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Small Valdivia compact spaces

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Abstract

We prove a preservation theorem for the class of Valdivia compact spaces, which involves inverse sequences of retractions of a certain kind. Consequently, a compact space of weight $\leqslant \aleph_1$ is Valdivia compact iff it is the limit of an inverse sequence of metric compacta whose bonding maps are retractions. As a corollary, we show that the class of Valdivia compacta of weight $\leqslant \aleph_1$ is preserved both under retractions and under open 0-dimensional images. Finally, we characterize the class of all Valdivia compacta in the language of category theory, which implies that this class is preserved under all continuous weight preserving functors.

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1. Introduction

A topological space is called *Valdivia compact*, if it is homeomorphic to the closure in a Tikhonov cube of a subset of a Σ-product. This is a natural generalization of *Corson compact spaces*, which are defined as compact subsets of Σ-products. The class of Valdivia compact spaces was introduced by Argyros, Mercourakis and Negrepontis in [1] and further investigated by Valdivia [16,17]. The name *Valdivia compact* was introduced by Deville and Godefroy in [3]. Valdivia compacta were extensively studied by Kalenda and we refer to his article [7] for a survey of results. It is well known that the class of Corson compacta is stable under continuous images. On the other hand, there are simple examples of continuous images of Valdivia compacta which are not Valdivia compact. An interesting result of Kalenda [8] says that every Valdivia compact space which is not Corson compact, has a two-to-one continuous map onto a non-Valdivia compact space.

The main subject of this paper is to study Valdivia compact spaces of weight $\leqslant \aleph_1$ (we call these spaces *small Valdivia compacta*). A typical phenomenon occurs when studying various classes of nonmetrizable compact spaces: namely, a class restricted to spaces of weight $\leqslant \aleph_1$ has a very simple structure and above $\aleph_1$ the structure becomes much more complicated. In this context, we prove some positive results about the stability of the class of small

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Valdivia compacta. It turns out that small Valdivia compacta are precisely those spaces which can be obtained as limits of inverse sequences of metric compacta whose bonding maps are retractions. In fact, we prove a more general preservation result which involves inverse sequences of *simple* retractions (the definition is given in Section 4). We apply this result to give positive answers to two questions on the stability of Valdivia compacta, in the case of spaces of weight $\leqslant \aleph_1$. The mentioned preservation result shows also that retracts of Tikhonov or Cantor cubes are Valdivia compact. A stronger result has recently been obtained by Leiderman and the first author in [12], namely retracts of products of metric compacta belong to the class of *semi-Eberlein spaces* and the latter class is properly contained in the class of Valdivia compacta.

A question of Kalenda [7] asks whether every open continuous image of a Valdivia compactum is Valdivia compact. He gives an affirmative answer in the case where the image has a dense set of G_δ points, see [9,7]. Uspenskij and the first author described in [13] an example of a compact connected Abelian group of weight $\aleph_1$ which is not Valdivia compact. Such a group is an open (epimorphic) image of a Valdivia compactum (a product of compact metric spaces), which gives a negative answer to Kalenda's question. On the other hand, in this paper we show that a small open 0-dimensional image of a Valdivia compact space is Valdivia compact. Another question from [7] asks whether X is Valdivia compact provided $X \times Y$ is Valdivia compact for some space Y. We give a partial positive answer to a more general question whether the class of Valdivia compacta is closed under retractions. Namely, we show that a retract of a Valdivia compactum is Valdivia compact provided its weight is $\leqslant \aleph_1$.

The last part of the paper explores the properties of inverse systems of retractions. As a result, we give a characterization of the class of all Valdivia compacta in terms of certain inverse systems of retractions. This characterization belongs to the language of category theory and therefore as an application we obtain that every continuous weight preserving functor on the class of compact spaces preserves Valdivia compacta.

2. Preliminaries

2.1. Notation and basic definitions

All topological spaces are assumed to be Hausdorff. By a "map" we mean a continuous map, unless otherwise indicated. The closure of a set A is denoted by $\operatorname{cl} A$ or $\operatorname{cl}_X A$, if it is not clear from the context, which topological space we have in mind. We say that A is *countably closed* if $\operatorname{cl} M \subseteq A$ whenever $M \subseteq A$ is countable. A subset D of a topological space X is *κ-monolithic* if for every $A \in [D]^{\leqslant\kappa}$, $\operatorname{cl}_X A$ is contained in D and its net-weight is $\leqslant \kappa$ ($[D]^{\leqslant\kappa}$ denotes the family of all subsets of D of cardinality $\leqslant \kappa$). As we shall consider κ-monolithic sets in compact spaces, "net-weight" can be replaced by "weight" in the above definition.

A Tikhonov cube of weight κ is denoted by $[0,1]^S$, where S is any set of cardinality κ. If $T \subseteq S$ then we denote by $x \mid T$ the element $(x \restriction T)^\frown 0_{S\setminus T} \in [0,1]^S$, where $0_{S\setminus T}$ is the constant 0 function in $[0,1]^{S\setminus T}$ and $^\frown$ denotes the concatenation of functions. We also write $X \mid T$ for the set $\{x \mid T\colon x \in X\}$, where $X \subseteq [0,1]^S$. The map $j^S_T : [0,1]^T \to [0,1]^S$ defined by $j^S_T(x) = x^\frown 0_{S\setminus T}$ is called the *canonical embedding* of $[0,1]^T$ into $[0,1]^S$. We shall often identify $[0,1]^T$ with a subspace of $[0,1]^S$, meaning that canonical embedding. The standard projection (i.e. the restriction map) from $[0,1]^S$ onto $[0,1]^T$ is denoted by π^S_T.

Given two maps $f : X \to Y$ and $g : X \to Z$, the *diagonal* of f, g is the map $f \Delta g : X \to Y \times Z$ defined by $(f \Delta g)(x) = \langle f(x), g(x)\rangle$. The disjoint topological sum of spaces X, Y is denoted by $X \oplus Y$. Ordinals will be denoted by Greek letters and treated as sets of their predecessors linearly ordered by the relation $\in$. Cardinals are particular examples of ordinals, although we usually denote them by using the $\aleph$-notation. Given an ordinal δ, the set $\delta + 1$ is a compact linearly ordered space. In particular, $\omega + 1$ (which is denoted also by $\aleph_0 + 1$) is a convergent sequence together with its limit. The cardinality of a set A is denoted by $|A|$.

A family $\mathcal{A}$ of subsets of a space X is T_0 *separating* if for every $x, y \in X$ with $x \neq y$ there exists $A \in \mathcal{A}$ such that $|\{x, y\} \cap A| = 1$. Every T_0 separating family $\mathcal{U}$ of open F_σ subsets of a compact space X induces an embedding $f : X \to [0,1]^{\mathcal{U}}$ defined by $f(x)(u) = h_u(x)$, where $h_u : X \to [0,1]$ is a fixed continuous function such that $u = {h_u}^{-1}[(0,1]]$, $u \in \mathcal{U}$.

A *retraction* is, by definition, a map $f : X \to Y$ which has a *right inverse* $g : Y \to X$, i.e. $fg = \mathrm{id}_Y$. In this case f is a surjection, g is an embedding and $gf : X \to X$ is a self-map which is identity on its range. Conversely, if X is compact and $r : X \to X$ is such that $r^2 = r$ then $r : X \to r[X]$ is a retraction in the above sense. We then say that r is an *internal retraction*. We shall use the following well-known fact: if $f : X \to Y$ is an open surjection, X is a complete metric space and Y is a 0-dimensional compact space then f is a retraction. The proof is a direct application of Michael's selection theorem for the multivalued map f^{-1}, which is lower semi-continuous in this case.

We shall use the method of elementary substructures, in order to avoid tedious "closing-off" constructions. We refer to [4] for a survey on the use of this method in set-theoretic topology. Given an uncountable cardinal χ we denote by $H(\chi)$ the class of all sets which are hereditarily of cardinality $< \chi$. That is, $x \in H(\chi)$ iff $|x| < \chi$, $|y| < \chi$ for every $y \in x$, $|z| < \chi$ for every $z \in y \in x$ and so on. It can be shown easily that $H(\chi)$ is a set and for every formula $\varphi(x_1, \dots, x_n)$ (with parameters $x_1, \dots, x_n$) which is satisfied in the universe of set theory we are working in, there exists a cardinal χ such that the structure $\langle H(\chi), \in\rangle$ contains the parameters $x_1, \dots, x_n$ and satisfies $\varphi(x_1, \dots, x_n)$. This is called the *reflection principle*. Given finitely many objects, say $a_1, \dots, a_n$ and having in mind finitely many formulas which speak about these objects, there is always a "big enough" cardinal χ such that $a_1, \dots, a_n \in H(\chi)$ and all the formulas are satisfied in $\langle H(\chi), \in\rangle$ provided they are satisfied in the universe. We shall ignore the details, saying "fix a big enough χ", having in mind a cardinal such that $H(\chi)$ contains all the relevant objects and reflects all the (finitely many) relevant formulas. We write $M \preceq H(\chi)$ for "M is an elementary substructure of $\langle H(\chi), \in\rangle$". Every $M \preceq H(\chi)$ has the following properties which we shall use throughout the paper:

(i) if $x \in M$ and $|x| \leqslant \aleph_0$ then $x \subseteq M$.
(ii) if $x_0, \dots, x_{n-1} \in M$ then $\{x_0, \dots, x_{n-1}\} \in M$.

A particular case of the well-known theorem of Skolem–Löwenheim says that for every infinite set $S \subseteq H(\chi)$ there exists $M \preceq H(\chi)$ such that $S \subseteq M$ and $|S| = |M|$. We shall demonstrate the use of elementary substructures in the next subsection, reproving some properties of σ-complete inverse systems.

2.2. *Inverse systems*

We shall consider inverse systems of compact spaces whose all bonding mappings are surjections. An inverse system indexed by a directed partially ordered set Σ will be denoted by $\mathbb{S} = \langle X_s; p_s^t; \Sigma\rangle$, where X_s is the sth space in the system and $p_s^t : X_t \to X_s$ is the bonding map, $s \leqslant t$. Formally, the limit of $\mathbb{S}$ is a pair $\langle X, \{p_s\}_{s\in\Sigma}\rangle$, where $p_s : X \to X_s$ and for every space Y and for every collection of maps $\{f_s\colon s \in \Sigma\}$ such that $f_s : Y \to X_s$ and $s \leqslant t \implies f_s = p_s^t f_t$, there exists a unique map $g : Y \to X$ such that $f_s = p_s g$ holds for every $s \in \Sigma$. Mappings p_s are called *projections*. In most cases it is clear from the context what the projections are; we then say that X is the limit of $\mathbb{S}$ and we write $X = \varprojlim \mathbb{S}$, instead of $\langle X, \{p_s\}_{s\in\Sigma}\rangle = \varprojlim \mathbb{S}$. If T is a directed subset of Σ then $\mathbb{S} \restriction T := \langle X_s; p_s^t; T\rangle$ is again an inverse system. $\mathbb{S}$ is *continuous* if $X_s = \varprojlim(\mathbb{S} \restriction T)$, whenever $T \subseteq \Sigma$ is directed and such that $s = \sup T$. It is well known that $\varprojlim \mathbb{S} = \varprojlim(\mathbb{S} \restriction T)$, whenever T is cofinal in Σ. We say that $\mathbb{S} = \langle X_s; p_s^t; \Sigma\rangle$ is *σ-complete* if every countable directed set $T \subseteq \Sigma$ has the least upper bound in Σ (i.e. Σ is *σ-complete*) and $X_t = \varprojlim(\mathbb{S} \restriction T)$, where $t = \sup T$. Given a σ-complete directed poset Σ, we say that $T \subseteq \Sigma$ is *σ-closed* in Σ if $\sup M \in T$ for every countable directed set $M \subseteq T$.

Given two maps $f : X \to Y$ and $g : X \to Z$, we say that f *factors through* g if there exists $h : Z \to Y$ such that $f = hg$. If both f, g are quotient maps then such a map g is unique and it is automatically continuous.

The following properties of σ-complete inverse systems will be used throughout the paper.

Proposition 2.1. *Let $\mathbb{S} = \langle X_s; p_s^t; \Sigma\rangle$ be a σ-complete inverse system of compact spaces and let $X = \varprojlim \mathbb{S}$. Then for every map $f : X \to Y$ into a second countable space there exists $\delta \in \Sigma$ such that f factors through p_δ, i.e. there exists $g : X_\delta \to Y$ such that $f = gp_\delta$.*

Proof. Fix a big enough cardinal χ and a countable $M \preceq H(\chi)$ which knows $\mathbb{S}$ and f. By elementarity, $\Sigma \cap M$ is directed. Let $\delta = \sup(\Sigma \cap M)$. We claim that f factors through p_δ.

Fix x_0, x_1 such that $f(x_0) \neq f(x_1)$. We have to show that $p_\delta(x_0) \neq p_\delta(x_1)$. Fix disjoint open sets $u_0, u_1 \subseteq Y$ such that $f(x_i) \in u_i$. We may assume that $u_0, u_1 \in M$, because M knows some countable base of Y. Now $f^{-1}[u_0]$ is an open F_σ subset of X and therefore by compactness it is the union of a countable family of open sets of the form ${p_s}^{-1}[w]$, where $s \in \Sigma$ and $w \subseteq X_s$. By elementarity, we may assume that such a family belongs to M and consequently is contained in M. In particular, there are $s_0 \in \Sigma \cap M$ and an open set $w_0 \subseteq X_{s_0}$ such that $x_0 \in {p_{s_0}}^{-1}[w_0] \subseteq f^{-1}[u_0]$. By the same argument, there are $s_1 \in \Sigma \cap M$ and an open set $w_1 \subseteq X_{s_1}$ such that

$x_1 \in {p_{s_1}}^{-1}[w_1] \subseteq f^{-1}[u_1]$. Modifying the sets if necessary and using the fact that $\Sigma \cap M$ is directed, we may assume that $s_0 = s_1 = s \in M$. Then $w_0 \cap w_1 = \emptyset$ and consequently $p_\delta(x_0)$, $p_\delta(x_1)$ are separated by disjoint sets $(p_s^\delta)^{-1}[w_i]$, $i = 0, 1$. Thus $p_\delta(x_0) \neq p_\delta(x_1)$. □

Proposition 2.2. *Assume $X = \varprojlim\langle X_s; p_s^{s'}; \Sigma\rangle$, $Y = \varprojlim\langle Y_s; q_s^{s'}; \Sigma\rangle$, where both inverse systems are σ-complete and X_s, Y_s are metric compacta for every $s \in \Sigma$. Let $f : X \to Y$. Then there are a σ-closed cofinal set $T \subseteq \Sigma$ and a family of maps $\{f_t\}_{t\in T}$ such that $f_t : X_t \to Y_t$ and $q_t f = f_t p_t$ holds for every $t \in T$. If moreover the map f is open then we may assume that f_t is open for every $t \in T$.*

Proof. Since the projections are quotient maps (recall that we consider inverse systems whose bonding maps are onto), we do not have to worry about the continuity of the maps f_t. Denote by T the set of all $t \in \Sigma$ such that $q_t f$ factors through p_t, i.e. such that there exists a (necessarily unique) map f_t satisfying $q_t f = f_t p_t$; moreover we require that f_t be open if f is so. Then T is σ-closed, by the continuity of both systems (recall that the limit of an inverse system of open maps is open). It remains to check that T is cofinal in Σ.

Fix a big enough cardinal χ, fix $s \in \Sigma$ and find a countable $M \preceq H(\chi)$ which contains enough information and such that $s \in M$. Let $\delta = \sup(\Sigma \cap M)$. Then $s < \delta$. We claim that $\delta \in T$. Fix $x_0, x_1 \in X$ such that $q_\delta f(x_0) \neq q_\delta f(x_1)$. Then $q_t f(x_0) \neq q_t f(x_1)$ for some $t \in \Sigma \cap M$ and by elementarity we can find $\varphi \in C(X_t) \cap M$ such that $\varphi q_t f(x_0) \neq \varphi q_t f(x_1)$. By Proposition 2.1, $\varphi q_t f$ factors through p_r for some r. Since $\varphi q_t f \in M$, we may assume that $r \in M$. Thus $r < \delta$ and $\varphi q_t f$ factors also through p_δ, i.e. there exists a map g such that $g p_\delta = \varphi p_t f$. Hence $g p_\delta(x_0) \neq g p_\delta(x_1)$, which shows that $p_\delta(x_0) \neq p_\delta(x_1)$. It follows that $q_\delta f$ factors through p_δ.

Assume now that f is open and denote by f_δ the map which realizes the factorization (where δ is as above). We claim that f_δ is open. Fix a basic open set $u = (p_t^\delta)^{-1}[w]$, where $w \subseteq X_t$ and $t \in \Sigma \cap M$. Since M knows a countable base of X_t, we may assume that $w \in M$. Let $v = f {p_t}^{-1}[w] = f {p_\delta}^{-1}[u]$. Then $v \in M$ is an open F_σ set and therefore by elementarity there is a countable collection of open sets $\mathcal{V} = \{{q_{r_n}}^{-1}[a_n]\colon n \in \omega\}$ whose union is v. By elementarity, we may assume that $\mathcal{V} \in M$. Hence $\{r_n\colon n \in \omega\} \subseteq M$ and consequently $r_n < \delta$ for every $n \in \omega$. Let $b_n = (q_{r_n}^\delta)^{-1}[a_n]$. Then $v = {q_\delta}^{-1}[b]$, where $b = \bigcup_{n\in\omega} b_n$. Finally, we have $b = q_\delta f {p_\delta}^{-1}[u] = f_\delta p_\delta {p_\delta}^{-1}[u] = f_\delta[u]$, therefore $f_\delta[u]$ is open. □

Proposition 2.3 *(Spectral Theorem).* (Cf. [15].) *Let $X = \varprojlim\langle X_s; p_s^{s'}; \Sigma\rangle$, $Y = \varprojlim\langle Y_s; q_s^{s'}; \Sigma\rangle$, where both inverse systems are σ-complete and X_s, Y_s are metric compact for every $s \in \Sigma$. Assume further that $f : X \to Y$ is a retraction. Then there are a σ-closed cofinal set $T \subseteq \Sigma$ and a family of maps $\{f_t\}_{t\in T}$ such that $q_t f = f_t p_t$ and f_t is a retraction for every $t \in T$.*

Proof. Let $i : Y \to X$ be a right inverse of f. By Proposition 2.2 applied to both f and i, there are a σ-closed cofinal set $T \subseteq \Sigma$ and families of maps $\{f_t\}_{t\in T}$, $\{i_t\}_{t\in T}$ such that $q_t f = f_t p_t$ and $p_t i = i_t q_t$ for every $t \in T$. Then, given $t \in T$, we have $f_t i_t q_t = f_t p_t i = q_t f i = q_t$ and therefore $f_t i_t = \mathrm{id}_{Y_t}$, because q_t is a surjection. □

2.3. Σ-products

Given a set κ (not necessarily a cardinal), we denote by $\Sigma(\kappa)$ the Σ-product modelled on κ, which by definition consists of all $x \in [0,1]^\kappa$ such that $\mathrm{suppt}(x) := \{\alpha \in \kappa\colon x(\alpha) \neq 0\}$ is countable. Clearly, $\Sigma(\kappa)$ is $\aleph_0$-monolithic in $[0,1]^\kappa$. It is well known that for every set κ, $\Sigma(\kappa)$ has a countable tightness. Another result, due to Corson [2, Theorem 1], says that every two disjoint relatively closed subsets of $\Sigma(\kappa)$ have disjoint closures in $[0,1]^\kappa$. This implies that $\Sigma(\kappa)$ is a normal space and for every relatively closed set $S \subseteq \Sigma(\kappa)$ the space $\mathrm{cl}_{[0,1]^\kappa} S$ is (homeomorphic to) the Čech–Stone compactification of S. The fact that $[0,1]^\kappa = \beta\Sigma(\kappa)$ was formally proved by Glicksberg [5, Theorem 2]. Below we state and prove a more general property of Σ-products which implies the above result (see Proposition 2.7 below) and which will be used throughout the paper.

Lemma 2.4. *Let $X \subseteq [0,1]^\kappa$ be such that $X \cap \Sigma(\kappa)$ is dense in X and let $f : X \to Y$ be a continuous map. Let $\mathcal{B}$ be an open base for $f[X]$. Assume χ is big enough and $M \preceq H(\chi)$ is such that $f, \kappa \in M$ and $\mathcal{B} \subseteq M$. Let $S = M \cap \kappa$.*

Then

(a) $x \mid S \in \mathrm{cl}_{[0,1]^\kappa} X$ *for every* $x \in X$.
(b) *If* $x \in X$ *and* $x \mid S \in X$ *then* $f(x) = f(x \mid S)$.

Proof. (a) Fix $x \in X$ and fix a basic neighborhood u of $x \mid S$. Then u is defined by a finite function, say ψ, whose values are open rational intervals. Let u' denote the neighborhood defined by the function ψ restricted to S. Then $u' \in M$. Now $M \models u' \cap X \neq \emptyset$ so there exists $y \in M$ with $y \in \Sigma(\kappa) \cap u' \cap X$. Thus $\mathrm{suppt}(y) \subseteq M$ (because it is countable) and therefore $\mathrm{suppt}(y) \subseteq S$. It follows that $y \in u$, so $u \cap X \neq \emptyset$. As u is arbitrary, we get $x \mid S \in \mathrm{cl}\, X$.

(b) Suppose $x, x \mid S \in X$ and $f(x) \neq f(x \mid S)$. Let $v, w \in \mathcal{B}$ be disjoint and such that $f(x) \in v$, $f(x \mid S) \in w$. There exist basic neighborhoods u_1, u_2 of x and $x \mid S$ respectively, such that $f[u_1] \subseteq v$ and $f[u_2] \subseteq w$. Assume each u_i is defined by a finite function ψ_i whose values are open rational intervals. Shrinking both neighborhoods if necessary, we may assume that $\psi = \psi_1 \restriction S = \psi_2 \restriction S$. Let u denote the basic open set defined by ψ. Then $u \in M$ and $M \models f^{-1}[v] \cap u \neq \emptyset$. Thus, there exists $y \in M$ such that $y \in \Sigma(\kappa) \cap X \cap u$ and $f(y) \in v$. But $\mathrm{suppt}(y) \subseteq M$; hence $y(i) = 0$ for $i \notin S$ and therefore $y \in u_2$, because $(x \mid S)(i) = 0 \in \psi_2(i)$ for $i \notin S$. It follows that $f(y) \in w$, a contradiction. □

2.4. Valdivia compacta

A space X is said to be *Valdivia compact* [3] if for some set T there exists an embedding $h : X \to [0,1]^T$ such that

$$h[X] = \mathrm{cl}_{[0,1]^T}\big(\Sigma(T) \cap h[X]\big).$$

Such a map h will be called a *Valdivia embedding*. More generally, we say that $f : X \to [0,1]^T$ is a *Valdivia map* if $\Sigma(T) \cap f[X]$ is dense in $f[X]$ (f does not have to be one-to-one).

Clearly, every Corson compact space is Valdivia compact. In fact, Corson compacta are precisely those Valdivia compact spaces which have a countable tightness (because Σ-products are countably closed). Below we recall some basic properties of Valdivia compacta. For completeness, we give the proofs.

Proposition 2.5. *Assume X is Valdivia compact and $\kappa = \mathrm{w}(X)$. Then there exists a Valdivia embedding of X into $[0,1]^\kappa$.*

Proof. Assume $X \subseteq [0,1]^\lambda$ and the inclusion is a Valdivia embedding. Let $D = \{d_\alpha\colon \alpha < \kappa\} \subseteq \Sigma(\lambda) \cap X$ be dense in X. Let $T = \bigcup_{\alpha<\kappa} \mathrm{suppt}(d_\alpha)$. Then $|T| \leqslant \kappa$ and, identifying $[0,1]^T$ with a subset of $[0,1]^\lambda$, we have $X \subseteq [0,1]^T$. □

Proposition 2.6. [1] *Assume X is Valdivia compact of weight κ. Then $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \kappa\rangle$ is a continuous inverse sequence such that each X_α is Valdivia compact of weight $\leqslant \alpha + \aleph_0$ and each r_α^β is a retraction.*

Proof. Assume $X \subseteq [0,1]^\kappa$ and the inclusion is a Valdivia embedding. Using Lemma 2.4, find a continuous increasing sequence $\langle S_\alpha\colon \alpha < \kappa\rangle$ of subsets of κ such that $|S_\alpha| \leqslant \alpha + \aleph_0$ and $x \mid S_\alpha \in X$ for every $x \in X$, $\alpha < \kappa$. Set $X_\alpha = X \mid S_\alpha$ and $r_\alpha^\beta = R_{S_\alpha} \restriction X_\beta$, where $R_{S_\alpha}(x) = x \mid S_\alpha$. Then $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \kappa\rangle$ is as required and $X = \varprojlim \mathbb{S}$. □

Proposition 2.7. [2,5] *Let $X \subseteq [0,1]^\kappa$ be a Valdivia inclusion and let $\Sigma = \Sigma(\kappa) \cap X$. Then Σ is a normal space and $X = \beta\Sigma$.*

Proof. It suffices to show that every two disjoint relatively closed sets $A, B \subseteq \Sigma$ have disjoint closures in X. Suppose $p \in \mathrm{cl}_X A \cap \mathrm{cl}_X B$ and fix a countable $M \preceq H(\chi)$ such that $A, B, p \in M$. Let $S = \kappa \cap M$. Lemma 2.4(a) applied both to $A \cup \{p\}$ and $B \cup \{p\}$ says that

$$p \mid S \in \mathrm{cl}\big(A \cup \{p\}\big) \cap \mathrm{cl}\big(B \cup \{p\}\big) = (\mathrm{cl}\, A \cap \mathrm{cl}\, B) \cup \{p\},$$

where cl denotes the closure in $[0,1]^\kappa$. Thus $p \mid S \in A \cap B$, because S is countable; a contradiction. □

Note that the set Σ from the above proposition is $\aleph_0$-monolithic. The existence of a dense $\aleph_0$-monolithic subset is a property preserved by continuous images. Thus, there are obvious examples of spaces which are not continuous images of any Valdivia compactum, like e.g. non-metrizable compactifications of the natural numbers.

3. Inverse systems with right inverses I

In this section we study inverse systems whose bonding mappings are retractions.

Let $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ be an inverse system. A collection of embeddings $\{i_s^t\colon s \leqslant t,\ s,t \in \Sigma\}$, where $i_s^t : X_s \to X_t$, satisfying $i_s^s = \mathrm{id}_{X_s}$, $r_s^t i_s^t = \mathrm{id}_{X_s}$ and $i_t^r i_s^t = i_s^r$ whenever $s \leqslant t \leqslant r$, will be called a *right inverse of* $\mathbb{S}$. We shall write briefly $\langle i_s^t; \Sigma\rangle$ instead of $\{i_s^t\colon s \leqslant t,\ s,t \in \Sigma\}$. Clearly, if $\mathbb{S}$ has a right inverse then all bonding maps in $\mathbb{S}$ are retractions. In fact, all projections from the limit are also retractions, by the following lemma.

Lemma 3.1. *Assume $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ is an inverse system with a right inverse $\langle i_s^t; \Sigma\rangle$. There are uniquely determined embeddings $i_s : X_s \to X$ such that $r_s i_s = \mathrm{id}_{X_s}$ and $i_t i_s^t = i_s$ for every $s \leqslant t$ in Σ.*

Proof. Fix $s \in \Sigma$. Take the usual representation of X as a subset of the product $\prod_{t \geqslant s} X_t$ and define

$$i_s(x)(t) = i_s^t(x), \quad x \in X_s,\ s \leqslant t.$$

Given $s < t < u$, we have $r_t^u(i_s(x)(u)) = r_t^u i_t^u i_s^t(x) = i_s(x)(t)$, so i_s is a well defined map $i_s : X_s \to X$. Clearly, i_s is continuous, because $r_t i_s = i_s^t$ is continuous for every $t \geqslant s$. Moreover $r_s i_s = \mathrm{id}_{X_s}$. Finally, if $s < t$ and $u \geqslant t$ then

$$\big(i_t i_s^t(x)\big)(u) = i_t^u\big(i_s^t(x)\big) = i_s^u(x) = i_s(x)(u).$$

Thus $i_t i_s^t = i_s$. □

Given an inverse system $\langle X_s; r_s^t; \Sigma\rangle$ with a right inverse $\langle i_s^t; \Sigma\rangle$, we shall use the induced embeddings i_s, without referring to the above lemma explicitly.

Lemma 3.2. *Let $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \kappa\rangle$ be a continuous inverse sequence such that each $r_\alpha^{\alpha+1}$ is a retraction. Then $\mathbb{S}$ has a right inverse.*

Proof. We construct a right inverse $\langle i_\alpha^\beta; \kappa\rangle$ using induction on $\delta < \kappa$. Assume that i_α^β have already been defined for $\alpha \leqslant \beta < \delta$. If δ is a limit ordinal then we use Lemma 3.1 and the continuity of the sequence. So suppose that $\delta = \varrho + 1$. As $r_\varrho^{\varrho+1}$ is a retraction, choose $i : X_\varrho \to X_{\varrho+1}$ which is a right inverse of $r_\varrho^{\varrho+1}$ and define $i_\alpha^{\varrho+1} = i i_\alpha^\varrho$ for $\alpha < \varrho + 1$. In particular, $i_\varrho^{\varrho+1} = i$. We have

$$r_\alpha^{\varrho+1} i_\alpha^{\varrho+1} = r_\alpha^{\varrho+1} i i_\alpha^\varrho = r_\alpha^\varrho r_\varrho^{\varrho+1} i i_\alpha^\varrho = r_\alpha^\varrho i_\alpha^\varrho = \mathrm{id}_{X_\alpha}$$

and

$$i_\beta^{\varrho+1} i_\alpha^\beta = i i_\beta^\varrho i_\alpha^\beta = i i_\alpha^\varrho = i_\alpha^{\varrho+1},$$

whenever $\alpha < \beta < \varrho + 1$. This completes the proof. □

In the context of Lemmas 3.1 and 3.2, let us mention that Shapiro [14] constructed (assuming $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$) an example of a 0-dimensional kappa-metrizable space of weight $\aleph_2$, which has no nontrivial convergent sequences. Such a space is the limit of a σ-complete inverse system of metric compact spaces whose all bonding mappings are open, therefore retractions. However, the projections from the limit are open mappings which are not retractions.

Lemma 3.3. *Let $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ be an inverse system with a right inverse $\langle i_s^t; \Sigma\rangle$ and let $X = \varprojlim \mathbb{S}$, $R_s = i_s r_s$. Then*

(1) $s \leqslant t \Longrightarrow R_s R_t = R_s = R_t R_s$.
(2) $x = \lim_{s \in \Sigma} R_s(x)$ *for every* $x \in X$.

Proof. Property (1) is clear. Fix a neighborhood u of x. We can assume that $u = (r_s)^{-1}[v]$ for some $s \in \Sigma$ and an open set $v \subseteq X_s$. Fix $t \geqslant s$. Then

$$r_s\big(R_t(x)\big) = r_s\big(i_t r_t(x)\big) = r_s^t r_t i_t r_t(x) = r_s^t r_t(x) = r_s(x) \in v.$$

Thus $R_t(x) \in u$ for every $t \geqslant s$. This shows (2). □

Lemma 3.4. *Assume $\{R_s\colon s \in \Sigma\}$ is a family of internal retractions of a compact space X such that Σ is a directed partially ordered set and conditions* (1), (2) *of Lemma* 3.3 *hold. Then $X = \varprojlim\langle X_s; R_s^t; \Sigma\rangle$, where $X_s = R_s[X]$ and $R_s^t = R_s \restriction X_t$.*

Proof. If $r < s < t$ then

$$R_r^s R_s^t = (R_r \restriction X_s)(R_s \restriction X_t) = (R_r R_s) \restriction X_t = R_r \restriction X_t = R_r^t,$$

thus $\mathbb{S} = \langle X_s; R_s^t; \Sigma\rangle$ is indeed an inverse system and if $s < t$ then $R_s^t[X_t] = X_s$, because $X_s \subseteq X_t$ by condition (1). Let $\widetilde{R}_s : \varprojlim \mathbb{S} \to X_s$ denote the projection. The collection $\{R_s\colon s \in \Sigma\}$ is compatible with $\mathbb{S}$ which determines a continuous map $f : X \to \varprojlim \mathbb{S}$ such that $\widetilde{R}_s f = R_s$ for every $s \in \Sigma$. Condition (2) implies that f is one-to-one. It remains to show that $f[X] = \varprojlim \mathbb{S}$. Fix $y \in \varprojlim \mathbb{S}$ and choose

$$x \in \bigcap_{t \in \Sigma} \operatorname{cl}\big\{\widetilde{R}_s(y)\colon s \geqslant t\big\}.$$

The above set is nonempty by the compactness of X and by the directedness of Σ. Observe that if $t < s$ then $R_t \widetilde{R}_s(y) = R_t^s \widetilde{R}_s(y) = \widetilde{R}_t(y)$, whence

$$R_t(x) \in \operatorname{cl} R_t\big[\big\{\widetilde{R}_s(y)\colon s \geqslant t\big\}\big] = \big\{\widetilde{R}_t(y)\big\}.$$

Thus $\widetilde{R}_t f(x) = R_t(x) = \widetilde{R}_t(y)$, which means that $f(x) = y$. □

4. Simple retractions

We define a notion of a simple retraction and we show that the class of Valdivia compact spaces is stable under limits of inverse sequences with simple retractions. As a consequence, we obtain a characterization of Valdivia compacta of weight $\aleph_1$, which allows to answer the question on retractions and open images in case where the image space has weight $\aleph_1$. Another corollary says that retracts of Cantor/Tikhonov cubes are Valdivia compact.

Lemma 4.1. *Assume $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \kappa\rangle$ is a continuous inverse sequence of compact spaces with a right inverse $\langle i_\alpha^\beta; \kappa\rangle$. Assume further that $\{h_\alpha : X_\alpha \to [0,1]^{S_\alpha}\}_{\alpha<\kappa}$ is a collection of Valdivia embeddings such that for every $\alpha < \beta < \kappa$ the following diagrams commute*:

$$\begin{array}{ccc} X_\beta & \xrightarrow{h_\beta} & [0,1]^{S_\beta} \\ {\scriptstyle r_\alpha^\beta}\downarrow & & \downarrow{\scriptstyle \pi_{S_\alpha}^{S_\beta}} \\ X_\alpha & \xrightarrow{h_\alpha} & [0,1]^{S_\alpha} \end{array} \qquad \begin{array}{ccc} X_\beta & \xrightarrow{h_\beta} & [0,1]^{S_\beta} \\ {\scriptstyle i_\alpha^\beta}\uparrow & & \uparrow{\scriptstyle j_{S_\alpha}^{S_\beta}} \\ X_\alpha & \xrightarrow{h_\alpha} & [0,1]^{S_\alpha} \end{array}$$

where $\pi_{S_\alpha}^{S_\beta}$ is the projection and $j_{S_\alpha}^{S_\beta}$ is the canonical embedding. Then the limit map $h : X \to [0,1]^S$ (where $S = \bigcup_{\alpha<\kappa} S_\alpha$) is a Valdivia embedding.

Proof. The limit of embeddings is an embedding, so it remains to show that $\Sigma(S) \cap h[X]$ is dense in $h[X]$. Fix a standard basic open set $v \subseteq [0,1]^S$ such that $h[X] \cap v \neq \emptyset$. Find $\alpha < \kappa$ such that $h_\alpha[X_\alpha] \cap v \neq \emptyset$ (identifying $[0,1]^{S_\alpha}$ with a subspace of $[0,1]^S$ via $j_{S_\alpha}^S$). Increasing α if necessary, we may assume that v depends on (finitely many) coordinates in S_α, i.e. $v = (\pi_{S_\alpha}^S)^{-1}[\pi_{S_\alpha}^S[v]]$. Since h_α is a Valdivia embedding, there is $y \in \Sigma(S_\alpha) \cap v$ such that $y = h_\alpha(x)$ for some $x \in X_\alpha$. For every $\beta \geqslant \alpha$ we have $j_{S_\alpha}^{S_\beta}(y) \in v$ and $\operatorname{suppt}(j_{S_\alpha}^{S_\beta}(y)) = \operatorname{suppt}(y)$. By the assumption, $j_{S_\alpha}^{S_\beta}(y) = h_\beta i_\alpha^\beta(x)$ and hence $j_{S_\alpha}^S(y) = h i_\alpha(x)$. Thus $h i_\alpha(x) \in v \cap \Sigma(S)$. □

A retraction $r: X \to Y$ is *simple* if

(1) for every nonempty open set $u \subseteq X$ the image $r[u]$ contains a nonempty G_δ set;
(2) there exists a map $g: X \to [0,1]^{\aleph_0}$ such that the diagonal map $r \Delta g: X \to Y \times [0,1]^{\aleph_0}$ is one-to-one (i.e. r has a *metrizable kernel*, see [6]).

Observe that every retraction between metrizable spaces is simple. Also, every open retraction with a metrizable kernel is simple. Fix a space X with a point $p \in X$ which is not G_δ and define $r: X \oplus 1 \to X$ by setting $r(x) = x$ for $x \in X$ and $r(*) = p$, where $*$ is the "new" isolated point in $X \oplus 1$. Then r is a retraction with a metrizable kernel, but it is not simple. Below we prove the announced preservation property for simple retractions.

Theorem 4.2. *Assume $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \kappa\rangle$ is a continuous inverse sequence of compact spaces such that X_0 is Valdivia compact and each $r_\alpha^{\alpha+1}$ is a simple retraction. Then X is Valdivia compact.*

Proof. Let $\langle i_\alpha^\beta; \kappa\rangle$ be a right inverse for $\mathbb{S}$ (Lemma 3.2). We shall construct inductively Valdivia embeddings $h_\alpha: X_\alpha \to [0,1]^{S_\alpha}$ so that the assumptions of Lemma 4.1 are satisfied. We start with any Valdivia embedding $h_0: X_0 \to [0,1]^{S_0}$.

Fix $\beta < \kappa$ and assume that h_α has been defined for each $\alpha < \beta$. If β is a limit ordinal then we can use Lemma 4.1. Assume $\beta = \alpha + 1$. Let $r = r_\alpha^{\alpha+1}$, $i = i_\alpha^{\alpha+1}$. Note that $x \in i[X_\alpha]$ iff $x = ir(x)$.

Fix a countable set T such that $S_\alpha \cap T = \emptyset$ and fix $g: X_{\alpha+1} \to [0,1]^T$ such that $g \Delta r$ is one-to-one. We may assume that $g(x) \neq 0$ for every $x \in X_{\alpha+1}$. Define $\varphi: X_{\alpha+1} \to [0,1]$ by setting

$$\varphi(x) = d\big(g(x), (gir)(x)\big),$$

where d is a fixed metric on $[0,1]^T$ which is bounded by 1. Observe that $\varphi^{-1}(0) = i[X_\alpha]$. Indeed, if $x \notin i[X_\alpha]$ then $i(r(x)) \neq x$ and since $r(i(r(x))) = r(x)$, it must be that $g(i(r(x))) \neq g(x)$ which implies $\varphi(x) > 0$. Let $\mathbb{Q}_0 = (0,1] \cap \mathbb{Q}$. For each $q \in \mathbb{Q}_0$ define $\varphi_q: X_{\alpha+1} \to [0,1]$ by setting $\varphi_q(x) = \max\{0, \varphi(x) - q\}$. Finally, set $S_{\alpha+1} = S_\alpha \cup (T \times \mathbb{Q}_0)$ and define $h_{\alpha+1}: X_{\alpha+1} \to [0,1]^{S_{\alpha+1}}$ by setting

$$h_{\alpha+1} = (h_\alpha r) \Delta f, \quad \text{where } f = \Delta_{q \in \mathbb{Q}_0}(\varphi_q \cdot g).$$

We claim that $h_{\alpha+1}$ is a Valdivia embedding. Fix $x, x' \in X_{\alpha+1}$. If $r(x) \neq r(x')$ then $h_\alpha r(x) \neq h_\alpha r(x')$. If $r(x) = r(x')$ then $g(x) \neq g(x')$ and $\{x, x'\} \nsubseteq i[X_\alpha]$, because otherwise $x = ir(x) = ir(x') = x'$. Assuming $\varphi(x) \leqslant q < \varphi(x')$ for some $q \in \mathbb{Q}_0$, we have $\varphi_q(x) \cdot g(x) = 0$ and $\varphi_q(x') \cdot g(x') \neq 0$. Thus in both cases we have $h_{\alpha+1}(x) \neq h_{\alpha+1}(x')$ which shows that $h_{\alpha+1}$ is an embedding.

Fix an open set v such that $h_{\alpha+1}[X_{\alpha+1}] \cap v \neq \emptyset$. Let $u = {h_{\alpha+1}}^{-1}[v]$. Since $r[u]$ contains a nonempty G_δ set and the set

$$M = \big\{x \in X_\alpha\colon\ h_\alpha(x) \in \Sigma(S_\alpha)\big\}$$

is dense and countably closed, there exists $x \in M \cap r[u]$. Let $x = r(y)$, $y \in u$. Then $h_{\alpha+1}(y) \in v \cap \Sigma(S_{\alpha+1})$, because $S_{\alpha+1} \setminus S_\alpha = T \times \mathbb{Q}_0$ is countable. It follows that $h_{\alpha+1}$ is a Valdivia map. By the definition of $h_{\alpha+1}$ we have

$$\pi_{S_\alpha}^{S_{\alpha+1}} h_{\alpha+1} = h_\alpha r_\alpha^{\alpha+1} \quad \text{and} \quad h_{\alpha+1} i_\alpha^{\alpha+1} = (h_\alpha r i) \Delta (f i) = h_\alpha \Delta 0_{T \times \mathbb{Q}_0} = j_{S_\alpha}^{S_{\alpha+1}} h_\alpha,$$

where $0_{T \times \mathbb{Q}_0}$ denotes the constant 0 function in $[0,1]^{T \times \mathbb{Q}_0}$. Finally, for $\xi < \alpha$ we have

$$\pi_{S_\xi}^{S_{\alpha+1}} h_{\alpha+1} = \pi_{S_\xi}^{S_\alpha} \pi_{S_\alpha}^{S_{\alpha+1}} h_{\alpha+1} = \pi_{S_\xi}^{S_\alpha} h_\alpha r_\alpha^{\alpha+1} = h_\xi r_\xi^\alpha r_\alpha^{\alpha+1} = h_\xi r_\xi^{\alpha+1}$$

and similarly $h_{\alpha+1} i_\xi^{\alpha+1} = j_{S_\xi}^{S_{\alpha+1}} h_\xi$. By Lemma 4.1, this completes the proof. □

Corollary 4.3. *Assume $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \omega_1\rangle$ is a continuous inverse sequence such that each X_α is a compact metric space and each $r_\alpha^{\alpha+1}$ is a retraction. Then X is Valdivia compact.*

Corollary 4.3 implies that a compact space of weight $\leqslant \aleph_1$ is Valdivia compact iff it is representable as the limit of an inverse sequence of metric spaces whose bonding mappings are retractions.

Below we give an answer to the questions on retracts and open images, in case where the image has weight $\aleph_1$.

Theorem 4.4. *Let X be a Valdivia compact and assume $f : X \to Y$ is a retraction or Y is 0-dimensional and f is an open surjection. If $\mathrm{w}(Y) \leqslant \aleph_1$ then Y is Valdivia compact.*

Proof. We assume that $\mathrm{w}(Y) = \aleph_1$. Assuming $X \subseteq [0,1]^\lambda$ and the inclusion is a Valdivia embedding, by Lemma 2.4, we get $S \subseteq \lambda$ such that $|S| = \aleph_1$ and $f(x) = f(x \mid S)$ for every $x \in X$. Thus, taking $X \mid S$ instead of X, we may assume that $\mathrm{w}(X) = \aleph_1$. Let $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \omega_1\rangle$ be a continuous inverse sequence of retractions such that $X = \varprojlim \mathbb{S}$ and each X_α is second countable. Let $\mathbb{S}' = \langle Y_\alpha; p_\alpha^\beta; \omega_1\rangle$ be a continuous inverse sequence of metric compacta such that $Y = \varprojlim \mathbb{S}'$ and Y_α's are 0-dimensional if Y is so. Applying Proposition 2.2 and possibly replacing ω_1 by a closed cofinal set, we can find maps $f_\alpha : X_\alpha \to Y_\alpha$ such that $p_\alpha f = f_\alpha r_\alpha$ for each $\alpha < \omega_1$.

Suppose first that f is a retraction. By the Spectral Theorem 2.3, we may assume that each f_α is a retraction. Hence $p_\alpha f = f_\alpha r_\alpha$ is a retraction, which implies that p_α is a retraction as well. Thus all the bonding maps of the sequence $\mathbb{S}'$ are retractions. By Corollary 4.3, Y is Valdivia compact.

Suppose now that f is an open surjection and Y is 0-dimensional. By Proposition 2.2, we may assume that each f_α is open. Thus, for every $\alpha < \omega_1$ the map f_α is an open surjection from a compact metrizable space onto a 0-dimensional space, therefore it is a retraction. We can repeat the above argument to finish the proof in this case. □

Recall that, according to [13], there exists a Valdivia compact space, namely a product of $\aleph_1$ many metric compact groups, which has an open map (in fact: an epimorphism) onto a compact group which is not Valdivia. Thus the assumption on 0-dimensionality is essential in Theorem 4.4.

By a (*generalized*) *cube* we mean an arbitrary product of metric compacta. It has been proved in [12] that retracts of cubes belong to the class of semi-Eberlein compacta. A compact space K is *semi-Eberlein* [12] if $K \subseteq [0,1]^\kappa$ so that $K \cap c_0(\kappa)$ is dense in K, where $c_0(\kappa)$ denotes the well-known Banach space of all κ-sequences convergent to zero. Clearly, every semi-Eberlein space is Valdivia compact. The mentioned result from [12] is obtained by applying a preservation theorem for semi-Eberlein spaces, similar to Theorem 4.2, with stronger assumptions on the successor bonding maps. Theorem 4.2 gives a weaker statement, namely that

Corollary 4.5. *Retracts of cubes are Valdivia compact.*

Proof. Let X be a retract of a generalized cube. By the result of Shchepin [15] (or rather by the methods from [15]), $X = \varprojlim \mathbb{S}$, where $\mathbb{S} = \langle X_\alpha; p_\alpha^\beta; \kappa\rangle$ is a continuous inverse sequence such that X_0 is a compact metric space, each $p_\alpha^{\alpha+1}$ is an open retraction with a metrizable kernel. For the details we refer to [11]. Thus, each $p_\alpha^{\alpha+1}$ is a simple retraction, so X is Valdivia by Theorem 4.2. □

We finish this section by showing that some obvious modifications of the assumptions in Theorem 4.2 lead to false statements.

Example 4.6. (a) Let $\mathbb{S} = \langle X_\alpha; r_\alpha^\beta; \aleph_2\rangle$ be such that $X_\alpha = \alpha + 1$ and $r_\alpha^\beta : \beta + 1 \to \alpha + 1$ is a retraction which maps every $\xi > \alpha$ to α. Then $\mathbb{S}$ is a continuous inverse sequence, each $r_\alpha^{\alpha+1}$ is a two-to-one retraction with a metrizable kernel and each X_α is Valdivia compact. On the other hand, it is easy to see (e.g. using Lemma 3.4) that $\varprojlim \mathbb{S}$ is homeomorphic to the linearly ordered space $\omega_2 + 1$, which is not a continuous image of any Valdivia compact (see [10]).

(b) Fix a countable dense set $D = \{d_n\colon n \in \omega\}$ in 2^{ω_1} and define

$$X = \big(2^{\omega_1} \times \{0_\omega\}\big) \cup \{e_n\colon n \in \omega\},$$

where 0_ω denotes the constant zero function on ω and $e_n = d_n{}^\frown \chi_{\{n\}}$ ($\chi_{\{n\}} \in 2^\omega$ denotes the characteristic function of $\{n\}$). It is easy to see that $X \subseteq 2^{\omega_1+\omega}$ is homeomorphic to the Alexandrov duplicate of D in 2^{ω_1}, i.e.

$$X \cong \big(2^{\omega_1} \times \{0\}\big) \cup \big(D \times \{1\}\big),$$

where $D \times \{1\}$ consists of isolated points and a basic neighborhood of $\langle p, 0\rangle \in X$ is of the form $(V \times \{0, 1\}) \setminus F$, where V is a neighborhood of p in 2^{ω_1} and $F \subseteq D \times \{1\}$ is finite. Thus, X is a non-metrizable compactification of the natural numbers, therefore it is not a continuous image of any Valdivia compact. On the other hand, $X = \varprojlim\langle X_n; r_n^m; \aleph_0\rangle$, where $X_n = X \mid (\omega_1 + n)$ and $r_n^m(x) = x \mid (\omega_1 + n)$ for $x \in X_m$ $(n < m)$. Each X_n is Valdivia compact (being the union of 2^{ω_1} and a finite set) and each r_n^{n+1} is a retraction with a metrizable kernel (in fact $X_{n+1} \setminus X_n = \{e_n\}$ and $r_n^{n+1}(e_n) = d_n$).

5. Inverse systems with right inverses II

In this section we investigate σ-complete inverse systems whose bonding mappings are retractions. The aim is to obtain the characterization of Valdivia compacta mentioned in the introduction.

An *r-skeleton* in a space X is a pair $\langle \mathbb{S}, \mathbb{I}\rangle$, where $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ is a σ-complete inverse system of second countable spaces such that $X = \varprojlim \mathbb{S}$ and $\mathbb{I} = \langle i_s^t; \Sigma\rangle$ is a right inverse of $\mathbb{S}$. An r-skeleton $\langle \mathbb{S}, \mathbb{I}\rangle$ is *commutative* if

$$(i_s r_s)(i_t r_t) = (i_t r_t)(i_s r_s)$$

holds *for every* $s, t \in \Sigma$. Note that, regardless of commutativity, we always have $(i_s r_s)(i_t r_t) = i_s r_s = (i_t r_t)(i_s r_s)$, whenever $s < t$.

Let $\langle \mathbb{S}, \mathbb{I}\rangle$ be an r-skeleton in a compact space X, where $\mathbb{S}$, $\mathbb{I}$ are as above. Define $R_s = i_s r_s$. By Lemma 3.3, $\{R_s\}_{s\in\Sigma}$ has the following properties:

(1) $s \leqslant t \Longrightarrow R_s R_t = R_s = R_t R_s$.
(2) $x = \lim_{s\in\Sigma} R_s(x)$ for every $x \in X$.
(3) If $T \subseteq \Sigma$ is countable and directed then $t = \sup T$ exists and $R_t(x) = \lim_{s\in T} R_s(x)$ for every $x \in X$.

Conversely, assume that Σ is a directed poset and $\{R_s\colon s \in \Sigma\}$ is a collection of selfmaps of X satisfying conditions (1)–(3) and such that $R_s[X]$ is second countable for each $s \in \Sigma$. Setting $X_s = R_s[X]$ and $r_s^t = R_s \restriction X_t$, we get a σ-complete inverse system $\mathbb{S} = \langle X_s; r_s^t; \Sigma\rangle$ such that $X = \varprojlim \mathbb{S}$ (see Lemma 3.4) and the inclusions provide a right inverse to $\mathbb{S}$. Thus $\{R_s\colon s \in \Sigma\}$ determines an r-skeleton on X. We shall say that $\{R_s\colon s \in \Sigma\}$ is an *internal r-skeleton* in X.

We are going to prove that a compact space is Valdivia if and only if it has a commutative r-skeleton. For this aim we need some auxiliary results concerning r-skeletons.

Lemma 5.1. *Let $\{R_s\colon s \in \Sigma\}$ be an internal r-skeleton in a compact space X, let $\Sigma_0 \subseteq \Sigma$ be σ-closed and let $D = \bigcup_{s\in\Sigma_0} X_s$. Then for every map $f : D \to Y$ into a second countable space Y there exists $t \in \Sigma_0$ such that $f = f R_t$. In particular* $\operatorname{cl}_X D = \beta D$.

Proof. Fix a big enough cardinal χ and fix a countable $M \preceq H(\chi)$ such that $f, \Sigma_0 \in M$ and $\{R_s\colon s \in \Sigma\} \in M$. Let $t = \sup(\Sigma_0 \cap M)$. Then $t \in \Sigma_0$. We claim that $f(x) = f(R_t(x))$ for every $x \in D$. Suppose otherwise and fix in M disjoint basic open sets $v, w \subseteq Y$ such that $f(x) \in v$ and $f(R_t(x)) \in w$ for some $x \in D$. Since $X_t = \varprojlim\langle X_s; R_s^{s'}; \Sigma_0 \cap M\rangle$, where $R_s^{s'} = R_s \restriction X_{s'}$, there are $s \in \Sigma_0 \cap M$ and an open set $u \subseteq X_s$ such that

$$R_t(x) \in {R_s}^{-1}[u] \quad \text{and} \quad X_t \cap {R_s}^{-1}[u] \subseteq f^{-1}[w]. \tag{$*$}$$

In particular $x \in {R_s}^{-1}[u]$, because $s < t$ and $R_s R_t(x) = R_s(x)$. We may assume that $u \in M$, because X_s is second countable. By elementarity, $M \models D \cap {R_s}^{-1}[u] \cap f^{-1}[v] \neq \emptyset$, because $x \in D \cap {R_s}^{-1}[u] \cap f^{-1}[v]$. Fix $y \in M \cap D \cap {R_s}^{-1}[u] \cap f^{-1}[v]$. Then $y \in X_{s'}$ for some $s' \in \Sigma_0 \cap M$ and therefore $y \in X_t \cap {R_s}^{-1}[u]$, which by $(*)$ implies $f(y) \in w$. Hence $v \cap w \neq \emptyset$, a contradiction. □

Lemma 5.2. *Assume that $\{R_s\colon s \in \Sigma\}$ is an internal r-skeleton in a compact space X and let $T \subseteq \Sigma$ be a non-empty σ-closed set. Define $R_T : X \to X$ by setting $R_T(x) = \lim_{t\in T} R_t(x)$. Then R_T is a well defined retraction onto $X_T = \operatorname{cl}_X(\bigcup_{t\in T} X_t)$ and $R_t R_T = R_t$ for every $t \in T$. Moreover $\langle X_T, \{R_t \restriction X_T\}_{t\in T}\rangle = \varprojlim\langle X_t; R_t^{t'}; T\rangle$, where $R_t^{t'} = R_t \restriction X_{t'}$.*

Proof. Fix $x \in X_T$ and its neighborhood v in X_T. Choose $f : X_T \to [0,1]$ such that $f(x) = 1$ and $f^{-1}[(0,1]] \subseteq v$. By Lemma 5.1, $f \restriction D = (f \restriction D)R_t$ for some $t \in T$, where $D = \bigcup_{t \in T} X_t$. Since D is dense in X_T, actually $f = fR_t$. Thus, given $t' \geqslant t$, $t' \in T$, we have

$$fR_{t'}(x) = fR_t R_{t'}(x) = fR_t(x) = f(x).$$

Hence $R_{t'}(x) \in v$ for every $t' \geqslant t$. Since v was arbitrary, this shows that

$(*)$ $x = \lim_{t \in T} R_t(x)$ for every $x \in X_T$.

By Lemma 3.4, $\langle X_T, \{R_t \restriction X_T\}_{t \in T}\rangle = \varprojlim \langle X_t; R_t^{t'}; T\rangle$. Now since for every $t < t'$ in T we have $R_t = R_t^{t'} R_{t'}$, there exists a unique map $g : X \to X_T$ such that $(R_t \restriction X_T)g = R_t$ holds for every $t \in T$. Setting $h = ig$, where i denotes the inclusion of X_T into X, we have $R_t h = R_t$ for every $t \in T$ and consequently, using $(*)$ and the fact that $h[X] \subseteq X_T$, we get

$$h(x) = \lim_{t \in T} R_t\big(h(x)\big) = \lim_{t \in T} R_t(x) = R_T(x)$$

for every $x \in X$. This shows that R_T is a continuous map which is identity on X_T. □

We shall need the following simple property of σ-complete inverse systems.

Lemma 5.3. *Assume that $\mathbb{S} = \langle X_s; p_s^t; \Sigma\rangle$ is a σ-complete inverse system of compact spaces and $T \subseteq \Sigma$ is a directed set such that Σ is the only σ-closed subset of Σ which contains T. Then $\varprojlim \mathbb{S} = \varprojlim(\mathbb{S} \restriction T)$.*

Proof. Fix x_0, x_1 in $X = \varprojlim \mathbb{S}$ and assume $p_t(x_0) = p_t(x_1)$ for every $t \in T$. We have to show that $x_0 = x_1$. Define $\Sigma' = \{s \in \Sigma\colon\ p_s(x_0) = p_s(x_1)\}$. Then $T \subseteq \Sigma'$ and by the σ-continuity of the system, Σ' is σ-closed. Thus $\Sigma' = \Sigma$ and consequently $x_0 = x_1$. □

6. A characterization of the class of Valdivia compacta

In this section we prove the announced categorical characterization of the class of Valdivia compact spaces:

Theorem 6.1. *A compact space is Valdivia compact if and only if it has a commutative r-skeleton.*

Let F be a covariant functor on the category of compact spaces (i.e. $F(K)$ is compact whenever K is compact and $F(f) : F(K) \to F(L)$ whenever $f : K \to L$). We say that F is *continuous* if it preserves inverse limits and we say that F is *weight preserving* if $\mathrm{w}(F(K)) \leqslant \mathrm{w}(K)$ for every compact K. Typical examples of continuous weight preserving functors are probability measures, Vietoris hyperspace, symmetric n-power (i.e. the hyperspace of at most n-element sets) and superextension.

Since the notion of a commutative r-skeleton is defined in the language of category theory, Theorem 6.1 implies the following

Corollary 6.2. *The class of Valdivia compact spaces is stable under continuous weight preserving covariant functors on compact spaces.*

The proof of Theorem 6.1 relies on the following technical lemma.

Lemma 6.3. *Let X be a compact space with a commutative internal r-skeleton $\mathbb{S} = \{R_s\colon\ s \in \Sigma\}$. Assume further that $\mathcal{F}$ is a finite (possibly empty) collection of retractions of X which commute with $\mathbb{S}$, i.e. $fR_s = R_s f$ for every $f \in \mathcal{F}$ and $s \in \Sigma$. Then there exists a collection of sets $\mathcal{U}$ such that, letting $X_s := R_s[X]$, the following holds*:

(1) *Each element of $\mathcal{U}$ is of the form ${R_s}^{-1}[V]$ for some $s \in \Sigma$ and an open set $V \subseteq X_s$.*
(2) *$\mathcal{U}$ is T_0 separating on $X \setminus \bigcup_{f \in \mathcal{F}} f[X]$ and $\bigcup \mathcal{U} = X \setminus \bigcup_{f \in \mathcal{F}} f[X]$.*
(3) *Each X_s intersects only countably many elements of $\mathcal{U}$.*

Proof. Note that, since each X_s is second countable (and therefore separable), condition (3) is equivalent to saying that every point of $\bigcup_{s\in\Sigma} X_s$ is covered by only countably many elements of $\mathcal{U}$.

We use induction on $\kappa = \mathrm{w}(X)$. The statement holds trivially if $\kappa = \aleph_0$. Assume $\kappa > \aleph_0$ and the statement holds for spaces of weight $< \kappa$. Let $Y = \bigcup_{f\in\mathcal{F}} f[X]$.

Recall that $X = \varprojlim\langle X_s; R_s^{s'}; \Sigma\rangle$, where $R_s^{s'} = R_s \restriction X_{s'}$. Let $\mathcal{B} = \{V_\alpha\}_{\alpha<\kappa}$ be a base of X such that $V_\alpha = {R_{s_\alpha}}^{-1}[U_\alpha]$, where $U_\alpha \subseteq X_{s(\alpha)}$ is open ($\alpha < \kappa$). It is straight to construct a continuous chain $\{T_\alpha\}_{\alpha<\kappa}$ of directed subsets of Σ such that $\{s_\xi\colon \xi < \alpha\} \subseteq T_\alpha$ and $|T_\alpha| < \kappa$ for every $\alpha < \kappa$. Let $T = \bigcup_{\alpha<\kappa} T_\alpha$. Then T is a directed subset of Σ. Let $p : X \to X_T$ be the canonical projection, where $X_T = \varprojlim\langle X_s; R_s^{s'}; T\rangle$. Then p is one-to-one, because $\{s_\alpha\colon \alpha < \kappa\} \subseteq T$. Thus $X = X_T$ and, by Lemma 5.3, we may assume that Σ is the smallest σ-closed set containing T. Let $\Sigma_\alpha \subseteq \Sigma$ be the smallest σ-closed set which contains T_α. Then $\{\Sigma_\alpha\}_{\alpha<\kappa}$ is a chain of σ-closed subsets of Σ such that

(i) $X_\alpha := \varprojlim\langle X_s; R_s^{s'}; \Sigma_\alpha\rangle$ has weight $< \kappa$ for each $\alpha < \kappa$.
(ii) If $\delta < \kappa$ is a limit ordinal then Σ_δ is the smallest σ-closed set containing $\bigcup_{\alpha<\delta} \Sigma_\alpha$.
(iii) Σ is the smallest σ-closed set containing $\bigcup_{\alpha<\kappa} \Sigma_\alpha$.

Property (i) follows from Lemma 5.3 and the fact that $|T_\alpha| < \kappa$. Property (ii) follows from the continuity of the chain $\{T_\alpha\}_{\alpha<\kappa}$.

By Lemma 5.2, we may assume that $X_\alpha = \mathrm{cl}(\bigcup_{s\in\Sigma_\alpha} X_s)$ and the projection $R_\alpha : X \to X_\alpha$ satisfies $R_\alpha(x) = \lim_{s\in\Sigma_\alpha} R_s(x)$ for every $x \in X$ (to avoid confusion, we assume that Σ does not contain ordinals). Given $t \in \Sigma$ we have

$$R_t R_\alpha(x) = R_t\big(\lim_{s\in\Sigma_\alpha} R_s(x)\big) = \lim_{s\in\Sigma_\alpha} R_t R_s(x) = \lim_{s\in\Sigma_\alpha} R_s R_t(x) = R_\alpha R_t(x).$$

Thus

(iv) $R_\alpha R_t = R_t R_\alpha$ for every $\alpha < \kappa$, $t \in \Sigma$.

Similarly, $R_\alpha R_\beta = R_\beta R_\alpha = R_\alpha$ whenever $\alpha < \beta < \kappa$. Now (ii) and (iii) together with Lemma 5.3 imply that

(v) $X = \varprojlim\langle X_\alpha; R_\alpha^{\alpha'}; \kappa\rangle$ and $X_\delta = \varprojlim\langle X_\alpha; R_\alpha^{\alpha'}; \delta\rangle$ for a limit ordinal $\delta < \kappa$, where $R_\alpha^{\alpha'} = R_\alpha \restriction X_{\alpha'}$.

We shall construct inductively an increasing sequence $\{\mathcal{U}_\alpha\colon \alpha \leqslant \kappa\}$ of families of subsets of X satisfying the following conditions (where we set $X_\kappa = X$):

(a) Each element of $\mathcal{U}_\alpha$ is of the form ${R_s}^{-1}[V]$ for some $s \in \Sigma_\alpha$ and an open set $V \subseteq X_s$.
(b) $\mathcal{U}_\alpha$ is T_0 separating on $X_\alpha \setminus Y$ and $X_\alpha \cap \bigcup\mathcal{U}_\alpha = X_\alpha \setminus Y$.
(c) For each $s \in \Sigma$, X_s intersects only countably many elements of $\mathcal{U}_\alpha$.
(d) $U \cap X_\alpha = \emptyset$ whenever $U \in \mathcal{U}_\beta \setminus \mathcal{U}_\alpha$.

Then $\mathcal{U} := \mathcal{U}_\kappa$ will be the desired collection. It remains to prove that the construction can be carried out.

Concerning condition (a), we shall use the following observation:

(vi) If $V \subseteq X_s$, where $s \in \Sigma$, then $V \cap Y = \emptyset$ iff ${R_s}^{-1}[V] \cap Y = \emptyset$.

To see (vi), suppose $y \in {R_s}^{-1}[V] \cap f[X]$, where $f \in \mathcal{F}$. Then $R_s(y) \in V$ and $f(R_s(y)) = R_s(f(y)) = R_s(y)$ which implies that $R_s(y) \in Y$, i.e. $R_s(y) \in V \cap f[X]$. Conversely, if $y \in V \cap f[X]$ then $y \in X_s$ and hence $R_s(y) = y$, which gives $y \in {R_s}^{-1}[V] \cap f[X]$. This shows (vi). These arguments use only the fact that each $f \in \mathcal{F}$ commutes with each R_s, $s \in \Sigma$. Thus, by (iv), we also have

(vii) If $V \subseteq X_s$, $s \in \Sigma$ and $\alpha < \kappa$, then $V \cap X_\alpha = \emptyset$ iff ${R_s}^{-1}[V] \cap X_\alpha = \emptyset$.

Fix $\beta \leqslant \kappa$ and suppose that $\mathcal{U}_\alpha$ have already been constructed for $\alpha < \beta$. In order to avoid repeating the same arguments twice, we set $X_{-1} = \emptyset = \mathcal{U}_{-1}$ and we treat 0 as a successor ordinal. We consider two cases.

Case 1. $\beta = \alpha + 1$. Since $\mathrm{w}(X_{\alpha+1}) < \kappa$, we can apply the inductive hypothesis to $X_{\alpha+1}$ and $\{f \restriction X_{\alpha+1}\}_{f \in \mathcal{F}} \cup \{R_\alpha \restriction X_{\alpha+1}\}$, obtaining a family $\mathcal{V}$ satisfying (1)–(3). Thus, each element of $\mathcal{V}$ is of the form $U = (R_s \restriction X_{\alpha+1})^{-1}[W] = {R_s}^{-1}[W] \cap X_{\alpha+1}$, where $s \in \Sigma_{\alpha+1}$, W is open in X_s and disjoint from Y (by (vi)). Moreover, $\mathcal{V}$ is T_0 separating on $X_{\alpha+1} \setminus (X_\alpha \cup Y)$ and $\bigcup \mathcal{V} = X_{\alpha+1} \setminus (Y \cup X_\alpha)$. Given $U = (R_s \restriction X_{\alpha+1})^{-1}[W] \in \mathcal{V}$, define $U^* = {R_s}^{-1}[W]$ and set

$$\mathcal{U}_{\alpha+1} = \mathcal{U}_\alpha \cup \{U^*\colon U \in \mathcal{V}\}.$$

Clearly, (a) is satisfied. Note that $U^* \cap X_{\alpha+1} = U$. Thus $X_{\alpha+1} \cap (\mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha) = X_{\alpha+1} \setminus (X_\alpha \cup Y)$ and $\mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha$ is T_0 separating on $X_{\alpha+1} \setminus (X_\alpha \cup Y)$. By the inductive hypothesis, $\mathcal{U}_\alpha$ is T_0 separating on $X_\alpha \setminus Y$. Hence $\mathcal{U}_{\alpha+1}$ is T_0 separating on $X_{\alpha+1} \setminus Y$, because the family $\mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha$ separates every point of $X_{\alpha+1} \setminus (X_\alpha \cup Y)$ from every point of $X_\alpha \setminus Y$. This shows (b).

In order to show (c), fix $s \in \Sigma$ and fix $x \in X_s$. By Lemma 5.1, applied to the r-skeleton $\{R_s \restriction X_{\alpha+1}\colon s \in \Sigma_{\alpha+1}\}$, we can find $t \in \Sigma_{\alpha+1}$ such that $R_s \restriction X_{\alpha+1} = (R_s \restriction X_t)(R_t \restriction X_{\alpha+1})$. Thus we get

$$R_s R_{\alpha+1}(x) = R_s R_t R_{\alpha+1}(x) = R_{\alpha+1} R_t R_s(x) = R_{\alpha+1} R_t(x) = R_t(x).$$

The last equality follows from the fact that $R_t(x) \in X_t \subseteq X_{\alpha+1}$. On the other hand, we have $R_s R_{\alpha+1}(x) = R_{\alpha+1} R_s(x) = R_{\alpha+1}(x)$. Hence $R_{\alpha+1}(x) = R_t(x) \in X_t$.

Now, if $x \in U^* = {R_r}^{-1}[W]$, where $U = (R_r \restriction X_{\alpha+1})^{-1}[W] \in \mathcal{V}$ and $r \in \Sigma_{\alpha+1}$ then, since $X_r \subseteq X_{\alpha+1}$, we have

$$R_r R_{\alpha+1}(x) = R_{\alpha+1} R_r(x) = R_r(x) \in W,$$

which shows that $R_{\alpha+1}(x) = R_t(x) \in U \cap X_t$. By assumption, the set $\{U \in \mathcal{V}\colon R_t(x) \in U\}$ is countable. Thus x belongs to countably many elements of $\mathcal{U}_{\alpha+1} \setminus \mathcal{U}_\alpha$. This, together with the inductive hypothesis, shows (c).

Condition (d) follows directly from (vii) and from the fact that $X_\alpha \cap \bigcup \mathcal{V} = \emptyset$.

Case 2. β is a limit ordinal. Define $\mathcal{U}_\beta = \bigcup_{\alpha<\beta} \mathcal{U}_\alpha$. It is necessary to check conditions (b) and (c) only.

We first check (b). Fix $x_0 \neq x_1$ in $X_\beta \setminus Y$. By (v), there is $\alpha < \beta$ such that $R_\alpha(x_0) \neq R_\alpha(x_1)$ and by Lemma 3.3 we may assume that $R_\alpha(x_i) \notin Y$ for $i = 0, 1$. Find $U \in \mathcal{U}_\alpha$ such that e.g. $R_\alpha(x_1) \in U$ and $R_\alpha(x_0) \notin U$. Since $U = {R_s}^{-1}[V]$ for some $s \in \Sigma_\alpha$, we have $R_s R_\alpha = R_s$ and therefore $x_1 \in (R_s R_\alpha)^{-1}[V] = {R_s}^{-1}[V] = U$. Similarly $x_0 \notin U$. This shows (b).

Condition (c) holds trivially if $\operatorname{cf} \beta = \aleph_0$, so assume $\operatorname{cf} \beta > \aleph_0$, which implies that $\Sigma_\beta = \bigcup_{\alpha<\beta} \Sigma_\alpha$. Fix $s \in \Sigma$ and fix $x \in X_s$. By Lemma 5.1, there exists $t \in \Sigma_\beta$ such that $R_s \restriction X_\beta = (R_s \restriction X_t)(R_t \restriction X_\beta)$. Hence

$$R_s R_\beta(x) = R_s R_t R_\beta(x) = R_\beta R_t R_s(x) = R_\beta R_t(x) = R_t(x),$$

where the last equality follows from the fact that $X_t \subseteq X_\beta$. On the other hand, $R_s R_\beta(x) = R_\beta R_s(x) = R_\beta(x)$. Thus $R_\beta(x) = R_t(x)$. Let $\alpha < \beta$ be such that $t \in \Sigma_\alpha$. Assume $x \in U = {R_r}^{-1}[V] \in \mathcal{U}_\xi$, where $r \in \Sigma_\xi$ and $\xi < \beta$. Then

$$R_r R_t(x) = R_r R_\beta(x) = R_\beta R_r(x) = R_r(x) \in V,$$

which means that $R_t(x) \in U \cap X_t \subseteq U \cap X_\alpha$. Thus $U \in \mathcal{U}_\alpha$, because part (d) of the inductive hypothesis says that $U \cap X_\alpha = \emptyset$ provided $U \in \mathcal{U}_\xi \setminus \mathcal{U}_\alpha$. It follows that $x \notin U$ whenever $U \in \mathcal{U}_\beta \setminus \mathcal{U}_\alpha$. This, together with the inductive hypothesis, shows (c) and completes the proof. □

Proof of Theorem 6.1. Assume X has an internal commutative r-skeleton $\{R_s\colon s \in \Sigma\}$. Applying Lemma 6.3 with $\mathcal{F} = \emptyset$, we obtain a T_0 separating collection $\mathcal{U}$ which consists of open F_σ sets and each point of $D = \bigcup_{s \in \Sigma} X_s$ belongs to only countably many elements of $\mathcal{U}$. Taking for each $u \in \mathcal{U}$ a function $h_u : X \to [0,1]$ such that $u = {h_u}^{-1}[(0,1]]$, the diagonal map $\Delta_{u \in \mathcal{U}} h_u$ provides a Valdivia embedding of X into $[0,1]^{\mathcal{U}}$.

Assume now that X is Valdivia compact and the inclusion $X \subseteq [0,1]^\kappa$ is a Valdivia embedding. Let $\Sigma = \{S \in [\kappa]^{\aleph_0}\colon X \mid S \subseteq X\}$ and define $R_S(x) = x \mid S$. By Lemma 2.4, $\{R_S\colon S \in \Sigma\}$ is an internal r-skeleton in X. It is clear that $R_S R_T = R_T R_S$ holds for every $S, T \in \Sigma$. □

The following example shows that the assumption on commutativity in Theorem 6.1 cannot be dropped.

Example 6.4. Let κ be an infinite cardinal. Denote by Σ the collection of all closed countable sets $A \subseteq \kappa + 1$ such that $0 \in A$ and every isolated point of A is isolated in $\kappa + 1$. Then $\langle \Sigma, \subseteq \rangle$ is a directed poset. For each $A \in \Sigma$ define $r_A : \kappa + 1 \to \kappa + 1$ by setting $r_A(x) = \max\{a \in A\colon\ a \leqslant x\}$. The assumption on isolated points implies that r_A is continuous, so it is a retraction onto A. Furthermore, if $A, B \in \Sigma$ and $A \subseteq B$ then $r_A r_B = r_A = r_B r_A$. Finally, $x = \lim_{A \in \Sigma} r_A(x)$ and also $r_S(x) = \lim_{A \in \mathcal{A}} r_A(x)$ whenever $S = \sup \mathcal{A}$ and $\mathcal{A} \subseteq \Sigma$ is countable and directed. Thus $\{r_A\colon\ A \in \Sigma\}$ is an internal r-skeleton in $\kappa + 1$. On the other hand, if $\kappa > \aleph_1$ then $\kappa + 1$ is not Valdivia compact.

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