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Star stable domains

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Abstract

We introduce and study the notion of $\star$-stability with respect to a semistar operation $\star$ defined on a domain R; in particular we consider the case where $\star$ is the w-operation. This notion allows us to generalize and improve several properties of stable domains and totally divisorial domains.

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0. Introduction

Star operations, such as the v-closure (or divisorial closure), the t-closure and the w-closure, are an essential tool in modern multiplicative ideal theory for characterizing and investigating several classes of integral domains. For example, in the last few decades a large amount of literature has appeared on *Mori domains*, that is domains satisfying the ascending chain condition on divisorial ideals, and *Prüfer v-multiplication domains*, for short *PvMDs*, that is domains in which each finitely generated ideal is t-invertible (or w-invertible). The consideration that some important operations on ideals, like the integral closure, satisfy almost all the properties of star operations led Okabe and Matsuda to introduce in 1994 the more general and flexible notion of semistar operation [26]. The class of semistar operations includes the classical star operations and often provides a more appropriate context for approaching several questions of multiplicative ideal theory; see for example [10,14–17,32]. In this paper, we introduce the notion of $\star$-stability with respect to a semistar operation $\star$.

Motivated by earlier work of Bass [4] and Lipman [25] on the number of generators of an ideal, in 1974 Sally and Vasconcelos defined a Noetherian ring R to be *stable* if each nonzero ideal of R is projective over its endomorphism ring $\mathrm{End}_R(I)$ [35]. In a note of 1987, Anderson, Huckaba and Papick considered the notion of stability for arbitrary integral domains [2]. When I is a nonzero ideal of a domain R, then $\mathrm{End}_R(I) = (I : I)$; thus a domain R is stable if each nonzero ideal I of R is invertible in the overring $(I : I)$. Since 1998, stable domains have been thoroughly investigated by Olberding in a series of papers [27–31].

Given a semistar operation $\star$ on a domain R, we say that a nonzero ideal I of R is $\star$-*stable* if $I^\star$ is $\dot{\star}$-invertible in $(I^\star : I^\star)$ and that R is $\star$-*stable* if each nonzero ideal of R is $\star$-stable. (Here we denote by $\dot{\star}$ the semistar operation

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induced by $\star$ on a fixed overring T of R.) This notion allows us to generalize and improve several properties of stable domains and totally divisorial domains. We also recover some results proven in [12, Section 2] for $\star = w$.

Even though many results are stated for a general semistar operation, for technical reasons, the most interesting consequences are obtained for (semi)star operations spectral and of finite type. In this case, we show that $\star$-stability implies that $\star$ is the w-operation on R; in particular, on stable domains the w-operation is the identity.

For a (semi)star operation spectral and of finite type, the main result of Section 1 is that a domain R is $\star$-stable if and only if R is $\star$-locally stable and has $\star$-finite character, if and only if R is $\star$-locally stable and each $\star$-ideal of R is $\dot{\star}$-finite in its endomorphism ring. This implies that if a domain is locally stable, then stability is equivalent to the property that each nonzero ideal I is finitely generated in the overring $(I : I)$.

In Section 2 we study the $\star$-stability of overrings and we show that, for semistar operations of finite type, the $\star$-integral closure of a $\star$-stable domain is a PvMD.

In Section 3 we extend some properties of totally divisorial domains in the setting of semistar operations. For $\star = w$, we prove that each t-linked overring T of R is $\dot{w}$-divisorial if and only if all the endomorphism rings of w-ideals are $\dot{w}$-divisorial, if and only if R is w-stable and w-divisorial. Under these conditions, $\dot{w}$ is the w-operation on T. As a consequence, we get that R is totally divisorial if and only if all the overrings of type $(I : I)$ are divisorial, if and only if each nonzero ideal I of R is m-canonical in $(I : I)$. The Mori case and the integrally closed case are of particular interest.

Finally, in Section 4 we show that w-stable w-divisorial domains are v-coherent and use this fact to show that w-stable w-divisorial (respectively, totally divisorial) domains share several properties with generalized Krull (respectively, Dedekind) domains. As a matter of fact, in the integrally closed case each one of these properties becomes equivalent to R being a generalized Krull (respectively, Dedekind) domain; so a w-stable w-divisorial (respectively, totally divisorial) domain can be viewed as a "non-integrally closed generalized Krull (respectively, Dedekind) domain".

Throughout this paper R will be an integral domain with quotient field K, $R \neq K$. We denote by $\boldsymbol{F}(R)$ the set of nonzero fractional ideals of R, by $\overline{\boldsymbol{F}}(R)$ the set of nonzero R-submodules of K and by $\boldsymbol{f}(R)$ the set of nonzero finitely generated R-submodules of K. Clearly $\boldsymbol{f}(R) \subseteq \boldsymbol{F}(R) \subseteq \overline{\boldsymbol{F}}(R)$.

A *semistar operation* on R is a map $\star : \overline{\boldsymbol{F}}(R) \to \overline{\boldsymbol{F}}(R)$ such that, for each $E, F \in \overline{\boldsymbol{F}}(R)$ and for each $x \in K$, $x \neq 0$, the following properties hold:

($\star_1$) $(xE)^\star = xE^\star$;
($\star_2$) $E \subseteq F$ implies $E^\star \subseteq F^\star$;
($\star_3$) $E \subseteq E^\star$ and $E^{\star\star} := (E^\star)^\star = E^\star$.

Recall that, for all $E, F \in \overline{\boldsymbol{F}}(R)$, we have:

$$(EF)^\star = (E^\star F)^\star = \left(EF^\star\right)^\star = \left(E^\star F^\star\right)^\star;$$
$$(E+F)^\star = \left(E^\star + F\right)^\star = \left(E + F^\star\right)^\star = \left(E^\star + F^\star\right)^\star;$$
$$(E : F)^\star \subseteq (E^\star : F^\star) = (E^\star : F) = \left(E^\star : F\right)^\star;$$
$$(E \cap F)^\star \subseteq E^\star \cap F^\star = \left(E^\star \cap F^\star\right)^\star, \quad \text{if } E \cap F \neq (0);$$

see for instance [14].

If $\star_1$ and $\star_2$ are semistar operations on R, we say that $\star_1 \leq \star_2$ if $E^{\star_1} \subseteq E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(R)$. This is equivalent to the condition that $(E^{\star_1})^{\star_2} = (E^{\star_2})^{\star_1} = E^{\star_2}$, for each $E \in \overline{\boldsymbol{F}}(R)$.

The identity is a semistar operation, denoted by d. It follows from ($\star_3$) that $d \leq \star$, for each semistar operation $\star$.

A semistar operation $\star$ is called a *semistar operation of finite type* if, for each $E \in \overline{\boldsymbol{F}}(R)$, we have

$$E^\star = \bigcup\{F^\star \mid F \in \boldsymbol{f}(R) \text{ and } F \subseteq E\}.$$

If $\star$ is any semistar operation, the semistar operation $\star_f$ defined by

$$E^{\star_f} := \bigcup\{F^\star \mid F \in \boldsymbol{f}(R) \text{ and } F \subseteq E\},$$

for each $E \in \overline{\boldsymbol{F}}(R)$, is a semistar operation of finite type and $\star_f \leq \star$.

A nonzero ideal I of R is $\star$-finite if there exists a finitely generated J such that $I^\star = J^\star = J^{\star_f}$.

When $R^\star = R$, $\star$ is called a *(semi)star operation* on R and its restriction to the set of nonzero fractional ideals $\boldsymbol{F}(R)$ is a star operation, still denoted by $\star$.

As usual, we denote by v the (semi)star operation defined by $E^v := (R : (R : E))$, for each $E \in \overline{\boldsymbol{F}}(R)$, and set $t := v_f$. As a star operation on R, v is called the *divisorial closure*. It is well known that $\star \leq v$ and $\star_f \leq t$, for each (semi)star operation $\star$ [14, Proposition 1.6].

We say that a nonzero ideal I of R is a *quasi-$\star$-ideal* if $I^\star \cap R = I$. A *quasi-$\star$-prime (ideal)* is a prime quasi-$\star$-ideal and a *quasi-$\star$-maximal ideal* is a quasi-$\star$-ideal maximal in the set of all proper quasi-$\star$-ideals. A quasi-$\star$-maximal ideal is a prime ideal [14, Lemma 4.20] and, when $\star$ is a semistar operation of finite type, each quasi-$\star$-ideal is contained in a quasi-$\star$-maximal ideal [14, Lemma 4.20]. The set of quasi-$\star_f$-maximal ideals of R will be denoted by $\star_f$-Max(R). We say that R has *$\star_f$-finite character* if each nonzero ideal of R is contained at most in a finite number of quasi-$\star_f$-maximal ideals of R.

When $\star$ is a (semi)star operation, an ideal I is a quasi-$\star$-ideal if and only if $I^\star = I$. In this case, like in the classical case of star operations, we say that I is a *$\star$-ideal* and, analogously, we call a quasi-$\star$-prime ideal a *$\star$-prime* and a quasi-$\star$-maximal ideal a *$\star$-maximal ideal*. A v-ideal of R is also called a *divisorial ideal*.

If $\star$ is a semistar operation on R, we denote by $\tilde{\star}$ the semistar operation defined by

$$E^{\tilde{\star}} := \bigcap_{M \in \star_f\text{-Max}(R)} ER_M = \bigcup_{F \in \boldsymbol{f}(R),\ F^{\star_f} = R} (E : F),$$

for each $E \in \overline{\boldsymbol{F}}(R)$. We have $I^{\tilde{\star}} R_M = IR_M$, for each nonzero ideal I of R and each quasi-$\star_f$-maximal ideal M of R [14, Lemma 4.1(2)]. Clearly $\tilde{\star} = \widetilde{\star_f}$.

The semistar operation $\tilde{\star}$ is of finite type and spectral (a semistar operation $\star$ is *spectral* if there exists $\Lambda \subseteq \text{Spec}(R)$ such that $E^\star = \bigcap\{ER_P \mid P \in \Lambda\}$, for each $E \in \overline{\boldsymbol{F}}(R)$). More precisely, $\star = \tilde{\star}$ if and only if $\star$ is spectral and of finite type, if and only if $\star$ is of finite type and $(E \cap F)^\star = E^\star \cap F^\star$, for $E, F \in \overline{\boldsymbol{F}}(R)$ such that $E \cap F \neq (0)$ [14, Corollary 3.9 and Proposition 4.23].

Always we have $\tilde{\star} \leq \star_f \leq \star$. In addition, setting $w := \tilde{v}$, if $\star$ is a (semi)star operation, we have $\tilde{\star} \leq w$.

A nonzero ideal I of R is *$\star$-invertible* if $(I(R : I))^\star = R^\star$. When $\star = \star_f$, this is equivalent to the fact that $I(R : I)$ is not contained in any quasi-$\star_f$-maximal ideal. Since quasi-$\star_f$-maximal ideals and quasi-$\tilde{\star}$-maximal ideals coincide [17, Corollary 3.5(2)], it follows that an ideal I is $\star_f$-invertible if and only if it is $\tilde{\star}$-invertible. When $\star = \star_f$, a $\star$-invertible ideal is $\star$-finite.

If $\star$ is a semistar operation on R and T is an overring of R, the restriction of $\star$ to the set of T-submodules of K is a semistar operation on T, here denoted by $\dot{\star}$. When $T^\star = T$, $\dot{\star}$ is a (semi)star operation on T [16, Proposition 2.8]. Note that $\dot{\star}$ shares many properties with $\star$ (see for instance [33, Proposition 3.1]); for example, if $\star$ is of finite type then $\dot{\star}$ is of finite type [16, Proposition 2.8].

1. $\star$-Stable domains

Let R be an integral domain and $\star$ a semistar operation on R. Given a nonzero fractional ideal I of R, consider the overring $T := (I^\star : I^\star)$ of R. It is easy to see that $T = T^\star$; hence the restriction of $\star$ to the set of the T-submodules of K is a (semi)star operation on T, denoted by $\dot{\star}$.

We say that a nonzero fractional ideal I of R is *$\star$-stable* if $I^\star$ is $\dot{\star}$-invertible in $(I^\star : I^\star)$ and that R itself is *$\star$-stable* if each nonzero (fractional) ideal of R is $\star$-stable. The notion of d-stable domain coincides with the notion of stable domain introduced in [35].

A $\star$-invertible ideal I of R is $\star$-stable. In fact, since $R^\star \subseteq T^\star = T$, if $(I(R : I))^\star = R^\star$, we have

$$T = R^\star T \subseteq ((I(R : I))^\star T)^\star = (I(R : I)T)^\star \subseteq (I^\star(T : I^\star))^\star \subseteq T.$$

Thus $(I^\star(T : I^\star))^\star = T$. It follows that a domain with the property that each nonzero ideal is $\star$-invertible is $\star$-stable. For example, any completely integrally closed domain is v-stable. Recalling that if I is v-invertible, we have $R = (I^t : I^t) = (I^v : I^v)$, we see that a completely integrally closed domain that is not a PvMD is v-stable but not t-stable. Since on Krull domains $t = v$ and any integrally closed stable domain is a Prüfer domain [34, Proposition 2.1], any Krull domain that is not Dedekind is t-stable but not stable.

Proposition 1.1. *Let $\star$ be a semistar operation on an integral domain R satisfying one of the following conditions:*

(1) $(E \cap F)^\star = E^\star \cap F^\star$, *for each* $E, F \in \overline{F}(R)$ *such that* $E \cap F \neq (0)$.
(2) $(R : R^\star) \neq (0)$.

Then R *is* $\star$*-stable if and only if* $R^\star$ *is* $\dot{\star}$*-stable.*

Proof. If I is a nonzero ideal of R, then $I^\star$ is an ideal of $R^\star$. Hence, if $R^\star$ is $\dot{\star}$-stable, R is $\star$-stable, without any condition on $\star$.

Conversely, let R be $\star$-stable and let J be a nonzero ideal of $R^\star$. Assume that condition (1) holds and consider the ideal $I := J \cap R$ of R. Then $I^\star = (J \cap R)^\star = J^\star \cap R^\star = J^\star$. It follows that J is $\star$-stable. If (2) holds, then J is a fractional ideal of R. Hence it is $\star$-stable. □

Since we will be mostly interested in the case where $\star$ is a semistar operation spectral and of finite type, that is where $\star = \tilde{\star}$, by the previous proposition often we will restrict ourselves to assume that $R = R^\star$, that is to consider (semi)star operations.

Our first result is a generalization of [31, Theorem 3.5(1)⇔(2)]:

Proposition 1.2. *The following conditions are equivalent for an integral domain* R *and a semistar operation* $\star$ *on* R*:*

(i) R *is* $\star$*-stable.*
(ii) *For each nonzero ideal* I *of* R, $I^\star$ *is a divisorial ideal of* $(I^\star : I^\star)$ *(that is,* $(I^\star)^{v'} = I^\star$*, where* v' *is the* v*-operation on* $(I^\star : I^\star)$*).*

Proof. (i)⇒(ii) Since $I^\star$ is $\dot{\star}$-invertible in $(I^\star : I^\star)$ we have $I^\star = (I^\star)^{\dot{\star}} = (I^\star)^{v'}$ [6, Lemma 2.1].

(ii)⇒(i) Let $T := (I^\star : I^\star)$ and let $J := (T : I^\star)$. We have to show that $(I^\star J)^\star = T^\star = T$.

First, we show that $(J : J) = T$. We have $(J : J) = ((T : I^\star) : (T : I^\star)) = ((T : (T : I^\star)) : I^\star) = ((I^\star)^{v'} : I^\star) = (I^\star : I^\star) = T$.

The next step is to show that $(T : I^\star J) = T$. We have $(T : I^\star J) = ((T : J) : I^\star) = ((T : (T : I^\star)) : I^\star) = ((I^\star)^{v'} : I^\star) = (I^\star : I^\star) = T$.

Now we prove that $((I^\star J)^\star : I^\star J) = T$. It is clear that $((I^\star J)^\star : I^\star J) \supseteq T$. Conversely, if $x \in ((I^\star J)^\star : I^\star J)$ then $x(I^\star J) \subseteq (I^\star J)^\star \subseteq T$. Hence, $((I^\star J)^\star : I^\star J) \subseteq (T : I^\star J) = T$.

Finally, since $((I^\star J)^\star : (I^\star J)^\star) = T$, by hypothesis, $(I^\star J)^\star = ((I^\star J)^\star)^{v'}$. Thus $T = (T : T) = (T : (T : I^\star J)) = (I^\star J)^{v'} = (I^\star J)^\star$, and I is $\dot{\star}$-invertible. □

If I is a nonzero ideal of R, we denote by $v(I)$ the semistar operation defined on R by $E \mapsto (I : (I : E))$, for each $E \in \overline{F}(R)$. When $(I : I) = R$, then $v(I)$ is a (semi)star operation. The ideal I is called *m-canonical* if $J^{v(I)} := (I : (I : J)) = J$, for each nonzero fractional ideal J of R [22].

Lemma 1.3. *Let* R *be an integral domain and let* I *be a nonzero ideal of* R*. Then:*

(1) $I^{v(I)} = I$.
(2) *If* $\star$ *is a semistar operation on* R *such that* $I^\star = I$*, then* $\star \leq v(I)$*.*

Proof. (1) This is an easy consequence of the fact that I is an ideal of $(I : I)$; thus $(I : (I : I)) = I$.

(2) Let $E \in \overline{F}(R)$. Since $I = I^\star$, then $(I : E^\star) = (I : E)$. Hence $(E^\star)^{v(I)} = (I : (I : E^\star)) = (I : (I : E)) = E^{v(I)}$ and so $\star \leq v(I)$. □

Proposition 1.4. *The following conditions are equivalent for an integral domain* R *and a (semi)star operation* $\star$ *on* R*:*

(i) R *is* $\star$*-stable.*
(ii) $v(I^\star) = v'$*, for each nonzero ideal* I *of* R *(where* $v(I^\star)$ *is defined on* $(I^\star : I^\star)$ *and* v' *is the* v*-operation of* $(I^\star : I^\star)$*).*
(iii) *If* I, J *are two nonzero ideals of* R *such that* $(I^\star : I^\star) = (J^\star : J^\star)$ *then* $v(I^\star) = v(J^\star)$ *(as (semi)star operations on* $(I^\star : I^\star) = (J^\star : J^\star)$*).*

Proof. (i)⇒(ii) Since $v(I^\star)$ is a (semi)star operation on $(I^\star : I^\star)$ we have $v(I^\star) \leq v'$. Conversely, since by Proposition 1.2 $I^\star$ is divisorial in $(I^\star : I^\star)$, as a consequence of Lemma 1.3(2), we have that $v' \leq v(I^\star)$.

(ii)⇒(i) We have $(I^\star)^{v'} = (I^\star)^{v(I^\star)} = I^\star$ by Lemma 1.3(1). So, $I^\star$ is divisorial in $(I^\star : I^\star)$ for each ideal I of R and R is $\star$-stable by Proposition 1.2.

(ii)⇒(iii) This is straightforward since both $v(I^\star)$ and $v(J^\star)$ coincide with v' in $(I^\star : I^\star) = (J^\star : J^\star)$.

(iii)⇒(ii) Note that $T := (I^\star : I^\star)$ is a fractional ideal of R, since $I^\star$ is an ideal of R. So, there exists a nonzero integral ideal J of R and a nonzero element $x \in R$ such that $T = x^{-1}J^\star$. Clearly, $T = (T : T) = (J^\star : J^\star)$. Thus, by hypothesis, $v(I^\star) = v(J^\star)$. Moreover, it is easy to see that $v(J^\star) = v(x^{-1}J^\star) = v(T) = v'$, the v-operation of T. It follows that $v(I^\star) = v'$. □

Proposition 1.5. *Let R be an integral domain and $\star$ a (semi)star operation of finite type on R. If R is $\star$-stable, then each $\star$-maximal ideal of R is divisorial. In particular, $\star$-Max$(R) = t$-Max$(R) = v$-Max(R).*

Proof. Let M be a $\star$-maximal ideal of R and suppose that M is not divisorial. Then, $M^v = R$, otherwise M^v would be a $\star$-ideal containing M. Hence $(M : M) = (R : M) = R$. It follows that M is $\star$-invertible in R and $M^\star = (M(M : M))^\star = (M(R : M))^\star = R$, a contradiction. The second statement follows easily. □

Corollary 1.6. *Let R be an integral domain and $\star$ a (semi)star operation of finite type on R. If R is $\star$-stable then $\tilde{\star} = w$.*

Remark 1.7. (1) It is possible to prove that, given two semistar operations $\star_1 \leq \star_2$ on R either of finite type or with the property that $(I \cap J)^{\star_i} = I^{\star_i} \cap J^{\star_i}$, for any pair of nonzero ideals I, J of R and $i = 1, 2$ (for example, two spectral semistar operations), then $\star_1$-stability implies $\star_2$-stability. Thus, for example, a w-stable domain is t-stable. We have no examples of t-stable domains that are not w-stable.

(2) In general it is not true that if $\star$ is a (semi)star operation of finite type and R is $\star$-stable then $\star = t$. However, we will show in Corollary 2.4 that this happens when R is $\star$-integrally closed.

For an example, in [35, Example 5.4] it is proved that the one-dimensional local domain A with maximal ideal 3-generated constructed in [13] is stable. It is clear that A is Noetherian. Hence $d \neq t$ on A, because the maximal ideal is not 2-generated [29, Lemma 3.5].

Lemma 1.8. *Let R be an integral domain and $\star$ a semistar operation on R. Let J be a nonzero ideal of R and assume that $J^\star$ is $\dot{\star}$-finite in $(J^\star : J^\star)$. Then, for each prime ideal P of R:*

(1) $(J^\star : J^\star)R_P = (J^\star R_P : J^\star R_P)$.
(2) $((J^\star : J^\star) : J^\star)R_P = ((J^\star : J^\star)R_P : J^\star R_P)$.

Proof. (1) Let $T := (J^\star : J^\star)$. Since $J^\star$ is $\dot{\star}$-finite, there exist $x_1, x_2, \ldots, x_n \in J^\star$, such that $J^\star = (x_1T + x_2T + \cdots + x_nT)^\star$. Let $H := x_1R + x_2R + \cdots + x_nR \subseteq J^\star$, so that $(HT)^\star = J^\star$. Then, $HR_P \subseteq J^\star R_P$ and $TR_P = (J^\star : J^\star)R_P \subseteq (J^\star R_P : J^\star R_P) \subseteq (J^\star R_P : HR_P) = (J^\star : (HT)^\star)R_P = (J^\star : J^\star)R_P = TR_P$. Hence, $(J^\star R_P : J^\star R_P) = TR_P = (J^\star : J^\star)R_P$.

(2) Since $T^\star = T$, we have $(T : J^\star)R_P \subseteq (TR_P : J^\star R_P) \subseteq (TR_P : HR_P) = (T : HT)R_P = (T : (HT)^\star)R_P = (T : J^\star)R_P$. Hence, $(T : J^\star)R_P = (TR_P : J^\star R_P)$. □

The next result shows in particular that the study of $\star$-stable domains can be reduced to the local case.

Theorem 1.9. *Let R be an integral domain and $\star$ a (semi)star operation on R. If $\star = \tilde{\star}$, the following conditions are equivalent:*

(i) *R is $\star$-stable.*
(ii) *R has $\star$-finite character and R_M is stable, for each $M \in \star$-Max(R).*
(iii) *$J^\star$ is $\dot{\star}$-finite in $(J^\star : J^\star)$, for each nonzero ideal J of R, and R_M is stable, for each $M \in \star$-Max(R).*

Under these conditions, $\star = w$.

Proof. (i)⇒(ii) First, we show that R_M is stable, for each $M \in \star$-Max(R). Let I be a nonzero ideal of R_M. There exists an ideal J of R such that $I = JR_M = J^\star R_M$. Since R is $\star$-stable, $J^\star$ is $\dot{\star}$-invertible in $T := (J^\star : J^\star)$, that is, $(J^\star(T : J^\star))^\star = (J(T : J))^\star = T$ (as usual, we denote by $\dot{\star}$ the restriction of $\star$ to the set of the fractional ideals of

T). In particular, $J^\star$ is $\dot{\star}$-finite in T. Hence, by Lemma 1.8(2), we have $(I : I) = (JR_M : JR_M) = TR_M = (J(T : J))^\star R_M = JR_M(T : J)R_M = JR_M(TR_M : JR_M) = I((I : I) : I)$. It follows that I is invertible in $(I : I)$ and so R_M is a stable domain.

To prove that R has $\star$-finite character, we prove that a family of $\star$-maximal ideals that has nonempty intersection is a finite family.

Let M be a $\star$-maximal ideal. Since R_M is stable, by [31, Lemma 3.1], MR_M is principal in $(MR_M : MR_M)$, that is, there exists $x \in MR_M$ such that $MR_M = x(MR_M : MR_M)$ and so $M^2R_M \subseteq xMR_M \subseteq xR_M \subseteq MR_M$. The ideal $I := xR_M \cap R$ is a t-ideal (it is the contraction of a t-ideal of R_M), and so a $\star$-ideal, since $\star \leq t$. Moreover, $IR_M = xR_M$. Note that $M^2 \subseteq I$ and so $IR_N = R_N$ for each $\star$-maximal ideal $N \neq M$. Since by Lemma 1.8(1) $(I : I)R_N = (IR_N : IR_N)$ for each $\star$-maximal ideal N, we have $(I : I) = \bigcap\{(I : I)R_N | N \in \star\text{-Max}(R)\} = \bigcap\{(IR_N : IR_N)|N \in \star\text{-Max}(R)\} = \bigcap\{(R_N : R_N)|N \in \star\text{-Max}(R), N \neq M\} \cap (xR_M : xR_M) = \bigcap\{(R_N : R_N)|N \in \star\text{-Max}(R)\} = \bigcap\{R_N|N \in \star\text{-Max}(R)\} = R$. It follows that, since R is $\star$-stable, I is $\star$-invertible in R.

Now, let $\{M_\alpha\}$ be a collection of $\star$-maximal ideals such that $\bigcap_\alpha M_\alpha \neq (0)$. For each M_α we have a $\star$-ideal I_α, constructed as above. If $y \in \bigcap_\alpha M_\alpha$, $y \neq 0$, then $y^2 \in \bigcap_\alpha {M_\alpha}^2 \subseteq \bigcap_\alpha I_\alpha$. Then, $I := \bigcap_\alpha I_\alpha \neq (0)$. Let $J = \sum_\alpha (R : I_\alpha)$. Since for each α, I_α is a $\star$-invertible $\star$-ideal, we have $I_\alpha = I_\alpha^v$ [6, Lemma 2.1], and so $(R : J) = \bigcap_\alpha (R : (R : I_\alpha)) = \bigcap_\alpha I_\alpha = I$. Note that $I_\alpha \nsubseteq M_\beta$ if $\beta \neq \alpha$ and that, by Lemma 1.8(2), $(R : I_\alpha)R_M = (R_M : I_\alpha R_M)$. Then, for each α, we have $JR_{M_\alpha} = \sum_\beta (R : I_\beta)R_{M_\alpha} = (R_{M_\alpha} : IR_{M_\alpha}) + \sum_{\beta\neq\alpha}(R_{M_\alpha} : R_{M_\alpha}) = (R_{M_\alpha} : IR_{M_\alpha})$. Similarly, for a $\star$-maximal $N \notin \{M_\alpha\}$, we have $JR_N = R_N$. Hence $(J^\star : J^\star) \subseteq \bigcap(JR_M : JR_M) = \bigcap(R_{M_\alpha} : I_\alpha R_{M_\alpha}(R_{M_\alpha} : I_\alpha R_{M_\alpha})) \cap (\bigcap\{R_N|N \in \star\text{-Max}(R), N \notin \{M_\alpha\}\}) = \bigcap R_M = R$. It follows that J is $\star$-invertible in R and so $\star$-finite (and so, t-finite). Then, there exists $I_{\alpha_1}, I_{\alpha_2}, \ldots, I_{\alpha_n}$ such that $J^\star = ((R : I_{\alpha_1}) + (R : I_{\alpha_2}) + \cdots + (R : I_{\alpha_n}))^v$. Thus, $I = (R : J) = (R : J^\star) = I_{\alpha_1} \cap I_{\alpha_2} \cap \cdots \cap I_{\alpha_n}$, and so the only M_α's containing I are $M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}$. Since $\bigcap_\alpha M_\alpha^2 \subseteq I$, we conclude that $\{M_\alpha\} = \{M_{\alpha_1}, M_{\alpha_2}, \ldots, M_{\alpha_n}\}$ is a finite family.

(ii)⇒(iii) First we show that $(J^\star : J^\star)R_M = (JR_M : JR_M)$, for each nonzero ideal J and for each $\star$-maximal ideal M of R. For this, it is enough to show that $(JR_M : JR_M) \subseteq (J^\star : J^\star)R_M$. Let $x \in (JR_M : JR_M)$ and let $M_1, M_2, \ldots, M_n$ be the $\star$-maximal ideals such that $xR_{M_i} \neq R_{M_i}$. Since R_{M_i} is stable, for each $i = 1, 2, \ldots, n$, there exists $y_i \in J$ such that $JR_{M_i} = y_i(JR_{M_i} : JR_{M_i})$ [31, Lemma 3.1]. Then, $xy_i \in xJR_M \subseteq JR_M$ and there exists $d_i \in R \smallsetminus M$ such that $d_ixy_i \in J$. Setting $d := d_1d_2 \ldots d_n$, we have $dxJR_{M_i} = dxy_i(JR_{M_i} : JR_{M_i}) \subseteq J(JR_{M_i} : JR_{M_i}) \subseteq JR_{M_i}$, for each $i = 1, 2, \ldots, n$. Moreover, if N is a $\star$-maximal ideal such that $N \notin \{M_1, M_2, \ldots, M_n\}$, then $xR_N = R_N$. Thus, $dxJR_N = dJR_N \subseteq JR_N$ for each $\star$-maximal ideal N of R and so, $dxJ^\star = (dxJ)^\star = \bigcap(dxJR_N) \subseteq \bigcap JR_N = J^\star$. It follows that $dx \in (J^\star : J^\star)$ and $x \in (J^\star : J^\star)R_M$, since $d \in R \smallsetminus M$. Thus, $(J^\star : J^\star)R_M = (JR_M : JR_M)$.

Now let $T := (J^\star : J^\star)$. We prove that there exists a finitely generated ideal $H \subseteq J$ of R such that $(HT)^\star = J^\star$. Let $N_1, N_2, \ldots, N_s$ be the $\star$-maximal ideals containing J. Since R_{N_i} is stable, there exists $x_i \in J$, such that $J^\star R_{N_i} = JR_{N_i} = x_i(JR_{N_i} : JR_{N_i}) = x_iTR_{N_i}$, for each $i = 1, 2, \ldots, s$ [31, Lemma 3.1]. Let $F := x_1R + x_2R + \cdots + x_sR$. Since $FT \subseteq JT \subseteq J^\star$, we have $JR_{N_i} = x_iTR_{N_i} \subseteq FTR_{N_i} \subseteq JR_{N_i}$. It follows that $JR_{N_i} = FTR_{N_i}$ for each $i = 1, 2, \ldots, s$. If F is not contained in any $\star$-maximal ideal distinct from $N_1, N_2, \ldots, N_s$, we have $JR_N = FTR_N$ for each $\star$-maximal ideal N of R and so $J^\star = (FT)^\star$. Otherwise, let $N_{s+1}, N_{s+2}, \ldots, N_t$ be the $\star$-maximal ideals of R containing F and not containing J. If $x \in J \smallsetminus (N_{s+1} \cup N_{s+2} \cup \cdots \cup N_t)$ and $H := F + xR$, as before we get that $JR_N = HTR_N$ for each $\star$-maximal ideal N of R and so $J^\star = (HT)^\star$.

(iii)⇒(i) We have to prove that $I := (J((J^\star : J) : J))^\star = (J^\star : J^\star)$. By Lemma 1.8, we have $I = \bigcap J((J^\star : J) : J)R_M = \bigcap JR_M((JR_M : JR_M) : JR_M) = \bigcap(JR_M : JR_M) = \bigcap(J^\star : J^\star)R_M = (J^\star : J^\star)^\star = (J^\star : J^\star)$ where the intersection varies over the set of all $\star$-maximal ideals of R.

The fact that $\star = w$ is a straightforward consequence of Corollary 1.6. □

We state explicitly the previous theorem for $\star = w$. A direct proof of (i)⇔(ii) is given in [12, Theorem 2.2].

Corollary 1.10. *The following conditions are equivalent for an integral domain R:*

(i) *R is w-stable.*
(ii) *R has t-finite character and R_M is stable, for each $M \in t\text{-Max}(R)$.*
(iii) *J^w is $\dot{w}$-finite in $(J^w : J^w)$, for each nonzero ideal J of R, and R_M is stable, for each $M \in t\text{-Max}(R)$.*

For $\star = d$, we obtain the following result, where (i)⇔(iii) is due to Olberding [31, Theorem 3.3].

Corollary 1.11. *The following conditions are equivalent for an integral domain R:*

(i) *R is stable.*
(ii) *R is w-stable and $d = w$.*
(iii) *R has finite character and R_M is stable, for each maximal ideal M of R.*
(iv) *Each nonzero ideal J of R is finitely generated in $(J : J)$ and R_M is stable for each maximal ideal M of R.*

In particular, a one-dimensional w-stable domain is stable.

We recall that a domain R with the property that $\bigcap_{M\in\Lambda_1} R_M \neq \bigcap_{N\in\Lambda_2} R_N$, for any two distinct subsets Λ_1 and Λ_2 of $\mathrm{Max}(R)$ is called a #-domain. If R has the same property for $\Lambda_1, \Lambda_2 \subseteq t\text{-}\mathrm{Max}(R)$, we say that R is a *t#-domain* [21].

Corollary 1.12. *Let R be a w-stable integral domain. Then:*

(1) *t-Spec(R) is treed.*
(2) *R satisfies the ascending chain condition on t-prime ideals.*
(3) *R is a t#-domain.*

Proof. (1) follows from Theorem 1.9 and the fact that the spectrum of a local stable domain is linearly ordered [31, Theorem 4.11(ii)].

(2) follows from Theorem 1.9 and the fact that a local stable domain satisfies the ascending chain condition on prime ideals [31, Theorem 4.11(ii)].

(3) follows from [21, Corollary 1.3], because each t-maximal ideal of R is divisorial by Proposition 1.5. □

2. Overrings of ⋆-stable domains

B. Olberding proved that overrings of stable domains are stable [31, Theorem 5.1]. This result was generalized by S. El Baghdadi, who showed that a t-linked overring T of a w-stable domain is w'-stable, where w' denotes the w-operation on T (see [12, Theorem 2.10]). Recall that an overring T of an integral domain R is called *t-linked* over R if $T^w = T$ [7]. Each overring of R is t-linked precisely when $d = w$ [7, Theorem 2.6]. El Baghdadi's proof works more in general for semistar operations spectral and of finite type.

Theorem 2.1. *Let R be a domain and $\star$ a semistar operation on R such that $\star = \tilde{\star}$. If R is $\star$-stable, then each overring T of R is $\dot{\star}$-stable.*

Proof. Since $R \subseteq T$ implies $R^\star \subseteq T^\star$, by Proposition 1.1 we can assume that $R^\star = R$ and $T^\star = T$, that is, that $\star$ is a (semi)star operation on R and $\dot{\star}$ is a (semi)star operation on T.

First we show that T is $\dot{\star}$-locally stable. Let $M = M^{\dot{\star}}$ be a $\dot{\star}$-maximal ideal of T. Then $(M \cap R)^\star \subseteq M^\star \cap R^\star = M \cap R$. Hence $M \cap R$ is a $\star$-prime ideal of R and $R_{M\cap R} \subseteq T_M$. Since each localization of R at a $\star$-maximal ideal is stable (Theorem 1.9) and overrings of stable domains are stable [31, Theorem 5.1], then T_M is stable.

In order to apply Theorem 1.9, we have to prove that T has $\dot{\star}$-finite character. Let N be a $\star$-maximal ideal of R and let $\{M_\alpha\}$ be a family of $\dot{\star}$-maximal ideals of T, such that $\bigcap_\alpha M_\alpha \neq (0)$ and $M_\alpha \cap R \subseteq N$. We want to show that $\{M_\alpha\}$ is a finite set. Let $S := \bigcap_\alpha T_{M_\alpha} \supseteq T$. Since $R_N \subseteq T_{M_\alpha}$ for each α, we have that $R_N \subseteq S$. Hence S is stable as an overring of the stable domain R_N. Let $P_\alpha := M_\alpha T_{M_\alpha} \cap S$, for each α. The P_α's are pairwise incomparable, because $S_{P_\alpha} = T_{M_\alpha}$. Since $\bigcap_\alpha M_\alpha$ is nonempty, also $\bigcap_\alpha P_\alpha$ is nonempty. Let $x \in \bigcap_\alpha P_\alpha$. If the P_α's are infinitely many, then x is contained in infinitely many maximal ideals of S, because Spec(S) is treed [31, Theorem 4.11(ii)]. This contradicts the finite character of S. It follows that the P_α's, and so the M_α's, are finitely many. It is easy to see that this implies the $\dot{\star}$-finite character for T. □

Corollary 2.2. *Let R be a w-stable domain and let T be a t-linked overring of R. Then $\dot{w} = w'$ is the w-operation on T and T is w'-stable.*

Proof. T is $\dot{w}$-stable by Theorem 2.1. Since $\dot{w}$ is a (semi)star operation on T (because T is t-linked over R) and $\dot{w} = \tilde{\dot{w}}$, it follows from Corollary 1.6 that $\dot{w} = \tilde{\dot{w}} = w'$. □

If $\star$ is a semistar operation on R, the *$\star$-integral closure* of R is the integrally closed overring of R defined by $R^{[\star]} := \bigcup\{(F^\star : F^\star) | F \in \boldsymbol{f}(R)\}$ [16]. Clearly $R^{[\star]} = R^{[\star_f]}$. The v-integral closure $R^{[v]}$ of a domain R is also called the *pseudo-integral closure of* R [3]. We say that R is *$\star$-integrally closed* if $R^{[\star]} = R$. In this case, it is easy to see that $\star$ is necessarily a (semi)star operation on R [10, p. 50]. Denoting by R' the integral closure of R, we have $R \subseteq R' \subseteq R^{[\star]}$. In addition, if $\star$ is a (semi)star operation and $\widetilde{R} := \bigcup\{(I^v : I^v) | I \in \boldsymbol{F}(R)\}$ is the complete integral closure of R, we have $R^{[\star]} \subseteq R^{[v]} \subseteq \widetilde{R}$.

It is known that the integral closure of a stable domain is a Prüfer domain [34, Proposition 2.1]. We now show that, when $\star$ is a semistar operation of finite type, the $\star$-integral closure of a $\star$-stable domain is a PvMD. Recall that an integrally closed domain is a PvMD if and only if $w = t$ [23, Theorem 3.5].

Theorem 2.3. *Let R be an integral domain and $\star$ a semistar operation on R. Assume that R is $\star$-stable. Then, denoting by t' and w' respectively the t-operation and the w-operation on $R^{[\star]}$:*

(1) *Each nonzero finitely generated ideal of $R^{[\star]}$ is $\dot{\star}$-invertible.*
(2) *If $\star$ is of finite type, then $R^{[\star]}$ is a PvMD and $\dot{\star} = t' = w'$.*
(3) *If $\star = \tilde{\star}$, then $R^{[\star]}$ is a w'-stable PvMD and $\dot{\star} = t' = w'$.*

Proof. (1) Let I be a nonzero finitely generated ideal of $R^{[\star]}$. We have to prove that $(I(R^{[\star]} : I))^\star = R^{[\star]}$.

There exist $x \in K$ and a finitely generated ideal J of R, such that $I = xJR^{[\star]}$. Since R is $\star$-stable, $J^\star$ is invertible in $(J^\star : J^\star) \subseteq R^{[\star]}$. Thus $R^{[\star]} = (J^\star : J^\star)R^{[\star]} = (J^\star((J^\star : J^\star) : J))^\star R^{[\star]} \subseteq (JR^{[\star]}(R^{[\star]} : JR^{[\star]}))^\star \subseteq R^{[\star]}$. It follows that $(I(R^{[\star]} : I))^\star = (JR^{[\star]}(R^{[\star]} : JR^{[\star]}))^\star = R^{[\star]}$.

(2) If $\star$ is of finite type, $\dot{\star}$ is a (semi)star operation on $R^{[\star]}$ [16, Proposition 4.5(3)]. By (1) and [6, Lemma 2.1] we have $I^{\dot{\star}} = I^{t'}$ for each finitely generated ideal I of $R^{[\star]}$. Hence, $\dot{\star} = t'$ and, again by (1), we conclude that $R^{[\star]}$ is a PvMD.

(3) follows from (2) and Theorem 2.1. □

Corollary 2.4. *Let R be an integral domain and $\star$ a semistar operation of finite type on R. If R is $\star$-stable and $\star$-integrally closed, then R is a PvMD and $\star = t = w$. In particular, a t-stable pseudo-integrally closed domain is a PvMD.*

Corollary 2.5. *Let R be an integral domain and $\star$ a (semi)star operation of finite type on R. If R is $\star$-stable and completely integrally closed, then R is a Krull domain and $\star = t = w$.*

Proof. If R is $\star$-stable, then each t-maximal ideal of R is divisorial (Proposition 1.5). Hence R, being completely integrally closed, is a Krull domain [18, Theorem 2.6]. Since R is $\star$-integrally closed (because $R^{[\star]} \subseteq \widetilde{R}$), then $\star = t = w$ (Corollary 2.4). □

Corollary 2.6 (*[12, Corollary 2.11]*). *Let R be a w-stable domain. Then the w-integral closure $R^{[w]}$ of R is a w'-stable PvMD and $\dot{w} = t' = w'$ (where t' and w' are respectively the t-operation and the w-operation on $R^{[w]}$).*

Remark 2.7. The integral closure R' of a w-stable domain is always $\dot{w}$-stable by Theorem 2.1. However, R' is not necessarily t-linked over R [8, Section 4] and so we cannot use Corollary 2.2 to conclude that R' is w'-stable. (In fact, when $\dot{w}$ is not a (semi)star operation, we cannot compare $\dot{w}$ and w'.)

We do not know whether, for $\star$ of finite type, a $\star$-stable integrally closed domain is a PvMD. However, when $\star$ is a (semi)star operation on R such that $\star = \tilde{\star}$, then R is $\star$-integrally closed if and only if R is integrally closed [10, Lemma 4.13]. Hence, from Corollary 2.4, we have:

Corollary 2.8. *Let R be a domain and $\star$ a (semi)star operation on R such that $\star = \tilde{\star}$. If R is $\star$-stable and integrally closed, then R is a PvMD and $\star = t = w$. In particular, a w-stable integrally closed domain is a PvMD.*

As a matter of fact, it is proved in [12] that a w-stable integrally closed domain is precisely a strongly discrete PvMD with t-finite character. Recall that a valuation domain V is called *strongly discrete* if PV_P is a principal ideal for each prime ideal P of V. We say that a PvMD (respectively, a Prüfer domain) R is *strongly discrete* if R_P is a strongly discrete valuation domain, for each $P \in t$-Spec(R) (respectively, for each $P \in$ Spec(R)). If R is a strongly discrete PvMD (respectively, a Prüfer domain) and each proper t-ideal (respectively, each nonzero proper ideal) of R

has only finitely many minimal primes, then R is called a *generalized Krull domain* [9] (respectively a *generalized Dedekind domain*).

The following characterization of w-stable integrally closed domain is given in [12]. For stable domains, an analogous result is due to Olberding [27,29].

Theorem 2.9 (*[12, Theorem 2.6]*). *The following conditions are equivalent for an integral domain R:*

(i) *R is integrally closed and w-stable;*
(ii) *R is a w-stable PvMD;*
(iii) *R is a strongly discrete PvMD with t-finite character;*
(iv) *R is a generalized Krull domain with t-finite character;*
(v) *R is a w-stable generalized Krull domain;*
(vi) *R is a PvMD with t-finite character and each t-prime ideal of R is w-stable;*
(vii) *R is w-stable and each t-maximal ideal of R is t-invertible.*

3. $\star$-Divisorial $\star$-stable domains

Following [1], we say that a nonempty family Λ of nonzero prime ideals of R is *of finite character* if each nonzero element of R belongs to at most finitely many members of Λ and we say that Λ is *independent* if no two members of Λ contain a common nonzero prime ideal. We note that, for a (semi)star operation $\star$ of finite type, the family $\star$-Max(R) is independent if and only if no two members of $\star$-Max(R) contain a common prime t-ideal, because a minimal prime of a principal ideal is a t-ideal. When the family of all maximal (respectively, t-maximal) ideals of R is independent of finite character, R is called an *h-local domain* (respectively, a *weakly Matlis domain*).

A domain such that each ideal is divisorial (that is $d = v$) is called a *divisorial domain* and a domain whose overrings are all divisorial is called *totally divisorial* [5]. We say that a domain R is *$\star$-divisorial* if $\star = v$.

Theorem 3.1. *Let R be an integral domain and $\star$ a (semi)star operation on R. If $\star = \tilde{\star}$, the following conditions are equivalent:*

(i) *R is $\star$-stable and $\star$-divisorial.*
(ii) *The family $\star$-*Max(R)* is independent of finite character and R_M is totally divisorial, for each $M \in \star$-*Max(R).
(iii) *Each overring T of R such that $T^\star = T$ is $\dot{\star}$-divisorial.*
(iv) *$(I^\star : I^\star)$ is $\dot{\star}$-divisorial, for each nonzero ideal I of R.*
(v) *$v(I^\star) = \dot{\star}$ on $(I^\star : I^\star)$, for each nonzero ideal I of R.*

Under these conditions, $\star = w$ on R and $\dot{\star} = \dot{w} = w'$ is the w-operation on each t-linked overring T of R.

Proof. (i)⇒(ii) Since $\star = \tilde{\star} \leq w \leq t \leq v$, if R is $\star$-divisorial, it is also w-divisorial. Hence the family $\star$-Max$(R) = t$-Max(R) is independent of finite character [11, Theorem 1.5].

If M is a t-maximal ideal of R, R_M is stable by Theorem 1.9. Hence, to show that R_M is totally divisorial, it is enough to show that R_M is divisorial [29, Theorem 3.12]. Let $I = JR_M$ be a nonzero ideal of R_M, with J an ideal of R. Since t-Max(R) is independent of finite character, we have $I^v = (JR_M)^{v'} = J^v R_M$, where v' is the v-operation on R_M [1, Corollary 5.3]. Since $J^\star = J^v$, then $I^v = J^\star R_M = JR_M = I$. Hence, R_M is divisorial.

(ii)⇒(iii) Let T be an overring of R such that $T^\star = T$. By applying [1, Corollary 5.2] for $\mathcal{F} = \star$-Max(R), if J is a nonzero ideal of T and M is a $\star$-maximal ideal, we have $(T : (T : J))R_M \subseteq (TR_M : (T : J)R_M) = (TR_M : (TR_M : JR_M)) = JR_M$, where the last equality holds because R_M is totally divisorial and $TR_M \supseteq R_M$. Thus, denoting by v' the v-operation of T, we have $J^{v'} R_M = JR_M$, for each $\star$-maximal ideal M of R. It follows that $(J^{v'})^{\dot{\star}} = J^{\dot{\star}}$. Since $\dot{\star}$ is a (semi)star operation on T, $\dot{\star} \leq v'$ and $J^{v'} = J^{\star'}$.

(iii)⇒(iv) This is straightforward, since $(I^\star : I^\star)^\star = (I^\star : I^\star)$.

(iv)⇒(v) Since $v(I^\star)$ is a (semi)star operation on $(I^\star : I^\star)$, we have $v(I^\star) \leq v'$, the v-operation of $(I^\star : I^\star)$. Moreover, $I^\star$ is a $\dot{\star}$-ideal, so by Lemma 1.3(2), $\dot{\star} \leq v(I^\star)$. Thus, since by hypothesis $\dot{\star} = v'$, we have $\dot{\star} = v(I^\star)$.

(v)⇒(i) R is clearly $\star$-divisorial, taking $I = R$. That R is $\star$-stable is a consequence of Proposition 1.4(iii)⇒(i). In fact, if I and J are ideals of R such that $(I^\star : I^\star) = (J^\star : J^\star)$ it is clear that $v(I^\star) = v(J^\star)$, since both coincide with $\dot{\star}$.

To finish, if R is $\star$-stable, we have $\star = w$ by Theorem 1.9 and $\dot{w} = w'$ is the w-operation on each t-linked overring T of R by Corollary 2.2. □

We state explicitly the theorem for $\star = w$.

Corollary 3.2. *The following conditions are equivalent for an integral domain R:*

(i) *R is w-stable and w-divisorial.*
(ii) *R is a weakly Matlis domain and R_M is totally divisorial, for each $M \in t$-Max(R).*
(iii) *Each t-linked overring T of R is $\dot{w}$-divisorial.*
(iv) *$(I^w : I^w)$ is $\dot{w}$-divisorial, for each nonzero ideal I of R.*
(v) *$v(I^w) = \dot{w}$ on $(I^w : I^w)$, for each nonzero ideal I of R.*

Under these conditions, $\dot{w} = w'$ is the w-operation on each t-linked overring T of R.

Since each t-linked overring of a w-stable domain is w'-stable (Corollary 2.2), we get:

Corollary 3.3. *Each t-linked overring of a w-stable w-divisorial domain is a w'-stable w'-divisorial domain.*

We do not know if in general the condition that each t-linked overring T of R is w'-divisorial implies that R is w-stable. However, we now show that this is true in the integrally closed case and in the Mori case.

The following result follows from Corollary 3.2 and the fact that a valuation domain V is totally divisorial if and only if it is strongly discrete [5, Proposition 7.6]. (Compare it with Theorem 2.9.) The equivalences (ii)⇔(iii)⇔(iv) are part of [11, Theorem 3.5]. (i)⇔(ii) is [12, Corollary 2.8].

Corollary 3.4. *The following conditions are equivalent for an integral domain R:*

(i) *R is an integrally closed w-stable w-divisorial domain.*
(ii) *R is integrally closed and each t-linked overring of R is w'-divisorial.*
(iii) *R is a strongly discrete PvMD and a weakly Matlis domain.*
(iv) *R is a w-divisorial generalized Krull domain.*

Corollary 3.5. *Let R be an integral domain. The following are equivalent:*

(i) *R is completely integrally closed and w-stable.*
(ii) *R is completely integrally closed and w-divisorial.*
(iii) *R is a Krull domain.*

Proof. (i)⇔(iii) follows from Corollary 2.5. (ii)⇔(iii) was proved in [11, Proposition 3.7]. □

Mori domains whose t-linked overrings are all w'-divisorial were studied in [11]. A Mori domain is w-divisorial if and only if R_M is a divisorial Noetherian domain, necessarily one-dimensional, for each t-maximal ideal M [11, Theorem 4.5 and Proposition 4.1].

Corollary 3.6. *The following conditions are equivalent for an integral domain R:*

(i) *R is a Mori w-stable w-divisorial domain.*
(ii) *R is a Mori domain and each t-linked overring is w'-divisorial.*
(iii) *R is a Mori domain and R_M is totally divisorial, for each $M \in t$-Max(R).*
(iv) *R has t-dimension one and each t-linked overring of R is w'-divisorial.*
(v) *R has t-dimension one and is w-stable and w-divisorial.*
(vi) *For each nonzero ideal I of R, there are $a, b \in R$ such that $I^w = (aR + bR)^w$.*

Proof. (i)⇒(ii) and (v)⇒(iv) by Corollary 3.2(i)⇒(iii).

(ii)⇔(iii)⇔(iv)⇔(vi) are proved in [11, Theorem 4.11].

We show that (ii) and (iii) imply (i). In fact, by (ii) R is w-divisorial and so weakly Matlis, and by (iii) R_M is totally divisorial for each t-maximal M. So, we can conclude by applying Corollary 3.2(ii)⇒(i). In the same way, we get that (iii) and (iv) imply (v). □

For $\star = d$ we recover the following characterizations of totally divisorial domains: (i)⇔(iii)⇔(iv) are due to Olberding [29, Theorem 3.12 and Corollary 3.13], (iv)⇔(v)⇔(vi) are due to Picozza [32, Theorem 2.57].

Corollary 3.7. *The following conditions are equivalent for an integral domain R:*

(i) *R is stable and divisorial.*
(ii) *R is stable and w-divisorial.*
(iii) *R is h-local and R_M is totally divisorial, for each maximal ideal M of R.*
(iv) *R is totally divisorial.*
(v) *$(I : I)$ is divisorial for each nonzero ideal I of R.*
(vi) *Each nonzero ideal I of R is m-canonical in $(I : I)$.*

For ease of reference, we state Corollary 3.7 in the integrally closed case [27,29] and in the Noetherian case [5] (see also [11, Corollary 3.6 and Proposition 4.8]).

Corollary 3.8. *The following conditions are equivalent for an integral domain R:*

(i) *R is an integrally closed stable divisorial domain.*
(ii) *R is an integrally closed totally divisorial domain.*
(iii) *R is an h-local strongly discrete Prüfer domain.*
(iv) *R is a divisorial generalized Dedekind domain.*

Corollary 3.9. *The following conditions are equivalent for an integral domain R:*

(i) *R is a Noetherian stable divisorial domain.*
(ii) *R is a Noetherian totally divisorial domain.*
(iii) *R is a one-dimensional totally divisorial domain.*
(iv) *R is 2-generated (that is, each ideal of R is generated by 2 elements).*

Remark 3.10. (1) It is easy to check that the overrings of R of the form $(I : I)$, where I is a nonzero ideal of R, are precisely the overrings with nonzero conductor in R. But, if R is totally divisorial, it is not true in general that all the overrings of R have nonzero conductor (that is, totally divisorial domains are not always *conducive*). For example, any conducive totally divisorial Prüfer domain is a strongly discrete valuation domain [33, Theorem 4.7], while Corollary 3.8 shows that there exist plenty of non-quasilocal totally divisorial Prüfer domains.

(2) A Noetherian stable domain is always one-dimensional [35, Proposition 2.1]. We do not know if a Mori w-stable domain need to have t-dimension one. However, we can say that it has t-dimension at most equal to 2. In fact, by Corollary 1.10, a Mori domain R is w-stable if and only if R_M is stable for each t-maximal ideal M. In addition, it is known that if P is a nonzero nonmaximal prime ideal of a stable domain R, then R_P is a strongly discrete valuation domain [31, Theorem 4.11]. Since the Mori property localizes and a Mori valuation domain is a DVR, we see that a quasilocal stable Mori domain has dimension at most equal to 2.

(3) Examples of Mori w-stable w-divisorial domains that are neither Noetherian nor Krull can be constructed by means of pullbacks, as in [11, Example 4.13]. We do not know any example of a one-dimensional stable Mori domain that is not Noetherian.

The next theorem shows that the study of w-stable w-divisorial domains can be reduced to the case where the domain R has t-dimension at least equal to two and has no t-invertible t-prime ideals. If Λ is a set of prime ideals of R, we set $R_{\mathcal{F}(\Lambda)} := \bigcap_{P\in\Lambda} R_P$.

Theorem 3.11. *Assume that R is a w-stable w-divisorial domain. Let Λ_1 be the set of the t-invertible t-maximal ideals of R, Λ_2 be the set of the height-one t-maximal ideals of R that are not t-invertible, $\Lambda_3 := t\text{-Max}(R) \setminus \{\Lambda_1 \cup \Lambda_2\}$ and set $R_i := R_{\mathcal{F}(\Lambda_i)}$, for $i = 1, 2, 3$. (If $\Lambda_i = \emptyset$, set $R_i := K$.) Then:*

(1) *If $\Lambda_1 \neq \emptyset$, R_1 satisfies the equivalent conditions of Corollary 3.4.*
(2) *If $\Lambda_2 \neq \emptyset$, R_2 satisfies the equivalent conditions of Corollary 3.6 and has no t'-invertible t'-prime ideals (where t' is the t-operation on R_2).*
(3) *If $\Lambda_3 \neq \emptyset$, R_3 is a w-stable w-divisorial domain of t'-dimension strictly greater than one with no t'-invertible t'-prime ideals (where t' is the t-operation on R_3).*
(4) *$R = R_1 \cap R_2 \cap R_3$.*

Proof. Let Λ be a nonempty set of t-maximal ideals of R and $T := R_{\mathcal{F}(\Lambda)}$. Since t-Max(R) has t-finite character, then t'-Max$(T) = \{PR_P \cap T\,;\, P \in \Lambda\}$ [19, Proposition 1.17]. In addition, for $M = PR_P \cap T \in t'$-Max(T), we have $T_M = R_P$. Recalling that an ideal of a domain with t-finite character is t-invertible if and only if it is t-locally principal, we get that M is t'-invertible in T if and only if P is t-invertible in R.

Since T is t-linked over R, T is w'-stable and w'-divisorial (Corollary 3.3). Hence (3) and (4) follow easily.

(1) If $P \in \Lambda_1$, then PR_P is principal. Hence R_P is a stable quasi-local domain (Theorem 1.9) with principal maximal ideal; whence it is a valuation domain [31, Lemma 4.5]. It follows that R_1 is an integrally closed w'-stable domain and then it satisfies the equivalent conditions of Corollary 3.4.

(2) R_2 has t'-dimension one and is w'-stable and w'-divisorial. Hence R_2 satisfies the equivalent conditions of Corollary 3.6. Since the t-maximal ideals in Λ_2 are not t-invertible, then R_2 has no t'-invertible prime ideals. □

Corollary 3.12. *Let R be a totally divisorial domain. With the notation of the previous theorem:*

(1) *If $\Lambda_1 \neq \emptyset$, R_1 satisfies the equivalent conditions of Corollary 3.8.*
(2) *If $\Lambda_2 \neq \emptyset$, R_2 satisfies the equivalent conditions of Corollary 3.9 and has no invertible prime ideals.*
(3) *If $\Lambda_3 \neq \emptyset$, R_3 is a totally divisorial domain of dimension strictly greater than one with no invertible prime ideals.*

4. v-Coherence

A domain is *coherent* if the intersection of any two finitely generated ideals is finitely generated. Olberding proved that a stable divisorial domain is coherent [29, Lemma 3.2], even though there are stable domains that are not coherent [30, Section 5].

We next show that w-stable w-divisorial domains are v-coherent. Recall that R is *v-coherent* if the intersection of any two v-finite ideals is v-finite; this is equivalent to saying that the ideal $(R : I)$ is v-finite for each nonzero finitely generated ideal I of R. The class of v-coherent domains properly includes PvMD's, Mori domains and coherent domains [20]. A divisorial v-coherent domain is coherent.

The following lemma is probably known; for completeness we include the proof.

Lemma 4.1. *An integral domain R with t-finite character is v-coherent if and only if R_M is v-coherent, for each t-maximal ideal M of R.*

Proof. If R is a v-coherent domain, then each generalized ring of quotients of R is v-coherent [19, Proposition 3.1].

Conversely, let J be a finitely generated nonzero ideal of R. If $J^v \neq R$, there are just finitely many t-maximal ideals $M_1, \dots, M_n$ containing J and, for each $i = 1, \dots, n$, there is a finitely generated ideal $H_i \subseteq (R : J)$ such that $(R : J)R_{M_i} = (R_{M_i} : JR_{M_i}) = (H_i R_{M_i})^{v'} = H_i^v R_{M_i}$ (where v' is the v-operation on R_M).

Let $H := H_1 + \cdots + H_n$. If $(R : J) \neq H^v$, let $N_1, \dots, N_m$ be the t-maximal ideals of R different from the M_i's such that $HR_{N_j} \neq R_{N_j}$, $j = 1, \dots, m$. If $x \in R \backslash \{N_1 \cup \cdots \cup N_m\}$, by checking t-locally, we get $(R : J) = (H + xR)^v$. □

Theorem 4.2. *A w-stable w-divisorial domain is v-coherent.*

Proof. By Corollary 3.2 and Lemma 4.1, because totally divisorial domains are coherent [29, Lemma 3.2]. □

For $d = w$, the following proposition recovers [30, Lemma 4.1].

Proposition 4.3. *A w-stable domain R is v-coherent if and only if each t-maximal ideal of R is v-finite.*

Proof. Assume that R is v-coherent and let $M \in t$-Max(R). Since M is divisorial (Proposition 1.5), then $M = xR \cap R = (R : R + x^{-1}R)$, for some $x \in R$. Thus M is v-finite.

Conversely, if R is w-stable, R has t-finite character and R_M is stable, for each t-maximal ideal M of R (Theorem 1.9). Thus, MR_M is divisorial in R_M. In addition, if $M = J^v$, for some finitely generated ideal J of R, then $MR_M = J^v R_M = (JR_M)^{v'}$ is v'-finite (where v' is the v-operation on R_M). Hence, by Lemma 4.1, we may assume that R is stable quasilocal and that its maximal ideal M is divisorial v-finite.

We have to show that $(R : I)$ is v-finite for each nonzero finitely generated ideal I of R. If I is (t-)invertible, this is true. Thus we may assume that $(R : I) = (M : I)$. Now, $(I : I)(M : I) = (M : I) = (M : M)(M : I)$ and, since R is stable quasilocal, there exist $x, y \in K$ such that $I = x(I : I)$ and $M = y(M : M)$ [31, Lemma 3.1]. Hence, setting

$\alpha := xy$, $IM(M : I) = \alpha(M : I) = \alpha(R : I)$. On the other hand, we have also $(R : I) = \beta((R : I) : (R : I))$, for some $\beta \in K$. Thus we get $\alpha^{-1}I^v = (R : IM(M : I)) = (R : IM(R : I)) = (((R : I) : (R : I)) : M) = \beta^{-1}(R : IM)$. It follows that $(R : I) = \alpha^{-1}\beta(IM)^v$ is v-finite. □

Remark 4.4. (1) A one-dimensional stable coherent domain is Noetherian, because its prime ideals are finitely generated. We do not know whether a one-dimensional stable v-coherent domain must be Mori (or even Noetherian).

(2) Generalized rings of quotients of v-coherent domains are v-coherent [19, Proposition 3.1], but it is not known whether t-linked overrings of v-coherent domains are v-coherent. However, a t-linked overring of a w-stable w-divisorial domain, being w'-stable and w'-divisorial (Corollary 3.3), is v-coherent.

By using v-coherence, we can see that w-stable w-divisorial domains (respectively, totally divisorial domains) share several properties with generalized Krull domains (respectively, generalized Dedekind domains). In fact, since an integrally closed w-stable w-divisorial domain (respectively, totally divisorial domains) R is a strongly discrete PvMD (respectively, Prüfer domain) (Theorem 2.9), in the integrally closed case each one of these properties becomes equivalent to R being a generalized Krull domain (respectively, generalized Dedekind domain) (see [9, Theorem 3.5, 3.9 and Lemma 3.7] and [21, Corollary 2.15]).

Recall that an overring T of R is said to be *t-flat* over R if $T_M = R_{M\cap R}$, for each t-maximal ideal M of T [24]. Flatness implies t-flatness, but the converse is not true [24, Remark 2.12].

Corollary 4.5. *Let R be a w-stable w-divisorial domain. Then:*

(1) *Each t-prime ideal P of R is the radical of a v-finite divisorial ideal.*
(2) *R satisfies the ascending chain condition on radical t-ideals.*
(3) *Each proper t-ideal has only finitely many minimal (t-)primes.*
(4) *For each $\Lambda \subseteq t$-Spec(R), the overring $R_{\mathcal{F}(\Lambda)} := \bigcap_{P\in\Lambda} R_P$ is a t-flat t#-domain.*

Proof. (1) Since R is v-coherent (Theorem 4.2), then R_P is v-coherent, w-stable (Corollary 2.2) and divisorial (Corollary 3.2). Hence PR_P is v'-finite in R_P(Proposition 4.3) and so $PR_P = (JR_P)^{v'} = J^vR_P$, for some finitely generated ideal J of R. Since t-Max(R) is independent of finite character (Corollary 1.10) and t-Spec(R) is treed (Corollary 1.12), the set of minimal primes of J^v is finite. Set Min$(J^v) = \{P = P_1, \dots, P_n\}$. If $n \geq 2$, let $x \in P \setminus (P_2 \cup \cdots \cup P_n)$ and $I = (J + xR)^v$. Then $P = \sqrt{I}$.

(1) and (2) are equivalent by [9, Lemma 3.7].

(3) follows from (1) by [9, Lemma 3.8].

(4) Each localization of R at a t-prime ideal is divisorial by Corollary 3.2. In addition, t-Spec(R) is treed and R satisfies the ascending chain condition on t-prime ideals by Corollary 1.12. Hence the overring $T := R_{\mathcal{F}(\Lambda)}$ is t-flat by [11, Corollary 2.12]. Since T is t-linked over R, T is w'-stable and w'-divisorial (Corollary 3.3). Thus T is a t#-domain by Corollary 1.12. □

Corollary 4.6. *Let R be a totally divisorial domain. Then:*

(1) *Each prime ideal of R is the radical of a finitely generated ideal.*
(2) *R satisfies the ascending chain condition on radical ideals.*
(3) *Each proper ideal has only finitely many minimal primes.*
(4) *For each set Λ of prime ideals, the overring $R_{\mathcal{F}(\Lambda)} := \bigcap_{P\in\Lambda} R_P$ is a flat #-domain.*

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