and the ansätze equations are two integrable Hamiltonian systems determined by  $H$  and  $G$ .

Moreover, our results generalize the method of symmetry constraints [1,4] in soliton theory. However, it is difficult to tackle the case of integrable decompositions whose ansätze equations are higher-order integrable Hamiltonian systems. This is because we did not use the nice properties such as Lax pairs and recursion operators which soliton equations have been already explored to possess.

There are other questions worth further investigating. For example, what about the case of higher-dimensional soliton equations such as the KP equation and the case of systems of soliton equations such as the AKNS systems? Any successful attempt on these problems will help achieve a deeper understanding of the integrability by quadratures for partial differential equations (PDEs). We hope that some formulation of the integrability by quadratures can be furnished for PDEs, similar to the Liouville–Arnold theorem for finite-dimensional Hamiltonian systems.

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