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Fixed-Point Theorems for Discontinuous Multivalued Operators on Ordered Spaces with Applications

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Abstract—In this paper, some fixed-point theorems for discontinuous multivalued operators on ordered spaces are proved. These theorems improve the earlier known fixed-point theorems of [1,2]. The main fixed-point theorems are applied to first-order discontinuous differential inclusions for proving the existence of extremal solutions under certain monotonicity conditions.

Keywords—Fixed-point theorem, Functional differential inclusion.

1. INTRODUCTION

It is known that the algebraic fixed-point theorems are useful in proving the existence theorems for extremal solutions of nonlinear differential and integral equations involving discontinuities. See [3,4] and the references therein. Similarly, the fixed-point theorems for multivalued operators on ordered Banach spaces are useful for proving the existence of extremal solutions of discontinuous differential and integral inclusions. Recently, some fixed-point theorems for discontinuous multivalued mappings are proved in [1,2,5], which are further applied to discontinuous differential inclusions for proving the existence of minimal and maximal solutions under certain monotonicity conditions. Note that all these fixed-point theorems are proved in complete lattices under strictly monotone increasing nature of the multivalued maps. It is worthwhile to mention that all the ordered Banach spaces are not complete lattice, therefore it is desirable to improve the above fixed-point theorems of Dhage [1], Dhage and O'Regan [2] to an arbitrary ordered Banach space under suitable conditions.

In this article, we present some algebraic fixed-point theorems for multivalued operators on ordered spaces and discuss some of their applications to operator inclusions involving two multivalued operators as well as to first-order boundary value problems of discontinuous differential

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inclusions for proving the existence theorems for extremal solutions under generalized monotonicity conditions.

2. AUXILIARY RESULTS

As our approach is more applied than mere theoretical, we rather restrict ourselves to the ordered metric spaces, however the results presented here can be extended to the abstract setting of ordered topological and ordered spaces in a natural way.

In what follows, let X denote an ordered metric space with a metric d and an order relation $\leq$. Then, X becomes a ordered topological space, where the topology on X is induced by the metric d on it. A sequence $\{x_n\}$ of points of X is called *monotone increasing* if

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots.$$

Similarly, a sequence $\{x_n\}$ of points of X is called *monotone decreasing* if

$$x_1 \geq x_2 \geq \cdots \geq x_n \geq \cdots.$$

Finally, a sequence $\{x_n\}$ is called *monotone* if it is either monotone increasing or monotone decreasing on X.

The following crucial result concerning the convergence of a monotone sequence is proved in [6].

LEMMA 2.1. *If a monotone increasing (resp., monotone decreasing) sequence $\{x_n\}$ of points in X has a cluster point, then it is a* $\sup_n x_n$ (resp., $\inf_n x_n$).

Let $\mathcal{P}(X)$ and $\mathcal{P}_p(X)$ denote, respectively, the class of all subsets and the class of all nonempty subsets of X with the property p. Thus, $\mathcal{P}_{\mathrm{cl}}(X)$, $\mathcal{P}_{\mathrm{bd}}(X)$ and $\mathcal{P}_{\mathrm{cp}}(X)$ denote, respectively, the classed of all closed, bounded, and compact subsets of X. A mapping $Q : X \to \mathcal{P}_p(X)$ is called a multivalued mapping or a multivalued operator on X and a point $u \in X$ is called a *fixed point* of Q if $u \in Qu$.

We consider the following notations in the sequel.

Let

$$M = \{x \in X \mid x \leq y \text{ for some } y \in Q(x)\}$$

and

$$L = \{x \in X \mid x \geq y \text{ for some } y \in Q(x)\}.$$

THEOREM 2.1. *Let X be an ordered metric space and let*

$$Q : X \to \mathcal{P}_{\mathrm{cp}}(X).$$

Assume that

- (Q_0) *the set $M \neq \emptyset$,*
- (Q_1) *$x_1 \leq y_1 \in Qx_1$ implies $y_1 \leq y_2$ for some $y_2 \in Qy_1$,* and
- (Q_2) *every monotone increasing sequence $\{y_n\}$ defined by $y_n \in Qx_n$, $n = 0, 2, \dots$; converges. whenever $\{x_n\}$ is a monotone increasing sequence in X.*

Then, Q has a fixed point.

Similarly, we have the following.

THEOREM 2.2. *Let X be an ordered metric space and let $Q : X \to \mathcal{P}_{\mathrm{cp}}(X)$. Assume that*

- (Q_0) *the set $L \neq \emptyset$,*
- (Q_1) *$x_1 \geq y_1 \in Qx_1$ implies $y_1 \geq y_2$ for some $y_2 \in Qy_1$, and*
- (Q_2) *every monotone decreasing sequence $\{y_n\}$ defined by $y_n \in Qx_n$, $n = 0, 2, \dots$; converges, whenever $\{x_n\}$ is a monotone decreasing sequence in X.*

Then, Q has a fixed point.

The proofs of Theorems 2.1 and 2.2 appear in [7] and are based on the well ordered chains of generalized Q-iterations on M and L, respectively. We omit the details.

3. ORDERED BANACH SPACES

Let $\mathbb{R}$ denote the real line and X a real Banach space. A closed subset K of X is called a *cone* if it satisfies

(i) $K + K \subseteq K$,
(ii) $\lambda K \subseteq K$ for all $\lambda \in \mathbb{R}^+$, and
(iii) $\{-K\} \cap K = \theta$, where θ is a zero element of X.

A cone K in X is said to be *normal* if the norm is semimonotone on X, that is, if $x, y \in X$, and $x \leq y$ imply $\|x\| \leq N\|y\|$ for some constant $N > 0$. A cone K is *regular* if every monotone and order bounded sequence in X is convergent in norm. Again a cone K is said to be *fully regular* if every monotone and norm-bounded sequence in X is convergent in norm. The details of cones and their properties may be found in [4,8]. We define an order relation $\leq$ with the help of the cone K in X as follows. Let $x, y \in X$. Then,

$$x \leq y \Leftrightarrow y - x \in K. \tag{3.1}$$

The Banach space X together with a order relation $\leq$ is called an ordered Banach space and it is denoted by $(X, \leq)$. Let $a, b \in (X, \leq)$ be such that $a \leq b$. Then, the order interval $[a, b]$ is a set in X to be defined by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}. \tag{3.2}$$

In the following, we define an order relation in $\mathcal{P}_p(X)$ useful in the sequel. Let $A, B \in \mathcal{P}_{\text{cl}}(X)$. Then, by $A \leq B$, we mean $a \leq b$, for all $a \in A$ and $b \in B$. In particular, $a \leq A$ implies $a \leq b$ for all $b \in B$, and if $A \leq A$, then it follows that A is a singleton set.

The above order relation in $\mathcal{P}_p(X)$ has been used in [2,5,9,10] in the study of extremal solutions for differential and integral equations.

DEFINITION 3.1. *A multivalued mapping $Q : X \to \mathcal{P}_p(X)$ is called strictly monotone increasing if $x < y$, then $Qx \leq Qy$ for all $x, y \in X$. Similarly, a multivalued mapping $Q : X \to \mathcal{P}_p(X)$ is called strictly monotone decreasing if $x < y$, then $Qx \geq Qy$ for all $x, y \in X$.*

THEOREM 3.1. *Let $[a, b]$ be a order interval in a subset Y of an ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{\text{cp}}([a, b])$ be a strictly monotone increasing mapping. If every monotone sequence $\{y_n\} \subset \cup Q([a, b])$ defined by $y_n \in Qx_n$, $n = 0, 2, \ldots$, converges, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then Q has a least fixed point x_* a greatest fixed point x^* in $[a, b]$. Moreover,*

$$x_* = \min\{y \in [a, b] \mid Qy \leq y\}$$

and

$$x_* = \max\{y \in [a, b] \mid y \leq Qy\}.$$

PROOF. We shall first show that the Hypotheses (Q_0) and (Q_1) of Theorems 2.1 and 2.2 are satisfied. Since $Qx \subset [a, b]$ for each $x \in [a, b]$, we have that $a \leq Qa$ and $Qb \leq b$. Hence, $M \neq \emptyset$ and $L \neq \emptyset$. Next, Q is strictly monotone increasing, therefore, the Hypothesis (Q_1) holds. Now, an application of Theorems 2.1 and 2.2 yield that the fixed-point set of Q is nonempty. Define a well-ordered chain $C(a)$ of generalized Q-iteration of a and the inversely well-ordered chain $C(b)$ of generalized Q-iteration of b in $[a, b]$. Then,

$$x_* = \sup C(a)$$

and

$$x^* = \inf C(b)$$

are the fixed points of Q in $[a,b]$ in view of Lemma 2.4 and Lemma 2.5 of [7]. We show shall that x_* and x^* are respectively, the least and the greatest fixed point of Q in $[a,b]$. Let $u \in [a,b]$ be an arbitrary fixed point of Q. Then, we have $a \leq u \leq b$ and $u \in Qu$. Now, consider the order interval $[a,u] \subset [a,b]$. Since Q is strictly monotone increasing on $[a,b]$, we have $C(a) \subset [a,u]$. To see this, let $x \in [a,u)$ be any point, then $a \leq x < u$. By strictly monotonic nature of Q, $a \leq Qx \leq u$. As a result, $C(a) \subset [a,u]$ and that $x_* = \sup C(a) \in [a,u]$. Similarly, consider the order interval $[u,b] \subset [a,b]$. Again, for any $x \in (u,b]$ one has $Qx \subset [u,b]$. Hence, $C(b) \subset [u,b]$ and consequently, $x^* = \inf C(b) \in [u,b]$. Thus, for any fixed point u of Q, we have that $x_* \leq u \leq x^*$. Hence, x_* and x^* are, respectively, the least and the greatest fixed point of Q in $[a,b]$.

Finally, let $y \in [a,b]$ be such that $Qy \leq y$ and consider the order interval $[a,y] \subset [a,b]$. Let $C(a)$ be a well-ordered chain of generalized Q-iteration of a in $[a,b]$. Then, $C(a) \subset [a,y]$. Therefore, $a \leq x_* \leq y$ for each $y \in [a,b]$ for which $y \leq Qy$. Hence,

$$x_* = \min\{y \in [a,b] \mid Qy \leq y\}.$$

Similarly, let $y \in [a,b]$ be such that $y \leq Qy$ and consider the order interval $[y,b] \subset [a,b]$. Let $C(b)$ be an inversely well-ordered chain of generalized Q-iteration of b. Then, $C(b) \subset [y,b]$. Therefore. $y \leq x^* \leq b$ for each $y \in [a,b]$ for which $y \leq Qy$. Hence,

$$x^* = \min\{y \in [a,b] \mid y \leq Qy\}.$$

This completes the proof.

Let X be an ordered metric space. A multimap $Q : X \to \mathcal{P}_p(X)$ is called *totally compact* if $\overline{\cup Q(X)}$ is a compact subset of X. Q is called *compact* if $\cup Q(S)$ is a relatively compact subset of X for all bounded subsets S of X. Again, Q is called *totally bounded* if for any bounded subset S of X, $\cup Q(S)$ is a totally bounded subset of X. It is clear that every compact multivalued map is totally bounded, but the converse may not be true. However, these two notions are equivalent on bounded subsets of a complete metric space X.

COROLLARY 3.2. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a strictly monotone increasing multivalued mapping. Then, Q has a least and a greatest fixed point in $[a,b]$ if any one of the following conditions is satisfied.*

(a) *Q is compact multimap.*
(b) *The cone K in X is normal and Q is compact.*
(c) *The cone K is regular.*

PROOF. Let $\{x_n\}$ be a monotone sequence in $[a,b]$ and let $\{y_n\}$ be a monotone sequence in $\cup Q([a,b])$ defined by $y_n \in Qx_n$ for each $n \in \mathbb{N}$. Clearly, such a sequence $\{y_n\}$ exists since the multimap Q is monotone increasing on $[a,b]$. Suppose that the hypothesis (a) holds. Then, $\overline{\cup Q([a,b])}$ is compact and the sequence $\{y_n\}$ has a convergent subsequence. Since $\{y_n\}$ is strictly monotone increasing, it converges to a point in $[a,b]$. Again, if the hypothesis (b) holds, then the order interval $[a,b]$ is bounded in norm and $\cup Q([a,b])$ is relatively compact set in X. Therefore, the sequence $\{y_n\} \subset \cup Q([a,b])$ has a convergent subsequence and so the whole sequence converges to a point in $\cup Q([a,b])$. Finally, if the hypothesis (c) holds, then by definition of the cone, the sequence $\{y_n\}$ converges to a point in $\cup Q([a,b])$. Thus, all the conditions of Theorem 3.1 are satisfied under every hypothesis (a) or (b) or (c). Hence, an application of it yields that Q has a least and a greatest fixed point in $[a,b]$. This completes the proof.

A special case of Theorem 3.1 under slightly stronger condition in its applicable form to differential and integral inclusions is as follows.

THEOREM 3.3. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach space X and let $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ be a strictly monotone increasing mapping. If every sequence*

$\{y_n\} \subset \cup Q([a,b])$ defined by $y_n \in Qx_n$, $n = 0, 2, \dots$; has a cluster point, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$, then Q has a least fixed point x_* a greatest fixed point x^* in $[a,b]$. Moreover,

$$x_* = \min\{y \in [a,b] \mid Qy \leq y\}$$

and

$$x_* = \max\{y \in [a,b] \mid y \leq Qy\}.$$

4. MULTIVALUED HYBRID FIXED-POINT THEORY

In this section, we shall prove some fixed-point theorems for the operator inclusions involving the sum and the product of two multivalued operators in a Banach space under the mixed compactness and monotonicity conditions. It seems that some of the results of this section are also new even to the single-valued analysis of mappings in abstract spaces. The results of this section also have a wide range of applications to perturbed differential equations and inclusions for proving the existence of extremal solutions.

4.1. Hybrid Fixed-Point Theory in Banach Spaces

In this section, we combine a topological fixed-point theorem with a algebraic fixed-point theorem to derive a hybrid fixed-point theorem called the topo-algebraic hybrid fixed-point theorem for multivalued operators in Banach spaces. Before going to the main results, we give some preliminaries needed in the sequel.

Let X be a ordered metric space and let $T : X \to \mathcal{P}_p(X)$. Then, T is called *upper semicontinuous* (u.s.c.) if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty and closed subset of X, and for each open set $N \subset X$ containing $T(x_0)$, there exists an open neighborhood M of x_0 such that $\cup T(M) \subset N$. If T is nonempty and compact-valued, then T is u.s.c. if and only if G has closed graph, i.e., given two sequences $\{x_n\}$ and $\{y_n\}$, such that $y_n \in G(x_n)$ for every $n = 1, 2, \dots$: converging to the points x_0 and y_0 respectively, then $y_0 \in G(x_0)$. Finally, a multivalued mapping T on X into itself is called *completely continuous* if it is upper semicontinuous and compact on Y. Note that the complete continuity of Q in a complete metric space X is equivalent to continuity together with the totally boundedness of Q on X.

THEOREM 4.1. *Let $[a,b]$ be a norm-bounded order interval in a subset Y of an ordered Banach space X and let $T : [a,b] \times [a,b] \to \mathcal{P}_{cp}([a,b])$ be a mapping satisfying,*

(a) *$y \mapsto T(x,y)$ is completely continuous and strictly monotone increasing for each $x \in [a,b]$,*
(b) *$x \mapsto T(x,y)$ is strictly monotone increasing for each $y \in [a,b]$ and*
(c) *every monotone sequence $\{y_n\} \subset \cup T([a,b] \times [a,b])$ defined by $y_n \in T(x_n, y)$, $n \in \mathbb{N}$ converges for each $y \in [a,b]$, whenever $\{x_n\}$ is a monotone sequence in $[a,b]$.*

Then, the operator inclusion $x \in T(x,x)$ has a least and a greatest solution in $[a,b]$.

PROOF. Define the multivalued operator $Q : [a,b] \to \mathcal{P}_p([a,b])$ by

$$Qx = \{y \in T(x,y) \mid y \text{ is greatest}\}. \tag{4.1}$$

Let $x \in [a,b]$ be fixed and define the operator $T_x(y) : [a,b] \to \mathcal{P}_{cp}([a,b])$ by $T_x(y) = T(x,y)$. Then, T_x is a completely continuous multivalued operator which maps a closed convex and bounded subset $[a,b]$ of the Banach space X into itself. Therefore, an application of a fixed-point theorem of [9] yields that T_x has a least and a greatest fixed point in $[a,b]$ and consequently, the set Qx is nonempty for each $x \in [a,b]$. Moreover, Qx is compact for each $x \in [a,b]$. Furthermore, hypothesis (c) implies that Q satisfies all the conditions of Theorem 3.3 on $[a, b]$ and hence, an application this theorem yields that Q has a least and a greatest fixed point. This further implies that the operator inclusion $x \in T(x, x)$ has a greatest solution in $[a, b]$. Similarly, by taking

$$Qx = \{y \in T(x, y) \mid y \text{ is least}\}, \tag{4.2}$$

it is proved that the operator inclusion $x \in T(x, x)$ has a least solution in $[a, b]$. This completes the proof.

As a consequence of Theorem 4.1, we obtain the following.

COROLLARY 4.2. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a mapping satisfying*

(a) *$y \mapsto T(x, y)$ is completely continuous and strictly monotone increasing for each $x \in [a, b]$ and*
(b) *$x \mapsto T(x, y)$ is strictly monotone increasing for each $y \in [a, b]$.*

Then, the operator inclusion $x \in T(x, x)$ has a least and a greatest solution if any one of the following conditions is satisfied.

(i) *$[a, b]$ is norm-bounded and T is compact multimap.*
(ii) *The cone K in X is normal and $x \mapsto T(x, y)$ is compact for each $y \in [a, b]$.*
(iii) *The cone K is regular.*

The origin of the fixed-point theorems involving the sum of two operators in a Banach spaces lies in the works of Russian mathematician Krasnoselskii [11]. In this case, one operator happens to be a contraction and another happens to be completely continuous on the domain of their definition. Since every contraction is continuous, both operators in such theorems are continuous. Below we relax the continuity of one of the mappings in such hybrid fixed-point theorems, instead we assume the monotonicity and prove a fixed-point theorem in ordered Banach spaces.

THEOREM 4.3. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(X)$ be two multivalued operators satisfying*

(a) *A is compact and strictly monotone increasing,*
(b) *B is completely continuous and strictly monotone increasing, and*
(c) *$Ax + By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx$ has a least and a greatest solution in $[a, b]$.

PROOF. Define an operator T on $[a, b] \times [a, b]$ by $T(x, y) = Ax + By$. From hypothesis (c), it follows that T defines a multivalued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. Now, the desired conclusion follows by an application of Corollary 4.2.

REMARK 4.1. Note that Hypothesis (c) holds if there exist elements a and b in X such that $a \le Aa + Ba$ and $Ab + Bb \le b$.

When A and B are single-valued operators, Theorem 4.3 reduces to the following.

COROLLARY 4.4. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X. Let $A, B : [a, b] \to X$ be two single-valued operators satisfying*

(a) *A is compact and monotone increasing,*
(b) *B is completely continuous and monotone increasing,*
(c) *$Ax + By \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $Ax + Bx = x$ has a least and a greatest solution in $[a, b]$.

4.2. Hybrid Fixed-Point Theory in Banach Algebras

The hybrid fixed-point theory involving the product of two operators in a Banach algebra was initiated by the present author in [12] and developed further in different directions in the due course of time. See [13,14] and the references therein. The main feature of these fixed-point theorems is again that both the operators are continuous on their domain of definition. Below, we remove the continuity of one of the operators and prove a fixed-point theorem involving the product of two operators in a Banach algebra. We need the following preliminaries in the sequel.

A cone K in a Banach algebra X is called positive if

(iv) $K \circ K \subseteq K$, where " $\circ$ " is a multiplicative composition in X.

LEMMA 4.1. *(See [14].) If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \leq v_1$ and $u_2 \leq v_2$, then $u_1 u_2 \leq v_1 v_2$.*

THEOREM 4.5. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ be two multivalued operators satisfying*

(a) *A is compact and strictly monotone increasing, and*
(b) *B is completely continuous and strictly monotone increasing, and*
(c) *$Ax.By \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax.Bx$ has a least and a greatest solution in $[a, b]$.

PROOF. Define an operator T on $[a, b] \times [a, b]$ by $T(x, y) = Ax.By$. From Hypothesis (c) it follows that T defines a multivalued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$. Now, the desired conclusion follows by an application of Corollary 4.2.

REMARK 4.2. Note that Hypothesis (c) holds if

(i) the cone K in X is positive and
(ii) there exist elements a and b in $[a, b]$ such that $a \leq Aa.Ba$ and $Ab.Bb \leq b$.

When A and B are single-valued operators, Theorem 4.5 reduces to the following.

COROLLARY 4.6. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X. Let $A, B : [a, b] \to K$ be two single-valued operators satisfying*

(a) *A is compact and monotone increasing,*
(b) *B is completely continuous and monotone increasing, and*
(c) *$Ax.By \in [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $Ax.Bx = x$ has a least and a greatest solution in $[a, b]$.

THEOREM 4.7. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ and $C : [a, b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three multivalued operators satisfying*

(a) *A and C are compact and strictly monotone increasing, and*
(b) *B is completely continuous and strictly monotone increasing, and*
(c) *$Ax.By + Cx \subset [a, b]$ for all $x, y \in [a, b]$.*

Further, if the cone K in X is normal, then the operator inclusion $x \in Ax.Bx + Cx$ has a least and a greatest solution in $[a, b]$.

PROOF. Define an operator T on $[a, b] \times [a, b]$ by $T(x, y) = Ax.By + Cx$. From hypothesis (c), it follows that T defines a multivalued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$. Now, the desired conclusion follows by an application of Corollary 4.2.

REMARK 4.3. Note that Hypothesis (c) holds if

(i) the cone K in X is positive and
(ii) there exist elements a and b in $[a, b]$ such that $a \leq Aa.Ba + Ca$ and $Ab.Bb + Cb \leq b$.

When A, B, and C are single-valued operators, Theorem 4.5 reduces to the following.

COROLLARY 4.8. *Let $[a,b]$ be an order interval in a subset Y of an ordered Banach algebra X with a cone K. Let $A, B : [a,b] \to K$ and $C : [a,b] \to X$ be three single-valued operators satisfying*

(a) *A and C are compact and monotone increasing,*
(b) *B is completely continuous and monotone increasing, and*
(c) *$Ax.By + Cx \in [a,b]$ for all $x, y \in [a,b]$.*

Further, if the cone K in X is normal, then the operator inclusion $Ax.Bx + Cx = x$ has a least and a greatest solution in $[a,b]$.

5. DISCONTINUOUS DIFFERENTIAL INCLUSIONS

The method of upper and lower solutions has been successfully applied to the problems of nonlinear differential equations and inclusions. For the first direction, we refer to [4] and for the second direction, we refer to [15]. In this section, we apply the results of previous sections to first-order periodic boundary value problems of ordinary discontinuous differential inclusions for proving the existence of the extremal solutions between the given upper and lower solutions under monotonicity conditions.

5.1. Periodic Boundary Value Problems

Given a closed and bounded interval $J = [0,T]$ in $\mathcal{R}$, consider a periodic boundary value problem of first-order discontinuous differential inclusion (in short DI),

$$\begin{aligned} x'(t) &\in F(t, x(t)), \qquad \text{a.e. } t \in J, \\ x(0) &= x(T), \end{aligned} \tag{5.1}$$

where $F : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$.

By a solution of DI (5.1), we mean a function $x \in AC(J, \mathbb{R})$ such that

$$x'(t) = v(t), \qquad t \in J, \quad x(0) = x(T),$$

for some $v \in L^1(J, \mathbb{R})$ satisfying $v(t) \in F(t, x(t))$, a.e., for $t \in J$, where $AC(J, \mathbb{R})$ is a space of all absolutely continuous real-valued functions on J.

The DI (5.1) and its generalizations have been discussed in the literature very extensively for different aspects of the solution under different continuity conditions. See [16,17] and the references therein. Notice that the DI (5.1) and its generalizations with discontinuous F can be discussed via lattice theoretic approach as given in [1,2,5,12] for the existence of extremal solutions. In this section, we shall prove the existence theorems for extremal solutions to DI (5.1) via functional theoretic approach embodied in Theorem 3.1 under suitable conditions.

Define a norm $\|\cdot\|$ and an order relation "$\leq$" in $AC(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{5.2}$$

and

$$x \leq y \iff x(t) \leq y(t), \qquad \text{for all } t \in J. \tag{5.3}$$

Here, the cone K in $AC(J, \mathbb{R})$ is defined by

$$K = \{x \in AC(J, \mathbb{R}) \mid x(t) \geq 0\},$$

which is obviously normal. See [3,4,8].

We need the following definition in the sequel.

DEFINITION 5.1. *A function $a \in AC(J,\mathbb{R})$ is called a lower solution of the DI (5.1) if $a'(t) \leq v(t)$. a.e., $t \in J$, for all $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,a(t))$ a.e., $t \in J$ and $a(0) \leq a(T)$. Similarly, a function $b \in AC(J,\mathbb{R})$ is called an upper solution of the DI (5.1) if $b'(t) \geq v(t)$, a.e., $t \in J$, for all $v \in L^1(J,\mathbb{R})$ with $v(t) \in F(t,b(t))$ a.e., $t \in J$ and $b(0) \geq b(T)$.*

We use the following notations in the sequel.

Denote

$$|F(t,x)| = \{|u| \mid u \in F(t,x)\}$$

and

$$\|F(t,x)\| = \sup\{|u| \mid u \in F(t,x)\}.$$

Let $\beta : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ be a multivalued function. Then, the set of all Lebesgue integrable selectors S^1_β of β is defined by

$$S^1_\beta(x) = \{v \in L^1(J,\mathbb{R}) \mid v(t) \in \beta(t,x(t)) \text{ a.e., } t \in J\},$$

for $x \in AC(J,\mathbb{R})$. The problem that $S^1_\beta(x) \neq \emptyset$ has been of great interest since long time. Some nice results concerning $S^1_\beta(x) \neq \emptyset$ have been given in [18]. See also [19–21] and the references therein.

We consider the following set of assumptions.

(A_1) There exists a Lebesgue integrable function $m \in L^1(J,\mathbb{R})$ such that

$$|F(t,x)| \leq m(t), \qquad \text{a.e., } t \in J,$$

for all $x \in \mathbb{R}$.

(A_2) $F(t,x)$ is is closed and bounded subset of $\mathbb{R}$ for each $(t,x) \in J \times \mathbb{R}$.

(A_3) $S^1_F(x) \neq \emptyset$ for each $x \in AC(J,\mathbb{R})$.

(A_4) There exists a Lebesgue integrable function $k \in L^1(J,\mathbb{R}^+)$ such that the multifunction $x \mapsto F(t,x) + k(t)x$ is strictly monotone increasing for a.e., $t \in J$.

(A_5) There exist a lower solution a and an upper solution b of the DI (5.1) on J such that $a \leq b$.

Now consider the DI,

$$\begin{aligned} x' + k(t)x(t) &\in F_k(t,x(t)), \qquad \text{a.e., } t \in J, \\ x(0) &= x(T), \end{aligned} \tag{5.4}$$

where $F_k : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ by

$$F_k(t,x) = F(t,x) + k(t)x. \tag{5.5}$$

REMARK 5.1. Note that the DI (5.1) is equivalent to the DI (5.4) and the lower solution a of the DI (5.1) is the lower solution for the DI (5.4) and the upper solution b of the DI (5.1) is the upper solution for the DI (5.4) on J and conversely.

REMARK 5.2. Assume that Hypotheses (A_1)–(A_3) hold. Then, $F_k(t,x)$ is compact for each $(t,x) \in J \times \mathbb{R}$ and $S^1_{F_k}(x) \neq \emptyset$, for each $x \in AC(J,\mathbb{R})$. Again,

$$\begin{aligned} |F_k(t,x)| &= |F(t,x) + k(t)x(t)| \\ &\leq m(t) + k(t)[\|a\| + \|b\|] \\ &= \gamma(t), \end{aligned}$$

for all $t \in J$ and $x \in [a,b]$. Note that $\gamma(\cdot) = m(\cdot) + k(\cdot)[\|a\| + \|b\|] \in L^1(J,\mathbb{R})$.

We need the following lemma in the sequel.

LEMMA 5.1. *For any $k, \sigma \in L^1(J, \mathbb{R})$, x is a solution to the differential equation,*

$$\begin{aligned} x' + k(t)x(t) &= \sigma(t), \qquad \text{a.e., } t \in J, \\ x(0) &= x(T) \in \mathbb{R}, \end{aligned} \tag{5.6}$$

if and only if it is a solution of the integral equation,

$$x(t) = \int_0^T g_k(t,s)\sigma(s)\,ds, \tag{5.7}$$

where

$$g_k(t,s) = \begin{cases} \dfrac{e^{K(s)-K(t)}}{1-e^{-k(T)}}, & 0 \le s \le t \le T, \\ \dfrac{e^{K(s)-K(t)-K(T)}}{1-e^{-K(T)}}, & 0 \le t < s \le T, \end{cases} \tag{5.8}$$

where $K(t) = \int_0^t k(s)\,ds$.

Notice that the Green's function g_k is nonnegative on $J \times J$ and the number,

$$M_k := \max\{|g_k(t,s)| : t, s \in [0,T]\},$$

exists.

THEOREM 5.1. *Assume that Hypotheses (A$_1$)–(A$_5$) hold. Then, the DI (5.1) has a minimal and a maximal solution on J.*

PROOF. Let $X = AC(J, \mathbb{R})$ and define an order interval $[a,b]$ in X, which does exist in view of Hypothesis (A$_5$). Define a multimap Q on $[a,b]$ by

$$\begin{aligned} Qx &= \left\{ u \in X \mid u(t) = \int_0^T g_k(t,s)\,v(s)\,ds,\ v \in S^1_{F_k}(x) \right\} \\ &= \mathcal{K} \circ S^1_{F_k}(x), \end{aligned} \tag{5.9}$$

where the continuous operator $\mathcal{K} : L^1(J, \mathbb{R}) \to C(J, \mathbb{R})$ is defined by

$$\mathcal{K}v(t) = \int_0^T g_k(t,s)v(s)\,ds. \tag{5.10}$$

Obviously the multivalued operator Q is well defined since $S^1_{F_k}(x) \neq \emptyset$ for all $x \in X$ in view of Remark 5.1. We shall show that the multivalued operator Q satisfies all the conditions of Theorem 3.1.

STEP I. First, we show that Q is strictly monotone increasing on $[a,b]$. Let $x, y \in [a,b]$ be such that $x \le y$, $x \neq y$, and let $u_1 \in Qx$ be arbitrary. Then, there exists an element $v_1 \in S^1_{F_k}(x)$, that is, $v_1(t) \in F_k(t, x(t))$ a.e., $t \in J$ such that

$$u_1(t) = \int_0^T g_k(t,s)v_1(s)\,ds. \tag{5.11}$$

Since (A$_4$) holds, for every element $v_1 \in S^1_{F_k}(x)$ we have that $v_1 \le v_2$ on J for all $v_2 \in S^1_{F_k}(y)$. Now, for each element $u_2 \in Qy$ there is a $v_2 \in S^1_{F_k}(y)$ such that

$$u_2(t) = \int_0^T g_k(t,s)v_2(s)\,ds. \tag{5.12}$$

Now, for any $t \in J$, we have

$$\int_0^T g_k(t,s)v_1(s)\,ds \le \int_0^T g_k(t,s)v_2(s)\,ds.$$

As a result we have from (5.7),(5.8),

$$\begin{aligned} u_1(t) &= \int_0^T g_k(t,s)v_1(s)\,ds \\ &\le \int_0^T g_k(t,s)v_2(s)\,ds \\ &= u_2(t), \end{aligned}$$

for all $t \in J$. Hence, $u_1 \le u_2$. Therefore, $Qx \le Qy$, that is, Q is strictly monotone increasing on X and in particular on $[a,b]$.

STEP II. Next, we claim that Q has compact-values and maps $[a,b]$ into itself. First, we show that Qx is a compact subset of X for each $x \in [a,b]$. To show Q has compact values, it then suffices to prove that the composition operator $\mathcal{K} \circ S^1_{F_k}$ has compact values on X. Let $x \in [a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S^1_{F_k}(x)$. Then, by the definition of $S^1_{F_k}$, $v_n(t) \in F_k(t,x(t))$ a.e., for $t \in J$. Since $F_k(t,x(t))$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F_k(t,x(t))$ a.e., for $t \in J$. From the continuity of $\mathcal{K}$, it follows that $\mathcal{K}v_n(t) \to \mathcal{K}v(t)$ pointwise on J as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{K}v_n\}$ is an equicontinuous sequence. Let t, $\tau \in J$; then

$$\begin{aligned} |\mathcal{K}v_n(t) - \mathcal{K}v_n(\tau)| &\le \left| \int_0^T g_k(t,s)v_n(s)\,ds - \int_0^T g_k(\tau,s)v_n(s)\,ds \right| && (5.13) \\ &\le \int_0^T |g_k(t,s) - g_k(\tau,s)||v_n(s)|\,ds. && (5.14) \end{aligned}$$

The function g_k is continuous on the compact set $J \times J$, so it is uniformly continuous there. In addition, $v_n \in L^1(J,\mathbb{R})$, so the right-hand side of (5.13) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{K}v_n\}$ is equicontinuous, and an application of the Arzelá-Ascoli theorem implies that there is a uniformly convergent subsequence. We then have $\mathcal{K}v_{n_j} \to \mathcal{K}v \in (\mathcal{K} \circ S^1_{F_k})(x)$ as $j \to \infty$, and so $(\mathcal{K} \circ S^1_{F_k})(x)$ is compact. Hence, Qx is a compact subset of X for each $x \in [a,b]$.

Again, let $u \in Qb$ be arbitrary. Then there is a $v \in S^1_{F_k}(b)$ such that

$$u(t) = \int_0^T g_k(t,s)v(s)\,ds, \qquad t \in J.$$

Since b is an upper solution of DI (5.1), we have

$$\begin{aligned} u(t) &= \int_0^T g_k(t,s)v(s)\,ds \le \int_0^T g_k(t,s)\left[b'(s) + k(s)b(s)\right] ds \\ &\le \int_0^T g_k(t,s)b'(s)\,ds + \int_0^T k(s)g_k(t,s)b(s)\,ds = b(t), \end{aligned}$$

for all $t \in J$. Hence, $u \le b$ and consequently, $Qb \le b$. Similarly, it is proved that $a \le Qa$. Since Q is strictly monotone increasing, we have for any x, $a \le x \le b$,

$$a \le Qa \le Qx \le Qb \le b.$$

Hence, Q defines a multimap $Q : [a,b] \to \mathcal{P}_{cp}([a,b])$ and the claim follows.

STEP III. Let $\{x_n\}$ be a monotone increasing sequence in $[a,b]$ and let $\{y_n\}$ be a sequence in $\cup Q([a,b])$ defined by $y_n \in Qx_n$, $n \in \mathbb{N}$. We shall show that $\{y_n\}$ is a uniformly bounded and equi-continuous set in $[a,b]$. Since $y_n \in Qx_n$, there exists a $v_n \in S^1_{F_k}(x_n)$ such that

$$y_n(t) = \int_0^T g_k(t,s)v_n(s)\,ds,$$

for all $t \in J$. Therefore, by Remark 5.1,

$$\begin{aligned} |y_n(t)| &= \left|\int_0^T g_k(t,s)v_n(s)\,ds\right| \le \int_0^T g_k(t,s)\,|v_n(s)|\,ds \\ &\le \int_0^T g_k(t,s)\,\gamma(s)\,ds \le M_k\|\gamma\|_{L^1}, \end{aligned}$$

for all $t \in J$ and so $\{y_n\}$ is uniformly bounded.

Next, we prove the equicontinuity of the sequence $\{y_n\}$ on J. To finish, it is enough to show that y_n' is bounded on $[0,T]$. Now, for any $t \in [0,T]$,

$$|y_n'(t)| \le \left|\int_0^T \frac{\partial}{\partial t} g_k(t,s)v(s)\,ds\right| = \left|\int_0^T (-k(s))g_k(t,s)v(s)\,ds\right| \le K\,M_k\|\gamma\|_{L^1} = c,$$

where $K = \max_{t\in J} k(t)$. Hence, for any $t,\tau \in [0,T]$ one has

$$|y_n(t) - y_n(\tau)| \le c|t-\tau| \to 0, \qquad \text{as } t \to \tau.$$

This shows that $\{y_n\}$ is a equicontinuous sequence of functions in $[a,b]$. Now, $\{y_n\}$ is a uniformly bounded and equicontinuous, so it has a convergent subsequence by Arzelà-Ascoli theorem. Hence, $\{y_n\}$ has a cluster point in $[a,b]$. Now, we apply Theorem 3.1 to yield that the operator inclusion $x \in Qx$ has a least and a greatest solution which correspond, respectively, to the minimal and maximal solutions of the DI (5.1) on J. This completes the proof.

5.2. Perturbed Periodic Boundary Value Problem

Given a closed and bounded interval $J = [0,T]$ in $\mathcal{R}$, consider the initial value problem of first-order perturbed differential inclusion (in short, PDI),

$$\begin{aligned} &x'(t) \in F(t,x(t)) + G(t,x(t)), \qquad \text{a.e., } t \in J, \\ &x(0) = x(T), \end{aligned} \tag{5.15}$$

where $F, G : J \times \mathbb{R} \to \mathcal{P}_{cp}(\mathbb{R})$.

By a solution of PDI (5.15), we mean a function $x \in AC(J,\mathbb{R})$ whose first derivative x' exists and is a member of $L^1(J,\mathbb{R})$ in $F(t,x)$, i.e., there exists a $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,x(t)) + G(t,x(t))$ a.e., $t \in J$, and $x'(t) = v(t)$, $t \in J$ and $x(0) = x(T) \in \mathbb{R}$, where $AC(J,\mathbb{R})$ is the space of all absolutely continuous real-valued functions on J.

The special cases of perturbed DI (5.15) have been studied in the literature very extensively. See [11] and the references therein. In this paper we shall prove the existence of the extremal solutions of perturbed DI (5.15) under the weaker continuity condition of one of the multifunctions F and G.

DEFINITION 5.2. *A multivalued map map $F : J \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for every $y \in X$, the function $t \to d(y, F(t)) = \inf\{\|y-x\| : x \in F(t)\}$ is measurable.*

DEFINITION 5.3. *A multivalued map $\beta : J \times \mathbb{R} \to \mathcal{P}_p(\mathbb{R})$ is said to be L^1-Carathéodory if*

(i) *$t \mapsto \beta(t,x)$ is measurable for each $x \in \mathbb{R}$,*
(ii) *$x \mapsto \beta(t,x)$ is upper semicontinuous for almost all $t \in J$, and*
(iii) *for each real number $r > 0$, there exists a function $h_r \in L^1(J,\mathbb{R})$ such that*

$$\|\beta(t,x)\| = \sup\{|u| : u \in G(t,x)\} \le h_r(t), \qquad \text{a.e., } t \in J,$$

for all $x \in \mathbb{R}$ with $|x| \le r$.

Then, we have the following lemmas due to Lasota and Opial [18].

LEMMA 5.2. *If $\dim(X) < \infty$ and $F : J \times X \to \mathcal{P}_{\mathrm{cp}}(X)$ is L^1-Carathéodory, then $S^1_F(x) \neq \emptyset$ for each $x \in X$.*

LEMMA 5.3. *Let X be a Banach space, F an L^1-Carathéodory multivalued map with $S^1_F \neq \emptyset$ and $K : L^1(J,\mathbb{R}) \to C(J,X)$ be a linear continuous mapping. Then, the operator,*

$$\mathcal{K} \circ S^1_F : C(J,X) \longrightarrow \mathcal{P}_{\mathrm{cp}}(X)$$

is a closed graph operator in $C(J,X) \times C(J,X)$.

REMARK 5.3. It is known that a multivalued map $Q : X \to \mathcal{P}_p(X)$ is upper semicontinuous if and only if it is a closed graph operator.

DEFINITION. *A function $a \in AC(J,\mathbb{R})$ is called a lower solution of PDI (5.4) if for all $v_1 \in L^1(J,\mathbb{R})$ with $v_1(t) \in F(t,a(t))$ and $v_2 \in L^1(J,\mathbb{R})$ with $v_2(t) \in G(t,a(t))$ a.e., $t \in J$, we have that $a'(t) \le v_1(t) + v_2(t)$ a.e., $t \in J$ and $a(0) \le a(T)$. Similarly, a function $b \in AC(J,\mathbb{R})$ is called an upper solution of PDI (5.15) if for all $v_1 \in L^1(J,\mathbb{R})$ with $v_1(t) \in F(t,b(t))$ and $v_2 \in L^1(J,\mathbb{R})$ with $v_2(t) \in G(t,b(t))$ a.e., $t \in J$, we have that $b'(t) \ge v_1(t) + v_2(t)$ a.e., $t \in J$ and $b(0) \ge b(T)$.*

We now introduce the following hypotheses in the sequel.

(B_1) $G : J \times \mathbb{R} \to \mathcal{P}_{\mathrm{cp,cv}}(\mathbb{R})$ is L^1-Carathéodory multifunction.
(B_2) The multifunction $x \mapsto G(t,x)$ is strictly monotone increasing almost everywhere for $t \in J$.
(B_3) The PDI (5.15) has a lower solution a and an upper solution b with $a \le b$.

Consider the periodic PDI,

$$\begin{aligned} x'(t) + k(t)x(t) &\in F_k(t,x(t)) + G(t,x(t)), \qquad \text{a.e., } t \in J, \\ x(0) &= x(T), \end{aligned} \tag{5.16}$$

where F_k is defined by (5.5).

REMARK 5.4. Note that the PDI (5.15) is equivalent to the PDI (5.16) and the lower solution a of the PDI (5.15) is the lower solution for the PDI (5.16) and the upper solution b of the PDI (5.15) is the upper solution for the PDI (5.16) on J and conversely.

THEOREM 5.2. *Assume that Hypotheses (A_1)–(A_4) and (B_1)–(B_3) hold. Then, the PDI (5.15) has a solution in $[a,b]$.*

PROOF. Define an order interval $[a,b]$ in $AC(J,\mathbb{R})$ which does exist in view of Hypothesis (B_2). Now, PDI (5.15) is equivalent to the integral inclusion,

$$x(t) \in \int_0^T g_k(t,s)F_k(s,x(s))\,ds + \int_0^T g_k(t,s)G(t,x(t))\,ds, \qquad t \in J, \tag{5.17}$$

where the Green's function $g_k(t,s)$ is given by (5.8).

Define two multivalued operators $A, B : [a,b] \to \mathcal{P}_p(AC(J,\mathbb{R}))$ by

$$Ax\,(t) = \int_0^T g_k\,(t,s)\,F_k\,(s,x\,(s))\,ds, \qquad t \in J, \tag{5.18}$$

and

$$Bx(t) = \int_0^T g_k(t,s)G(s,x(s))\,ds, \qquad t \in J. \tag{5.19}$$

Clearly, the multivalued operators A and B are well defined on $[a,b]$ in view of Hypotheses (A_2) and (B_1). We shall show that A and B satisfy all the conditions of Theorem 4.3 on $[a,b]$.

STEP I. Since (A_1)–(A_4) hold, it can be shown as in the proof of Theorem 4.1 that A is a compact-valued, totally bounded and strictly monotone increasing multivalued map on $[a,b]$.

STEP II. Next, we show that B is completely continuous strictly monotone increasing multivalued operator on $[a,b]$. Let $x,y \in [a,b]$ be such that $x < y$ and let $u_1 \in Bx$ be arbitrary. Then, there is a $v_1 \in S_G^1(x)$ such that $u_1(t) = \int_0^t v_1(s)\,ds$. Since (H_4) holds, we have that $v_1 \le v_2$ for all $v_2 \in S_G^1(y)$. Therefore,

$$u_1(t) = \int_0^T g_k(t,s)v_1(s)\,ds \le \int_0^T g_k(t,s)v_2(s)\,ds = u_2(t),$$

for all $t \in J$, where $u_2 \in By$. This shows that B is strictly monotone increasing on $[a,b]$.

STEP III. Since (B_2) holds, from Remark 5.3 it follows that

$$a(t) \le \int_0^T g_k(t,s)v_1(s)\,ds + \int_0^T g_k(t,s)v_2(s))\,ds,$$

for all $t \in J$ and for all $v_1 \in S_{F_k}^1(a)$, $v_2 \in S_G^1(a)$. This further in view of the definitions of multivalued operators A and B implies that $a \le Aa + Ba$. Similarly, it is shown that $Ab + Bb \le b$. As A and B are strictly monotone increasing, we have that $Ax + By \in [a,b]$ for all $x,y \in [a,b]$.

STEP IV. Finally, we show that B is completely continuous on $[a,b]$. From the definition of B, it follows that

$$Bx(t) = \int_0^T g_k(t,s)G(s,x(s))\,ds = \left(\mathcal{K} \circ S_G^1\right)(x)(t),$$

where $\mathcal{K}$ is continuous linear operator on $L^1(J,\mathbb{R})$ into $C(J,\mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^T g_k(t,s)v(s)\,ds.$$

It is clear from Lemma 5.3 that $\mathcal{K} \circ S_G^1$ is a closed graph operator. Let $\{x_n\}$ be a sequence in $L^1(J,\mathbb{R})$ such that $x_n \to x_*$ as $n \to \infty$. Consider a sequence $\{y_n\}$ in $C(J,\mathbb{R})$ defined by $y_n \in \mathcal{K} \circ S_G^1(x_n)$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. But then $\{y_n - x_0\} \in \mathcal{K} \circ S_G^1(x_n)$ and $(y_n - x_0) \to (y_* - x_0)$. Since $\mathcal{K} \circ S_G^1$ is a closed graph operator, one has $y_* - x_0 \in \mathcal{K} \circ S_G^1(x_*)$ and consequently, $y_* \in \mathcal{K} \circ S_G^1(x_*)$. As a result B is a closed graph operator and which is further upper semicontinuous in view of Remark 5.1.

Next, we show that B is totally bounded on $[a,b]$. Let S be a subset of $[a,b]$. Since the cone K is normal in $AC(J,\mathbb{R})$, S is bounded in norm, and so there is a constant $r = \|a\| + \|b\|$ such that $\|x\| \le r$ for all $x \in S$. To conclude, it enough to show that $\cup B(S)$ is uniformly bounded and equicontinuous set in $AC(J,\mathbb{R})$. Let $y \in \cup B(S)$ be arbitrary. Then, there is a $v \in S_G^1(x)$ such that

$$y(t) = \int_0^T g_k(t,s)v(s)\,ds, \quad t \in J,$$

for some $x \in S$. Now by (H_2),

$$\begin{aligned}
|y(t)| &= \left|\int_0^T g_k(t,s)v(s)\,ds\right| \\
&\leq \int_0^T g_k(t,s)|v(s)|\,ds \\
&\leq M_k \int_0^T \|G(s,x)\|\,ds \\
&\leq M_k \int_0^T h_r(s)\,ds \\
&\leq M_k \|h_r\|_{L^1}.
\end{aligned}$$

This shows that the set $\cup B(S)$ is uniformly bounded in $AC(J,\mathbb{R})$. Now for any $t \in [0,T]$,

$$\begin{aligned}
|y'(t)| &\leq \left|\int_0^T \frac{\partial}{\partial t} g_k(t,s)v(s)\,ds\right| \\
&= \left|\int_0^T (-k(s))g_k(t,s)v(s)\,ds\right| \\
&\leq K\, M_k \|\gamma\|_{L^1} \\
&= c.
\end{aligned}$$

where $K = \max_{t\in J} k(t)$. Hence for any $t,\tau \in [0,T]$ one has

$$|y(t) - y(\tau)| \leq c|t-\tau| \to 0, \qquad \text{as } t \to \tau.$$

Therefore, for any $t,\tau \in J$, we have

$$|y(t) - y(\tau)| \to 0, \qquad \text{as } t \to \tau,$$

for all $y \in \cup B(S)$. Hence $\cup B(S)$ is an equicontinuous set in $AC(J,\mathbb{R})$. Thus, $\cup B(S)$ is a relatively compact subset of $AC(J,\mathbb{R})$ in view of Arzela-Ascoli theorem. Therefore, B is a completely continuous multivalued operator on $[a,b]$. As $Bx \subset B(S)$ for all $x \in S$, B is a compact-valued multivalued operator on $[a,b]$.

Thus, A and B satisfy all the conditions of Theorem 4.2 and hence, an application of it yields that the operator inclusion $x \in Ax+Bx$ has a least and a greatest solution in $[a,b]$. Consequently, the PDI (5.15) has a minimal and a maximal solution in $[a,b]$. This completes the proof.

REMARK 5.5. In a recent paper [22], the present author has proved some fixed-point theorems for discontinuous multivalued mappings on ordered Banach spaces under weaker monotonicity conditions as in [7] concerning the existence of a least and a greatest fixed points. But in that case the multivalued mappings are required to satisfy a stronger hypothesis that the images of the multivalued mappings are compact chains, a condition which is rather difficult to verify in the practical applications to differential and integral inclusions.

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