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Growth Properties of Semigroups Generated by Fractional Powers of Certain Linear Operators

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Certain semigroups are generated by powers $-(-A)^a$, for closed operators A in Banach space and $0 < a < 1$. Properties of extent of the resolvent set and size of the resolvent operator of A correspond to properties relating to the sectors of holomorphy of the semigroups, and their growth near the origin and infinity. In this paper, we deal with semigroups having two different types of growth properties. In the first instance, the semigroup grows near the origin as r^{-t}, $0 < t < 1$. We show that such semigroups are fractional-power semigroups of operators A, whose resolvents decay as r^{-s}, $0 < s < 1$, in subsectors of the right-hand half-plane. In the second instance, the semigroups are bounded near the origin, and admit special estimates on growth at the periphery of their sectors of definition. We show that for the corresponding A, the resolvent is defined and admits special growth estimates in a region which contains every subsector of the right half-plane; and in these subsectors, the resolvent decays as r^{-1}.

INTRODUCTION

There are closed operators A in Banach spaces with the property that fractional powers $-(-A)^a$ generate holomorphic semigroups. Some properties of such semigroups are related to properties of A: the region of holomorphy of the semigroup and properties of growth and convergence within that region, compared to the location of the resolvent set and size of the resolvent operator of A. Previous papers (i.e., [4, 5]) have considered the problem of deriving the properties of the semigroup from assumptions about A. In this paper, we consider the converse problem of reconstructing the operator A from a semigroup having known properties of a fractional-power semigroup, establishing a correspondence between the latter properties and those of A.

In Sect. 1, we deal with semigroups holomorphic in certain sectors of the plane, and having a slow growth near the origin; that is, growing no faster than r^{-s}, where r represents distance from the origin and $0 < s < 1$. We show that such growth corresponds to a relatively slow decay near infinity—as with r^{-t}, $0 < t < 1$—in the resolvent of A, which exists in a subsector of the right-hand half-plane. These conditions on the growth of the resolvent are weaker than those in Kato [4] and Komatsu [5]. In [4], the resolvent is also defined in a sector, but the resolvent is assumed to decay as r^{-1}; these assumptions lead to semigroups which are bounded near the origin. Komatsu retained the condition of decay as r^{-1}, but weakened the condition on existence of the resolvent; he assumed merely that the resolvent be defined along the positive real ray.

In Sect. 2, the semigroups are considerably more special. They converge in the C_0 sense at the origin, and admit special estimates on their asymptotic growth in their sectors of definition. Correspondingly, A itself has properties very close to those of a semigroup generator. The resolvent set contains every proper subsector of the right half-plane, and the resolvent decays almost as the reciprocal of distance from the imaginary axis. Here it does not suffice to apply the previous methods—of Sect. 1, or Kato—to A in each subsector of the half-plane. Beals [1] has shown that A need not have the desired properties of semigroup generators, unless the resolvent set is asymptotic to the half-plane, in a special sense. Our conditions on the fractional-power semigroups correspond to conditions on A in line with, but stronger than those of Beals.

Certain notation will remain constant throughout. We shall deal with operators in a fixed Banach space X. For a complex z, $\arg z$ will denote the value of the argument having $-\pi < \arg z \leqslant \pi$. For a real r, $-(-z)^r$ will represent that branch which is negative whenever z is a negative real. It will be convenient to write $|\arg z| = \pi - \varphi$, $0 \leqslant \varphi \leqslant \pi$. Thus if $\arg z = \pm(\pi - \varphi)$, then it is simple to see that $-(-z)^r = |z|^r \exp(\pm i[\pi - r\varphi])$, accordingly. Finally, $\mathbb{R}^+ \exp(i\varphi)$ will represent the ray

$$\{z = t \exp(i\varphi) \mid t \geqslant 0\}.$$

1

Let T be a closed, densely defined linear operator in a Banach space X. Assume that the resolvent set of T contains a sector of the form $|\arg z| < \pi - \theta$, as well as a neighborhood of the origin, and that for some $k > 0$, the resolvent of T satisfies $\|(zI - T)^{-1}\| = O(|z|^{-k})$ as $z \to \infty$, in each smaller sector $|\arg z| \leqslant \pi - \theta - \epsilon$. For convenience, let us say the resolvent contains the sector $|\arg(z + 1)| < \pi - \theta$. Suppose $f(z)$ is a function holomorphic along and to the left of $|\arg z| = \pi - \theta - \epsilon$, at least for small ϵ, and $|f(z)| = O(|z|^{-m})$ as $z \to \infty$, where $m + k > 1$. Then the integral

$$f(T) = (1/2\pi i) \int_{\Gamma_\epsilon} f(z)(zI - T)^{-1} \, dz \tag{1.1}$$

with $\Gamma_\epsilon = \{z \mid |\arg(z + \frac{1}{2})| = \pi - \theta - \epsilon\}$ oriented toward increasing imaginary part, is independent of ϵ, and defines a bounded operator in X. In particular, for small enough $a > 0$ and for complex w close enough to the positive real axis, $f_w(z) = \exp(-w(-z)^a)$ satisfies the requirements, and we have the resulting operator, $\exp(-w(-T)^a)$.

Our two main results are as follows:

THEOREM 1.1. *Let A be a closed, densely defined operator. Suppose there are numbers θ and b_0, $\pi/2 \leqslant \theta < \pi$ and $0 < b_0 \leqslant 1$, such that:*

(a) *The resolvent set of A contains the sector given by $|\arg z| < \pi - \theta$, together with a neighborhood of the origin.*

(b) *In any sector $|\arg z| < \pi - \theta - \epsilon$, $\epsilon > 0$, we have $\|(zI - A)^{-1}\| = O(|z|^{-b})$ as $z \to \infty$, for each $b < b_0$.*

(c) $(1 - b_0)\theta < \pi/2$.

Then for any fixed a with $1 - b_0 < a < \pi/2\theta$, we have:

(i) *The family $\{\exp(-w(-A)^a)\}$ is a holomorphic semigroup in w, defined for $|\arg w| < (\pi/2) - a\theta$.*

(ii) *Each operator $\exp(-w(-A)^a)$ is one-to-one, and the union of their ranges is dense in X.*

(iii) *For each $\epsilon > 0$ and $k > (1 - b_0)/a$, there are positive constants C and D, such that in the sector given by $|\arg w| < (\pi/2) - a(\theta + \epsilon)$, we have*

$$\| \exp(-w(-A)^a)\| \leqslant C |w|^{-k} \exp(-D |w|).$$

THEOREM 1.2. *Let $\{B(w)\}$ be a holomorphic semigroup in w, with numbers φ and k_0, $0 < \varphi < \pi/2$ and $0 \leqslant k_0 < 1$, such that:*

(a) *$B(w)$ is defined for $|\arg w| < (\pi/2) - \varphi$.*

(b) *Each $B(w)$ is one-to-one, and the union of the ranges $B(w)X$ is dense in X.*

(c) *For each* $\epsilon > 0$ *and* $k > k_0$, *there are constants* C *and* D, *such that* $\| B(w)\| \leqslant C \mid w \mid^{-k} \exp(-D \mid w \mid)$ *whenever* $\mid \arg w \mid < (\pi/2) - \varphi - \epsilon$.

Then for any a such that $\varphi/\pi < a \leqslant 2\varphi/\pi$, *there exists an operator* S, *satisfying the hypothesis of Theorem* 1.1, *such that* $B(w) = \exp(-w(-S)^a)$ *for* $\mid \arg w \mid < (\pi/2) - \varphi$.

Proof of Theorem 1.1. As previously mentioned, it will be convenient to assume that the resolvent set of A contains the sector $\mid \arg(z + 1)\mid < \pi - \theta$ (cf. condition 1.1(a).)

(i) Assume w is complex, $\mid \arg w \mid < (\pi/2) - a\theta$. Pick ϵ so small that $a(\theta + \epsilon) < \pi/2$ and $\mid \arg w \mid < (\pi/2) - a\theta - \epsilon$, and let Γ_ϵ be the angle given by $\mid \arg(z + \frac{1}{2})\mid = \pi - \theta - \epsilon$. Then $\mid \arg z \mid > \pi - (\theta + \epsilon)$ for $z \in \Gamma_\epsilon$, so that $\mid \arg - (-z)^a \mid > \pi - a(\theta + \epsilon)$. Hence $\mid \arg(-w(-z)^a)\mid > \pi/2$, and we may set $f(z) = \exp(-w(-z)^a)$, $T = A$ in (1.1). This defines $\exp(-w(-A)^a)$ as a holomorphic function of w. The semigroup property follows by a standard argument from the operator calculus ([2, Lemma 2]).

(ii) (See [2, Lemmas 1, 3].)

(iii) Given $\epsilon > 0$, we may write

$$\exp(-w(-A)^a) = (1/2\pi i) \int_{\Gamma_{\epsilon/2}} \exp(-w(-z)^a)(zI - A)^{-1}\, dz \qquad (1.2)$$

for any w satisfying

$$\mid \arg w \mid < (\pi/2) - a(\theta + \epsilon). \qquad (1.3)$$

Now if $k > (1 - b_0)/a$, we have $1 - ak < b_0$, so that there is a constant $C_1 > 0$ such that $\|(zI - A)^{-1}\| < C_1 r^{-(1-ak)}$ for $z \in \Gamma_{\epsilon/2}$, where (say) $r = \mid z + 1 \mid$.

We next wish to estimate $\mid \exp(-w(-z)^a)\mid$. For $z \in \Gamma_{\epsilon/2}$, $-(-z)^a$ is to the left of the angle $\mathbb{R}^+ \exp(\pm i[\pi - a\theta - a(\epsilon/2)])$. Consequently, there is a $D_1 > 0$ such that $-(-z)^a + D_1$ lies to the left of $\mathbb{R}^+ \exp(\pm i[\pi - a\theta - a(3\epsilon/4)])$, that is, $\mid \arg(D_1 - (-z)^a)\mid > \pi - a\theta - (3a\epsilon/4)$, and so for w as in (1.3),

$$\mid \arg(wD_1 - w(-z)^a)\mid > (\pi/2) + (a\epsilon/4). \qquad (1.4)$$

Moreover, there is a C_2 such that $\mid D_1 - (-z)^a \mid > C_2 r^a$. In view of (1.4), for $z \in \Gamma_{\epsilon/2}$, we have

$$\operatorname{Re}(wD_1 - w(-z)^a) < -\mid w \mid C_2 r^a \sin(a\epsilon/4) = -C_3 \mid w \mid r^a.$$

Therefore

$$\exp(\operatorname{Re}(-w(-z)^a)) \leqslant \exp(-\operatorname{Re} wD_1)\exp(-C_3 \mid w \mid r^a)$$
$$\leqslant \exp(-D \mid w \mid)\exp(-C_3 \mid w \mid r^a),$$

with $D = D_1 \sin a(\theta + \epsilon)$. Equation (1.2) then gives

$$\| \exp(-w(-A)^a)\| \leqslant C_4 \exp(-D \mid w \mid) \int_0^\infty \exp(-C_3 \mid w \mid r^a)\, r^{ka-1}\, dr,$$

and the last integral is $C_5 \mid w \mid^{-k}$. Q.E.D.

Proof of Theorem 1.2. Because $B(t)$ is integrable in real t at zero and of exponential decay at infinity,

$$R(v) = \int_0^\infty \exp(-vt)\, B(t)\, dt$$

defines a holomorphic function of v for $\operatorname{Re} v \geqslant -D_0$, for some $D_0 > 0$. We shall first extend $R(v)$ analytically to the sector defined by $\mid \arg v \mid < \pi - \varphi$.

Let $0 \leqslant \arg v < \pi - \varphi - \epsilon$; a similar proof goes if $0 \geqslant \arg v > -(\pi - \varphi - \epsilon)$. Let L be the ray $\mathbb{R}^+ \exp(-i[(\pi/2) - \varphi - (\epsilon/2)])$, oriented out from the origin. If $w \in L$, then $0 \leqslant \arg vw < (\pi/2) - (\epsilon/2)$. In view of condition (c) of the hypothesis, the integral

$$\tilde{R}(v) = \int_L \exp(-vw)\, B(w)\, dw \tag{1.5}$$

is absolutely convergent, and so defines an analytic function of v, for $0 \leqslant \arg v < \pi - \varphi - \epsilon$. Due to the rapid decay of the integrand, it is clear that $\tilde{R}(v)$ coincides with $R(v)$ whenever $\operatorname{Re} v > 0$. Hence we see that $R(v)$ can be continued to the sector $\mid \arg v \mid < \pi - \varphi - \epsilon$, with $\epsilon > 0$ arbitrary. Thus $R(v)$ extends to $\mid \arg v \mid < \pi - \varphi$.

LEMMA 1.3. *Given* $\epsilon > 0$ *and* $k > k_0$, *in the sector* $\mid \arg v \mid < \pi - \varphi - \epsilon$, *we have* $\| R(v)\| \leqslant C \mid v \mid^{k-1}$ *for some constant* C.

Proof. Again, assume $0 \leqslant \arg v < \pi - \varphi - \epsilon$, and define L as for (1.5). We may write $R(v) = \tilde{R}(v) = \int_L \exp(-wv)\, B(w)\, dw$. For $w \in L$, $0 \leqslant \arg vw < (\pi/2) - (\epsilon/2)$ implies

$$\| R(v)\| \leqslant \int_0^\infty \exp(-\mid v \mid \mid w \mid \sin(\epsilon/2))\, C_1 \mid w \mid^{-k}\, d \mid w \mid$$
$$= C_1(\sin(\epsilon/2))^{k-1} \mid v \mid^{k-1}.$$

Q.E.D.

For small $\epsilon > 0$, let M_ϵ be the contour

$$M_\epsilon = \{v \mid \text{either } 0 < |\arg v| = \pi - \varphi - \epsilon \text{ and } \operatorname{Re} v \leqslant -D_0 \\ \text{or } |\arg v| \geqslant \pi - \varphi - \epsilon \text{ and } \operatorname{Re} v = -D_0\}, \tag{1.6}$$

oriented in the direction of increasing imaginary part. Along this contour, by Lemma 1.3, $\| R(v)\| = O(|v|^{k-1})$ as $v \to \infty$, for any k with $1 > k > k_0$. Hence if n is a natural number and p is real, $p \leqslant n - 1$, then the integral

$$K(p, n) = (1/2\pi i) \int_{M_\epsilon} -(-v)^{p-n} R(v)\, dv$$

is absolutely convergent and independent of ϵ, and defines a bounded operator.

Lemma 1.4. *$R(0)$ is one-to-one.*

Proof. For any $x \in X$ and $t > 0$,

$$\begin{aligned} B(t)x &= \lim_{h \to 0^+} h^{-1} \int_t^{t+h} B(s)x\, ds \\ &= \lim h^{-1} \left(\int_t^\infty B(s)x\, ds - \int_{t+h}^\infty B(s)x\, ds \right) \\ &= \lim h^{-1}(B(t) - B(t+h)) \int_0^\infty B(s)x\, ds \\ &= -B'(t)\, R(0)x, \end{aligned}$$

and the conclusion follows from condition (b). Q.E.D.

In particular, we may speak of $R(0)^{-1}$.

Let now $(\pi/2\varphi) \leqslant p < \pi/\varphi$. Henceforth we take $n > (\pi/\varphi) + 1$. Then

$$\begin{aligned} R(0)\, K(p, n) &= (1/2\pi i) \int_{M_\epsilon} -(-v)^{p-n} R(0)\, R(v)\, dv \\ &= (1/2\pi i) \int_{M_\epsilon} -(-v)^{p-n}(-v)^{-1}[R(v) - R(0)]\, dv \\ &= K(p, n+1). \end{aligned}$$

Hence, for a given $x \in X$, we have $K(p, n)x \in \mathscr{D}(R(0)^{-n})$ iff $K(p, n+1)x \in \mathscr{D}(R(0)^{-n-1})$, and if both are true, then $R(0)^{-n}K(p, n)x =$

$R(0)^{-n-1} K(p, n+1)x$. Therefore we may define the operator S as follows:

$$x \in \mathscr{D}(S) \quad \text{iff} \quad K(p, n)x \in \mathscr{D}(R(0)^{-n}) \quad \text{for large } n,$$

and

$$Sx = R(0)^{-n} K(p, n)x \quad \text{for} \quad x \in \mathscr{D}(S).$$

We shall show that S satisfies the hypothesis of Theorem 1.1.

LEMMA 1.5. (i) *S is closed and densely defined.*

(ii) *The resolvent set of S contains the sector $|\arg z| < \pi - p\varphi$, along with a neighborhood of the origin.*

(iii) *In any smaller sector $|\arg z| < \pi - p\varphi - \epsilon$, we have $\|(zI - S)^{-1}\| = O(|z|^{-b})$ as $z \to \infty$, for any $b < 1 - (k_0/p)$.*

Proof. (i) $R(0)^{-n}$ is closed, because it has a bounded inverse. Because $K(p, n)$ is bounded, $S = R(0)^{-n} K(p, n)$ is closed. Further, as in Lemma 1.4, for $x \in X$ and $t > 0$,

$$\begin{aligned} B(t)x &= B(t/n)^n x \\ &= (-R(0)\, B'(t/n))^n x \\ &= \pm R(0)^n\, B'(t/n)^n x, \end{aligned}$$

so that $K(p, n)\, B(t)x = \pm R(0)^n\, K(p, n)\, B'(t/n)^n x$. It follows $B(t)x \in \mathscr{D}(S)$. By condition (b), $\mathscr{D}(S)$ is dense.

(ii) If $|\arg z| < \pi - p\varphi$, assume $|\arg z| < \pi - p\varphi - \epsilon$. Define the contour $M_{\epsilon/2p}$ as in (1.6). As before, the integral

$$F(z) = (1/2\pi i) \int_{M_{\epsilon/2p}} (z + (-v)^p)^{-1} R(v)\, dv$$

defines a holomorphic function of z. Because $f(v) = z + (-v)^p$ does not have zeroes to the left of $M_{\epsilon/2p}$, we can show by the usual operator argument that $F(z)$ satisfies the resolvent equation. To prove that $F(z)$ is the resolvent of S, it then suffices to show that $F(0)$ is exactly $(0 - S)^{-1}$. To this end, since $F(0)$ commutes with $R(0)$, it will suffice to show that $K(p, n)\, F(0) = F(0)\, K(p, n) = -R(0)^n$.

Now

$$\begin{aligned} K(p, n)\, F(0) &= F(0)\, K(p, n) \\ &= (1/2\pi i) \int_{M_{\epsilon/2p}} -(-v)^{-n} R(v)\, dv. \end{aligned}$$

In the integral, we may change to the contour $\operatorname{Re} v = -D_0$, and we have

$$(1/2\pi i)\int_{-D_0-i\infty}^{-D_0+i\infty} -(-v)^{-n} R(v)\,dv$$

$$= (1/2\pi i)\int_{-D_0-i\infty}^{-D_0+i\infty} -(-v)^{-n} \int_0^\infty \exp(-vt)\,B(t)\,dt\,dv$$

$$= \int_0^\infty (1/2\pi i)\int_{-D_0-i\infty}^{-D_0+i\infty} (-1)^n v^{-n} \exp(-vt)\,dv\,B(t)\,dt$$

$$= -\int_0^\infty t^{n-1}(1/(n-1)!)\,B(t)\,dt,$$

and the last is $-R(0)^n$, by repeated integration by parts. The same argument applies if z is close enough to the origin—specifically, if $|z| < D_0{}^p$. This establishes (ii).

(iii) Again, we assume $|\arg z| < \pi - p\varphi - \epsilon$. We have

$$(zI - S)^{-1} = (1/2\pi i)\int_{M_{\epsilon/2p}} (z + (-v)^p)^{-1} R(v)\,dv. \tag{1.7}$$

Suppose $0 < b < 1 - (k_0/p)$. Since $k_0 < p(1 - b)$, we may choose k such that $k_0 < k < p(1 - b)$. By Lemma 1.3, there is a constant C_1 such that $\| R(v)\| \leqslant C_1 |v|^{-(1-k)}$ for $v \in M_{\epsilon/2p}$. Further $|z + (-v)^p|$ is at least as large as the distance from $-(-v)^p$ to the sector $|\arg w| < \pi - p\varphi - \epsilon$ and the distance from z to the image $\{-(-w)^p \mid w \in M_{\epsilon/2p}\}$. The former distance is at least $|v|^p \sin(\epsilon/2)$, the latter at least $|z| \sin(\epsilon/2)$. Hence

$$|z + (-v)^p| \geqslant (|v|^p \sin(\epsilon/2))^{1-b}(|z| \sin(\epsilon/2))^b.$$

Equation (1.7) then yields

$$\|(zI - S)^{-1}\| \leqslant (1/2\pi)(\sin(\epsilon/2))^{-1}\int_{M_{\epsilon/2p}} |v|^{-p(1-b)} |z|^{-b} C_1 |v|^{k-1}\,d|v|$$

$$\leqslant C_2 |z|^{-b}\int_1^\infty |v|^{-(1+p(1-b)-k)}\,d|v|$$

$$= C(\epsilon) |z|^{-b}. \qquad \text{Q.E.D.}$$

In view of Lemma 1.5, S satisfies the hypothesis of Theorem 1.1, where $b_0 = 1 - (k_0/p)$, $\theta = p\varphi$. Here we note that $(1 - b_0)\theta =$

$k_0\varphi < \pi/2$, as required. Because $1 - b_0 = k_0/p < 1/p < (\pi/2p\varphi) = (\pi/2\theta)$, the semigroup $\{\exp(-w(-S)^{1/p})\}$ is defined according to Theorem 1.1, for $|\arg w| < (\pi/2) - (1/p)\theta = (\pi/2) - \varphi$, the domain of definition of $B(w)$.

Define the contour Γ_ϵ by

$$\Gamma_\epsilon = \{z \mid |\arg(z + D)| = \pi - p\varphi - \epsilon\}$$

for small enough $D > 0$ (cf. the remarks leading to (1.1).) As in Theorem 1.1, for any $t > 0$,

$$\exp(-t(-S)^{1/p}) = (1/2\pi i) \int_{\Gamma_\epsilon} \exp(-t(-z)^{1/p})(zI - S)^{-1}\, dz. \tag{1.8}$$

Defining $M_{\epsilon/2p}$ as in (1.6), we have

$$(zI - S)^{-1} = (1/2\pi i) \int_{M_{\epsilon/2p}} (z + (-v)^p)^{-1} R(v)\, dv, \qquad z \in \Gamma_\epsilon\,. \tag{1.9}$$

Using (1.9) in (1.8) and reversing the order of integration, we find

$$H(t) \equiv \exp(-t(-S)^{1/p}) \tag{1.10}$$
$$= (1/2\pi i) \int_{M_{\epsilon/2p}} [(1/2\pi i) \int_{\Gamma_\epsilon} \exp(-t(-z)^{1/p}(z + (-v)^p)^{-1}\, dz]\, R(v)\, dv.$$

Because $-(-v)^p$ is to the left of Γ_ϵ when $v \in M_{\epsilon/2p}$, Cauchy's formula applies to the integral within the integral in (1.10), and also $[(-v)^p]^{1/p} = -v$. Hence (1.10) becomes

$$H(t) = (1/2\pi i) \int_{M_{\epsilon/2p}} \exp(tv)\, R(v)\, dv.$$

Then (compare with the proof of Lemma 1.5(ii))

$$\begin{aligned}
H(t)\, R(0)^2 &= (1/2\pi i) \int_{M_{\epsilon/2p}} \exp(tv)(-v)^{-2}\, R(v)\, dv \\
&= (1/2\pi i) \int_{-D_0 - i\infty}^{-D_0 + i\infty} \exp(tv)(-v)^{-2} \int_0^\infty \exp(-vs)\, B(s)\, ds\, dv \\
&= \int_0^\infty (1/2\pi i) \int_{-D_0 - i\infty}^{-D_0 + i\infty} \exp(-(s - t)v)(-v)^{-2}\, dv\, B(s)\, ds \\
&= \int_t^\infty (s - t)\, B(s)\, ds = B(t) \int_0^\infty uB(u)\, du \\
&= B(t)\, R(0)^2.
\end{aligned}$$

Because $R(0)^2$ has dense range, we conclude $\exp(-t(-S)^{1/p}) = H(t) = B(t)$ for all $t > 0$. Finally, since both functions are holomorphic, it follows that $B(w) = \exp(-w(-S)^{1/p})$ throughout their common domain. This completes the proof of Theorem 1.2.

If we start with A as in Theorem 1.1, then each $\exp(-w(-A)^a)$ satisfies the hypothesis of Theorem 1.2. The construction in Theorem 1.2 may be carried out for $\exp(-w(-A)^a)$ with $p = 1/a$ so that we find S with $\exp(-w(-A)^a) = \exp(-w)-S)^a)$. Then we have the expected result:

THEOREM 1.6. *Suppose $B(w) = \exp(-w(-A)^a)$, the latter defined as in Theorem 1.1. If we apply Theorem 1.2 to $B(w)$ to produce an operator S, such that $B(w) = \exp(-w(-S)^a)$, then $S = A$.*

Proof. From the proof of Lemma 1.5 (ii), S can be characterized by

$$S^{-1} = (1/2\pi i) \int_{M_{\epsilon a/2}} (-v)^{-p} R(v)\, dv, \tag{1.11}$$

where $R(v) = \int_0^\infty \exp(-vt)\, B(t)\, dt$ for $\operatorname{Re} v > 0$, and $R(v)$ extends analytically to the sector $|\arg v| < \pi - \varphi = \pi - a\theta$.

Now if $\operatorname{Re} v > 0$,

$$\begin{aligned} R(v) &= \int_0^\infty \exp(-vt) \exp(-t(-A)^a)\, dt \\ &= \int_0^\infty \exp(-vt)(1/2\pi i) \int_{\Gamma_{\epsilon/3}} \exp(-t(-z)^a)(zI - A)^{-1}\, dz\, dt. \end{aligned}$$

For small enough ϵ, $-(-z)^a$ has negative real part for every $z \in \Gamma_{\epsilon/3}$, so that

$$\begin{aligned} R(v) &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} \int_0^\infty \exp(-vt - (-z)^a t)\, dt\, (zI - A)^{-1}\, dz \\ &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} (v + (-z)^a)^{-1} (zI - A)^{-1}\, dz. \end{aligned}$$

This last integral converges, because as $z \to \infty$, $|v + (-z)^a|^{-1} = O(|z|^{-a})$ and $\|(zI - A)^{-1}\| = O(|z|^{-b})$ for any b with $1 - a < b < b_0$. Substituting in (1.11), we have

$$\begin{aligned} S^{-1} &= (1/2\pi i) \int_{M_{\epsilon a/2}} (-v)^{-p} (1/2\pi i) \int_{\Gamma_{\epsilon/3}} (v + (-z)^a)^{-1} (zI - A)^{-1}\, dz\, dv, \\ S^{-1} &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} \left[(1/2\pi i) \int_{M_{\epsilon a/2}} (-v)^{-p} (v + (-z)^a)^{-1}\, dv \right] (zI - A)^{-1}\, dz. \end{aligned} \tag{1.12}$$

If $z \in \Gamma_{\epsilon/3}$, then $|\arg z| > \pi - \theta - (\epsilon/3)$, and $|\arg(-(-z)^a)| > \pi - a\theta - (a\epsilon/3)$, so that $-(-z)^a$ is to the left of $M_{a\epsilon/2}$. Therefore Cauchy's formula applies to the integral within the integral in (1.12), and we have (since $p = 1/a$)

$$S^{-1} = (1/2\pi i) \int_{\Gamma_{\epsilon/3}} (-z)^{-1}(zI - A)^{-1}\, dz. \tag{1.13}$$

The operator on the right is exactly A^{-1}, because it coincides with A^{-1} on the ranges of $\exp(-w(-A)^a)$:

$$\begin{aligned} A^{-1}B(w)x &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} \exp(-w(-z)^a)\, A^{-1}(zI - A)^{-1}x\, dz \\ &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} \exp(-w(-z)^a)\, z^{-1}[A^{-1} + (zI - A)^{-1}]x\, dz \\ &= (1/2\pi i) \int_{\Gamma_{\epsilon/3}} z^{-1}(zI - A)^{-1}\, dz\, (1/2\pi i) \int_{\Gamma_{\epsilon/3}} \exp(-w(-z)^a)x\, dz. \end{aligned}$$

This completes the proof.

We shall add a number of observations. The proofs are fairly straightforward operator and contour integral arguments.

Observations.

(a) The operator-valued function

$$\begin{aligned} R(v) &= \int_0^\infty e^{-vt}B(t)\, dt \\ &= \int_{\Gamma_\epsilon} (v + (-z)^a)^{-1}(zI - A)^{-1}\, dz \end{aligned} \tag{1.14}$$

has played a key role in several arguments. $R(v)$ turns out to be the resolvent of an operator T, defined by

$$Tx = A^2(1/2\pi i) \int_{\Gamma_\epsilon} -(-z)^{a-2}(zI - A)^{-1}x\, dz \tag{1.15}$$

for exactly those x for which the integral in (1.15) belongs to $\mathscr{D}(A^2)$; note that the integral is convergent in the norm. We shall write $T = -(-A)^a$. Then $-(-A)^a$ is exactly the smallest closed extension of the infinitesimal generator of $\{B(t) \mid t > 0\}$. In this sense, $\{B(t)\}$ is generated by a fractional power of A.

(b) Although $B(w) = \exp(-w(-A)^a)$ is defined only for

$1 - b_0 < a < (\pi/2\theta)$, it is clear that the complex integral in (1.14) converges also for $(\pi/2\theta) \leqslant a \leqslant 1$. We may therefore extend the definition of $-(-A)^a$ to $a \in [(\pi/2\theta), 1]$, and (1.15) still holds with $T = -(-A)^a$.

(c) We have the expected properties of powers. If $a_1, a_2 > 1 - b_0$ and $a_1 + a_2 \leqslant 1$, then $(-A)^{a_1}(-A)^{a_2} = (-A)^{a_1+a_2}$. Further, if $1 \geqslant a_1 > 1 - b_0$ and $1 \geqslant a_2 > (1 - b_0)/a_1$, then the power $-((-A)^{a_1})^{a_2}$ of $-(-A)^{a_1}$ can be defined by the methods above, and we find $-((-A)^{a_1})^{a_2} = -(-A)^{a_1 a_2}$.

(d) Finally, if $b_0 = 1$, so that $-(-A)^a$ is defined for $0 < a \leqslant 1$, then $\mathscr{D}((-A)^{a_1})$ contains $\mathscr{D}((-A)^{a_2})$ whenever $a_1 < a_2$ and $(-A)^{a_1}x \to (-A)^{a_2}x$ as a_1 increases to a_2, for any $x \in \mathscr{D}((-A)^{a_2})$.

2

Beals [1, 2] has described a class of operators A which, although they cannot generate semigroups in the usual sense, nevertheless have the property of semigroup generators, that the Abstract Cauchy Problem $u'(t) = Au(t)$ is solvable for a dense set of initial conditions $u_0 = u(0)$. This set is precisely the union of the ranges of operators which we may view as $\exp(-t(-A)^a)$, $t > 0$, and these semigroups play a crucial role in establishing the ACP property of A. It is therefore natural to attempt to charactize the semigroups which arise in this fashion, as we shall do in this section, under certain strong conditions on A.

As in the previous section, we shall deal with integrals of the form $f(A) = (1/2\pi i) \int_\Gamma f(z)(zI - A)^{-1}\, dz$, with appropriate conditions on f and on $(zI - A)^{-1}$, to make the integrals absolutely convergent. Moreover, to the right of Γ will lie a region larger, in a special sense, than any proper subsector of the right half-plane, so that $f_w(z) = \exp(-w(-z)^a)$ will be applicable for all a, $0 < a < 1$. Thus $\{\exp(-w(-A)^a)\}$ will be defined for all a, for appropriate w.

Once more, the characterization is given in two principal results.

THEOREM 2.1. *Let A be a closed, densely defined linear operator, and assume*:

(a) *There are numbers $p_0 > 1$, $C_0 > 0$ such that the resolvent set of A contains the region $S_0 = \{z \mid \operatorname{Re} z \geqslant C_0 \mid \operatorname{Im} z \mid^{1/p_0}\}$, along with a neighborhood of the origin.*

(b) *In each sector $\mid \arg z \mid < (\pi/2) - \epsilon$, $\epsilon > 0$, we have*

$\|(zI - A)^{-1}\| = O(|z|^{-1})$ *as* $z \to \infty$. *Further, for any positive number* $b < 1$, *there are constants* $C > 0$ *and* $p < p_0$ *such that* $\|(zI - A)^{-1}\| = O(|z|^{b-1})$, *as* $z \to \infty$ *with* $\operatorname{Re} z \geqslant C |\operatorname{Im} z|^{1/p}$.

Then for any fixed a, $0 < a < 1$, $B(w) = \exp(-w(-A)^a)$ *has the following properties*:

(i) $\{B(w)\}$ *is a holomorphic semigroup, defined for* $|\arg w| < (1 - a)(\pi/2)$.

(ii) *For each* $\epsilon > 0$,

$$\sup\{\|B(w)\|: 0 < |w| \leqslant 1, |\arg w| \leqslant (1 - a)(\pi/2) - \epsilon\} < \infty.$$

In particular, $\{B(t): t > 0\}$ *is a* C_0*-semigroup.*

(iii) *Given any number* c, $0 < c < 1$, *there are positive constants* k, D_1 *and* D_2 *such that*

$$\|B(w)\| \leqslant D_1 |w|^{-c} \exp(D_2 |w| (\sin \varphi)^{1-ak})$$

whenever $|\arg w| = (\pi/2) - a(\pi/2) - \varphi$, $\varphi > 0$.

THEOREM 2.2. *Suppose* $\{E(w)\}$ *is a semigroup with the properties* (i)–(iii) *above, for some particular* $a \in (0, 1)$. *Then there exists an operator* S *satisfying the hypothesis of Theorem* 2.1, *and a real number* r, *such that*

$$E(w) \equiv \exp(rw) \exp(-w(-S)^a).$$

Proof of Theorem 2.1. (i) Let $0 < b < 1$, and let p and C be the corresponding constants from condition (b). Consider a curve Γ_b given by $\operatorname{Re} z + \delta = C |\operatorname{Im} z|^{1/p}$, oriented in the direction of increasing imaginary part. We may assume $C > C_0$, so that for small enough $\delta > 0$, Γ_b lies in S_0. As $z \in \Gamma_b$ approaches infinity, we have $\|(zI - A)^{-1}\| = O(|z|^{b-1})$, $|\arg z| \to (\pi/2)$, and $\arg(-(-z)^a) \to \pi - a(\pi/2)$. Hence we may write

$$B(w) = (1/2\pi i) \int_{\Gamma_b} \exp(-w(-z)^a)(zI - A)^{-1} \, dz$$

for $|\arg w| < (\pi/2) - a(\pi/2)$, where the integral is independent of b. The holomorphic semigroup property follows as in [2].

(ii) As observed in Sect. 1, the infinitesimal generator of $\{B(t): t > 0\}$ has a closure T, defined by

$$Tx = A^2(1/2\pi i) \int_{\Gamma_b} -(-z)^a z^{-2}(zI - A)^{-1}x \, dz$$

whenever the right side is meaningful. Moreover, by the methods of the previous section, particularly in Lemma 1.5, we can show that any complex v lying to the right of the image $\{-(-z)^a: z \in \Gamma_b\}$ is in the resolvent set of T, and we have

$$(vI - T)^{-1} = (1/2\pi i) \int_{\Gamma_b} (v + (-z)^a)^{-1}(zI - A)^{-1} \, dz. \tag{2.1}$$

It follows that the resolvent set of T contains all but a bounded subset of any sector of the form $|\arg v| < \pi - a(\pi/2) - \epsilon$, $\epsilon > 0$.

Suppose that $|\arg v| < \pi - a\pi - \epsilon$. Then in (2.1), we may deform Γ_b so that it closes from above and below onto the nonnegative real ray (see [4].) Equation (2.1) then becomes

$$(vI - T)^{-1} = \frac{\sin a\pi}{\pi} \int_0^\infty r^a(v^2 - 2vr^a \cos a\pi + r^{2a})^{-1}(rI - A)^{-1} \, dr,$$

and it follows that $\|(vI - T)^{-1}\| \leqslant D\,|v|^{-1}$ for large $|v|$, $D = D(\epsilon)$.

Suppose, next, that $\pi - a\pi + \epsilon < |\arg v| < \pi - a(\pi/2) - \epsilon$, with v still to the right of $\{-(-z)^a: z \in \Gamma_b\}$. Then the process of deforming Γ_b to the real ray involves crossing a singularity of $(v + (-z)^a)^{-1}$. However, the singularity is a simple pole, and so by the residue theorem, we obtain

$$\begin{aligned}(vI - T)^{-1} = (1/2\pi i) &\int_{\mathbb{R}^+\exp(\pm 0i)} (v + (-z)^a)^{-1}(zI - A)^{-1} \, dz \\ &- \operatorname{Res}[(v + (-z)^a)^{-1}(zI - A)^{-1}]_{z=-(-v)^{1/a}},\end{aligned}$$

or

$$\begin{aligned}(vI - T)^{-1} = \frac{\sin a\pi}{\pi} &\int_0^\infty r^a(v^2 - 2vr^a \cos a\pi + r^{2a})^{-1}(rI - A)^{-1} \, dr \\ &+ (1/a)(-v)^{1/a} v^{-1}((-v)^{1/a} I + A)^{-1}\end{aligned} \tag{2.2}$$

Because $|\arg(-(-v)^{1/a})| < (\pi/2) - (\epsilon/a)$, conditions (a) and (b) guarantee that $\| v^{-1}(-v)^{1/a}(A + (-v)^{1/a}I)^{-1} \|$ is dominated by $D(\epsilon)\,|v|^{-1}$. The same is, therefore, true for the entire right side of equation (2.2).

Finally, if $\pi - a\pi - \epsilon < |\arg v| < \pi - a\pi + \epsilon$, then equation (2.1) guarantees, at least, that $\|(vI - T)^{-1}\| = O(|v|^{-k})$ for some $k > 0$.

In view of the preceding three paragraphs, by the Phragmen-Lindelof theorem, we have $\|(vI - T)^{-1}\| = O(|v|^{-1})$, as $v \to \infty$ in any sector $|\arg v| < \pi - a(\pi/2) - \epsilon$. By a theorem of Hille ([3, Theorem 12.8.1]), it follows that T generates a semigroup of class $H(-(\pi/2)(1 - a), (\pi/2)(1 - a))$ (in Hille's notation). This semigroup must be $\{B(w)\}$, because T contains the generator of $\{B(w)\}$; then (ii) is true by Hille's definition.

(iii) Henceforth, for any number $q > 1$, we shall use the familiar notation $q' = q/(q - 1)$. q and q' are related by $(1/q) + (1/q') = 1$.

Let $0 < c < 1$. Referring to condition (b), we may take $b = ac$; let p and C be the corresponding constants. We may write

$$B(w) = (1/2\pi i) \int_{\Gamma_b} \exp(-w(-z)^a)(zI - A)^{-1}\, dz.$$

We have $\|(zI - A)^{-1}\| = O(|z|^{ac-1})$ as $z \to \infty$ along Γ_b.

Assume $0 \leqslant \arg w = (\pi/2) - a(\pi/2) - \varphi$; similar reasoning applies if $\arg w$ is negative. Write $\Gamma_b = \Gamma^+ \cup \Gamma^-$, with $\Gamma^+ = \{z \in \Gamma_b : \operatorname{Im} z \geqslant 0\}$, $\Gamma^- = \Gamma_b - \Gamma^+$. Let $C_1 > \max\{\operatorname{Re}(-(-z)^a) : z \in \Gamma^+\}$. If $z \in \Gamma^+$, then $|\arg w[-(-z)^a - C_1]| > (\pi/2) + \epsilon$, ϵ depending on C_1, but independent of z and w. Therefore

$$|\exp(w[-(-z)^a - C_1])| \leqslant \exp(-|w|\,|z|^a \sin\epsilon),$$

and

$$\begin{aligned} &\left\| (1/2\pi i) \int_{\Gamma^+} \exp(-w(-z)^a)(zI - A)^{-1}\, dz \right\| \\ &\quad = (1/2\pi) \exp(C_1 \operatorname{Re} w) \left\| \int_{\Gamma^+} \exp(-w(-z)^a - wC_1)(zI - A)^{-1}\, dz \right\| \\ &\quad \leqslant \exp(C_1 \operatorname{Re} w)\, C_3 \int_0^\infty \exp(-|w|\, r^a \sin\epsilon)\, r^{ac-1}\, dr \\ &\quad = D_3 |w|^{-c} \exp(D_4 |w|). \end{aligned} \tag{2.3}$$

Suppose $z \in \Gamma^-$. We may write $\arg(-(-z)^a) = -\pi + a(\pi/2) + \theta$, where $|\theta| = |\theta(z)| < (\pi/2) - \epsilon$, for some ϵ independent of z. Then

$$\begin{aligned}\cos(\arg w + \arg(-(-z)^a)) &= \cos((\pi/2) - a(\pi/2) - \varphi - \pi + a(\pi/2) + \theta) \\ &= \sin(\theta - \varphi) = \sin\theta\cos\varphi - \cos\theta\sin\varphi \\ &\leqslant \sin\theta - \cos((\pi/2) - \epsilon)\sin\varphi.\end{aligned}$$

Writing $C_4 = \cos(\pi/2 - \epsilon)$, we find

$$\exp(\mathrm{Re}(-w(-z)^a)) \leqslant \exp(|w|\,|z|^a[\sin\theta - C_4\sin\varphi]). \tag{2.4}$$

Now $\arg z = -\pi + (\pi/2) + (\theta/a) = -(\pi/2) + (\theta/a)$. Hence $\sin(\theta/a) = \cos(\arg z) = \mathrm{Re}\, z/|z| = O(|z|^{1/p-1}) = O(|z|^{-1/p'})$ as $z \to \infty$. But as $z \to \infty$, $\theta \to 0^+$. Hence for large $|z|$, $\sin\theta$ is approximable by $a\sin(\theta/a) = O(|z|^{-1/k})$, where $k = p'$. We conclude that there is C_5 such that $\sin\theta \leqslant C_5|z|^{-1/k}$. Equation (2.4) then yields

$$\exp(\mathrm{Re}(-w(-z)^a)) \leqslant \exp(C_5|w|\,|z|^{a-1/k} - C_4|w|\,|z|^a\sin\varphi).$$

Because $C_5|w|\,t^{a-1/k} - \frac{1}{2}C_4|w|\,t^a\sin\varphi$ has maximal value, for $t > 0$, of the form $C_6|w|(\sin\varphi)^{1-ak}$, we find

$$\exp(\mathrm{Re}(-w(-z)^a)) \leqslant \exp(C_6|w|(\sin\varphi)^{1-ak})\exp(-\tfrac{1}{2}C_4|w|\,|z|^a\sin\varphi),$$

and finally

$$\begin{aligned}&\left\|(1/2\pi i)\int_{\Gamma^-}\exp(-w(-z)^a)(zI - A)^{-1}\,dz\right\| \\ &\quad\leqslant C_3\exp(C_6|w|(\sin\varphi)^{1-ak})\int_0^\infty \exp(-C_7|w|\,r^a\sin\varphi)\,r^{ac-1}\,dr \\ &\quad= C_8(|w|\sin\varphi)^{-c}\exp(C_6|w|(\sin\varphi)^{1-ak}).\end{aligned}$$

Since we may assume $ak = ap' > 1$ (by taking p, originally, close to 1), it follows

$$\left\|\int_{\Gamma^-}\cdots\right\| \leqslant C_9|w|^{-c}\exp(C_6|w|(\sin\varphi)^{1-ak}).$$

We combine this with (2.3), and the proof is complete.

Proof of Theorem 2.2. Assuming that $\{E(w)\}$ has the properties (i), (ii), (iii), then so does $\{\bar{E}(w)\}$, where $\bar{E}(w) = \exp(-rw)\,E(w)$, for any fixed real r. In addition to (iii), therefore, replacing $E(w)$ if necessary by $\bar{E}(w)$ (with some large r), we may assume

$$\| E(t)\| = O(\exp(-\delta t)) \quad \text{as} \quad t \to \infty, t \text{ real, for some positive } \delta. \tag{2.5}$$

From (ii) and (2.5), we know that the infinitesimal generator T of $E(w)$ is a closed, densely defined operator. Further, the resolvent set of T contains the right-hand half-plane, and

$$(vI - T)^{-1} = \int_0^\infty \exp(-vt)\, E(t)\, dt \qquad \text{for} \quad \operatorname{Re} v > 0.$$

LEMMA 2.3. *Suppose* $0 < c < 1$. *Then there are constants* M_1, M_2, *and* k, *such that whenever* $| \arg v | = \pi - a(\pi/2) - \varphi$ *with* $\varphi > 0$ *and* $| v | (\sin \varphi)^{ak} \geqslant M_1$, *then* $(vI - T)^{-1}$ *is defined, and*

$$\|(vI - T)^{-1}\| \leqslant M_2(| v | \sin \varphi)^{c-1}.$$

Proof. Given c, let k, D_1, and D_2 be the constants arising from property (iii). We take $M_1 = 2^{ak+1}D_2$. Assume $0 \leqslant \arg v \leqslant \pi - a(\pi/2)$ and $| v | (\sin \varphi)^{ak} \geqslant M_1$; again, the treatment is analogous if $\arg v$ is negative.

Let L be the ray $\arg z = -(\pi/2) + a(\pi/2) + (\varphi/2)$, oriented away from the origin, and consider

$$R(v) \equiv \int_L \exp(-vz)\, E(z)\, dz. \tag{2.6}$$

If $z \in L$, $\arg vz = (\pi/2) - (\varphi/2)$, so that

$$| \exp(-vz)| = \exp(-| z | \, | v | \sin(\varphi/2)).$$

By (iii),

$$\| E(z)\| \leqslant D_1 \, | z |^{-c} \exp(| z | \, D_2(\sin \varphi)^{1-ak}).$$

Since

$$| v | \sin(\varphi/2) > \tfrac{1}{2} \, | v | \sin \varphi \geqslant 2^{ak}D_2(\sin \varphi)^{1-ak} > 2D_2(\sin(\varphi/2))^{1-ak},$$

the integral in (2.6) is absolutely convergent. Consequently, equation (2.6) defines a holomorphic function $R(v)$, for v as stated. Clearly $R(v)$ coincides with $\int_0^\infty \exp(-vt)\, E(t)\, dt = (vI - T)^{-1}$ if $\operatorname{Re} v > 0$. Thus the existence of $(vI - T)^{-1}$ is established.

Returning to (2.6), we have seen

$$| \exp(-vz)| = \exp(-| z | | v | \sin(\varphi/2)),$$

$$\| E(z)\| \leqslant D_1 | z |^{-c} \exp(| z | D_2(\sin(\varphi/2))^{1-ak}),$$

and

$$D_2(\sin(\varphi/2))^{1-ak} < \tfrac{1}{2} | v | \sin(\varphi/2).$$

Hence

$$\begin{aligned} \| R(v)\| &\leqslant \int_0^\infty D_1 | r |^{-c} \exp(-\tfrac{1}{2}r | v | \sin(\varphi/2))\, dr \\ &= M_3(| v | \sin(\varphi/2))^{c-1} \\ &= M_2(| v | \sin \varphi)^{c-1}, \end{aligned}$$

with M_2 dependent on c, but not upon $| v |$ or φ. Q.E.D.

From the lemma, it is clear that the resolvent set of T contains all but a bounded subset of any sector of the form $| \arg v | < \pi - a(\pi/2) - \varphi_0$. We shall fix φ_0 so that $\pi - a(\pi/2) - \varphi_0 > \pi/2$. Replacing T, if necessary, by $T - sI$ (with a corresponding change to $\exp(-sw)\, E(w)$), for some real s, we may assume that $(vI - T)^{-1}$ is defined for all v in a sector of the form

$$| \arg(v + \delta)| \leqslant \pi - a(\pi/2) - \varphi_0\,, \qquad \text{some} \quad \delta > 0.$$

Given $0 < c < 1$, define a contour $G = G(c)$ as follows: Let k and M_1 be the corresponding constants in Lemma 2.3; then G is the union of the three sets

$$\{v\colon \operatorname{Re} v = -\delta,\ | \arg v | \geqslant \pi - a(\pi/2) - \varphi_0\},$$

$$\{v\colon \operatorname{Re} v < -\delta,\ | \arg v | = \pi - a(\pi/2) - \varphi_0\,, \text{ and } | v |(\sin \varphi_0)^{ak} \leqslant M_1\},$$

$$\{v\colon \operatorname{Re} v < -\delta,\ | \arg v | = \pi - a(\pi/2) - \varphi, \text{ where } 0 < \varphi < \varphi_0 \text{ and } | v |(\sin \varphi)^{ak} = M_1\}.$$

We orient G toward increasing imaginary part.

As in the previous section (see the remarks preceding Lemma 1.5), we can define an operator by writing

$$S(n, c)x \equiv T^n(1/2\pi i) \int_G -(-v)^{1/a}\, v^{-n}(vI - T)^{-1}x\, dv. \tag{2.7}$$

Here we make $n > (1/a) + 1$, and $S(n, c)x$ is defined for exactly those x for which the integral in the right side of (2.7) belongs to $\mathcal{D}(T^n)$. Then $S(n, c)$ will be independent of n and c, and represents a closed, densely defined operator S. The same methods show that S has a resolvent, given by

$$(zI - S)^{-1} = (1/2\pi i) \int_G (z + (-v)^{1/a})^{-1}(vI - T)^{-1}\, dv \qquad (2.8)$$

whenever z is to the right of the image $\{-(-v)^{1/a}: v \in G\}$. Clearly the resolvent set contains a neighborhood of the origin.

LEMMA 2.4. (i) *There exist constants* $p_0 > 1$ *and* $C_0 > 0$, *such that* $(zI - S)^{-1}$ *exists for* $\operatorname{Re} z \geqslant C_0 \mid \operatorname{Im} z \mid^{1/p_0}$.

(ii) *Given* $0 < b < 1$, *there are constants* C *and* p, $C > 0$ *and* $p_0 > p > 1$, *such that* $\|(zI - S)^{-1}\| = O(\mid z \mid^{b-1})$ *as* $z \to \infty$ *with* $\operatorname{Re} z \geqslant C \mid \operatorname{Im} z \mid^{1/p}$.

(iii) *In each sector* $\mid \arg z \mid < (\pi/2) - \epsilon$, $\epsilon > 0$, *we have*

$$\|(zI - S)^{-1}\| = 0(\mid z \mid^{-1}) \qquad as \quad z \to \infty.$$

Proof. (i) Recalling the notation of Lemma 2.3, choose any fixed $c = c_0$, and let $k = k_0$ be the corresponding constant. We may assume $k_0 > 1$; it then suffices to take $p_0 = k_0'$.

For v approaching infinity along $G = G(c_0)$, we have $\mid \arg v \mid = \pi - a(\pi/2) - \varphi$, where $\sin \varphi = O(\mid v \mid^{-1/ak_0})$. Then $\mid \arg(-(-v)^{1/a})\mid = (\pi/2) - (\varphi/a)$, and again $\sin(\varphi/a) = O(\mid v \mid^{-1/ak_0})$. Therefore $\operatorname{Re}(-(-v)^{1/a}) = O(\mid v \mid^{1/a} \mid v \mid^{-1/ak_0}) = O(\mid v \mid^{1/ap_0})$. It follows that some set $\operatorname{Re} z \geqslant C_0 \mid \operatorname{Im} z \mid^{1/p_0}$ lies entirely to the right of the image $\{-(-v)^{1/a}: v \in G\}$.

(ii) Given b, take c so that $c < (b/4a)$. For k corresponding to c, we may assume $ak > 1/c$ and $k > p_0'$.

Because $k' < p_0$, by part (i) of the lemma, $(zI - S)^{-1}$ exists whenever $\operatorname{Re} z \geqslant C_1 \mid \operatorname{Im} z \mid^{1/k'}$ and is given by

$$(zI - S)^{-1} = (1/2\pi i) \int_{G(c)} (z + (-v)^{1/a})^{-1}(vI - T)^{-1}\, dv.$$

If $v \in G = G(c)$, then

$$\|(vI - T)^{-1}\| \leqslant C_2(\mid v \mid \sin \varphi)^{c-1} \leqslant C_3 \mid v \mid^{(c-1)/(ak)'}$$

Choose p such that $1 < p < k'$ and $d \equiv p(1 - b) < 1 - (2b/3)$.

Take C so large that if $\operatorname{Re} z \geqslant C \mid \operatorname{Im} z \mid^{1/p}$, then the distance between z and $\{-(-w)^{1/a}: w \in G\}$ exceeds $\frac{1}{2}\operatorname{Re} z \geqslant C_4 \mid z \mid^{1/p}$, and the distance between $-(-v)^{1/a}$ and z exceeds $\operatorname{Re}(-(-v)^{1/a}) \geqslant C_5(\mid v \mid^{1/a})^{1/k'}$. Then there is C_6 such that

$$\mid z + (-v)^{1/a} \mid \geqslant C_6(\mid z \mid^{1/p})^d(\mid v \mid^{1/ak'})^{1-d},$$

and so

$$\|(zI - S)^{-1}\| \leqslant C_7 \mid z \mid^{-d/p} \int_0^\infty \mid v \mid^{(d-1)/ak'} \mid v \mid^{(c-1)/(ak)'} d \mid v \mid. \quad (2.9)$$

For the exponents in the integrand, we have

$$\frac{1-d}{ak'} = \frac{1-d}{a}\left(1 - \frac{1}{k}\right) > \frac{2b}{3a}(1 - ac) > \frac{b}{2a},$$

$$\frac{1-c}{(ak)'} = 1 - c - \frac{1-c}{ak} > 1 - \frac{b}{4a} - \frac{1}{ak} > 1 - \frac{b}{2a}.$$

Therefore $(1 - d)/ak' + (1 - c)/(ak)' > 1$, and (2.9) yields

$$\|(zI - S)^{-1}\| \leqslant C_8 \mid z \mid^{-d/p} = 0(\mid z \mid^{b-1}).$$

(iii) Here the argument parallels one used in proving Theorem 2.1 (ii). For a given $\epsilon > 0$, there is $K > 0$ such that

$$(zI - S)^{-1} = (1/2\pi i) \int_G (z + (-v)^{1/a})^{-1}(vI - T)^{-1}\, dv \qquad (2.10)$$

whenever $\mid \arg z \mid = (\pi/2) - \varphi < (\pi/2) - \epsilon$ and $\mid z \mid > K$. Here G is $G(c)$ for any fixed c. If z is not on either of the rays $\mathbb{R}^+ \exp(\pm i[\pi - (\pi/a)])$, then we may deform G onto the nonnegative real ray. Then

$$(1/2\pi i) \int_G (z + (-v)^{1/a})^{-1}(vI - T)^{-1}\, dv \qquad (2.11)$$
$$= -\sum + \frac{\sin \pi/a}{\pi} \int_0^\infty r^{1/a} \left(r^{2/a} - 2r^{1/a} \cos \frac{\pi}{a} + z^2\right)^{-1} (rI - T)^{-1}\, dr,$$

where Σ represents the sum of the residues of the integrand $(z + (-v)^{1/a})^{-1} (vI - T)^{-1}$ at the finite number of points where $-(-v)^{1/a} = z$. If v_0 is such a point, the residue is $v_0(-v_0)^{1/a}(v_0I - T)^{-1} = -z^{-1}v_0(v_0I - T)^{-1}$. All such points v_0 are contained in the sector $\mid \arg v \mid < \pi - a(\pi/2) - a\epsilon$. By Hille's theorem ([5, Theorem 12.8.1]), owing to property (ii) of Theorem 2.1, $\| v_0(v_0I - T)^{-1} \| < K_1 = K_1(\epsilon)$. It follows that $\| \Sigma \|$ is dominated by

$K_1(\epsilon) \mid z \mid^{-1}$. The same is true of the integral in the right side of (2.11). Combining (2.10) and (2.11), we have $\|(zI - S)^{-1}\| = O(| z |^{-1})$ as $z \to \infty$, except possibly near the two rays $\mathbb{R}^+ \exp(\pm i[\pi - (\pi/a)])$. The conclusion of (iii) now follows by an application of the Phragmen-Lindelof principle. Q.E.D.

From Lemma 2.4, it is clear that S satisfies the hypothesis of Theorem 2.1. Therefore we can define $\exp(-w(-S)^a)$, $| \arg w | < (\pi/2) - a(\pi/2)$. From the proof of Theorem 2.1(ii), $\{\exp(-w(-S)^a)\}$ is generated by $-(-S)^a$, where

$$(-S)^a x = S^2(1/2\pi i) \int_{\Gamma_b} (-z)^a z^{-2}(zI - S)^{-1} x \, dz \tag{2.12}$$

whenever the right side has meaning. By choosing Γ_b and G appropriately, we find (note equation (2.8))

$$\begin{aligned}
&(1/2\pi i) \int_{\Gamma_b} (-z)^a z^{-2}(zI - S)^{-1} \, dz \\
&= (1/2\pi i) \int_{\Gamma_b} (-z)^a z^{-2}(1/2\pi i) \int_G (z + (-v)^{1/a})^{-1}(vI - T)^{-1} \, dv \, dz \\
&= (1/2\pi i) \int_G (1/2\pi i) \int_{\Gamma_b} (-z)^a z^{-2}(z + (-v)^{1/a})^{-1} \, dz \, (vI - T)^{-1} \, dv \\
&= (1/2\pi i) \int_G (-v)^{1-2/a}(vI - T)^{-1} \, dv \\
&= -T(1/2\pi i) \int_G (-v)^{2/a}(vI - T)^{-1} \, dv \\
&= -TS^{-2}.
\end{aligned}$$

By virtue of (2.12), we conclude $-(-S)^a = T$.But T generates $E(w)$. More precisely, in view of the remarks preceding equation (2.5) and following the proof of Lemma 2.3, T generates $\exp(-rw) E(w)$, for a real r. It follows $\exp(-w(-S)^a) = \exp(-rw) E(w)$, and the proof of Theorem 2.2 is complete.

Observations similar to those at the end of Section 1 hold true for the family of operators $-(-A)^a$, $0 < a < 1$. Thus, the operators have the expected properties of powers. We also have the result corresponding to Theorem 1.6, with an analogous proof. That is, if we start with an operator A and define $\exp(-w(-A)^a)$ as in Theorem 2.1, then the process of Theorem 2.2 applied to $E(w) = \exp(-w(-A)^a)$ recovers the operator $S = A$.

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