Existence of three positive solutions for boundary value problems with \( p \)-Laplacian

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Abstract

By using fixed point theorem, we study the following equation
\[
g(u'(t))' + a(t)f(u) = 0
\]
subject to boundary conditions, where \( g(v) = |v|^{p-2}v \) with \( p > 1 \); the existence of at least three positive solutions is proved.

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1. Introduction

The study of multiple positive solutions on three-point boundary value problems for ordinary differential equations has aroused extensive interest, one may see [1–6] and references therein. In [1,2], the authors considered the three-point boundary value problems for the one-dimensional \( p \)-Laplacian. In [1], Liu and Ge considered the three point boundary value problems

\[
(g(u'))' + a(t)f(u) = 0, \quad 0 < t < 1,
\]

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\[ u(0) - B_0(u'(\eta)) = 0, \quad u'(1) = 0, \quad (1.2) \]

or
\[ u'(0) = 0, \quad u(1) + B_1(u'(\eta)) = 0, \quad (1.3) \]

where \( g(v) = |v|^{p-2}v, \ p > 1, \) is called a \( p \)-Laplacian operator, \( \eta \in [0, 1] \) is a constant, \( B_0, B_1 \) satisfy that there are nonnegative numbers \( B, A \) such that
\[
Bx \leq B_i(x) \leq Ax, \quad x \in \mathbb{R}, \ i = 0, 1.
\]

Under some assumptions of \( f \) and \( a(t) \), the authors obtained at least two positive solutions (1.1) and (1.2) or (1.3), by an application of a fixed point theorem due to Avery and Henderson.

In [2], Avery and Henderson, studied the three point boundary problem
\[
\left( g(u') \right)' + a(t)f(u) = 0 \quad \text{for } 0 < t < 1,
\]
\[ u(0) = 0 \quad \text{and} \quad u'(v) = u'(1), \quad (1.4) \]

where \( v \in (0, 1) \) is a constant. By applying the Avery Five Functionals Fixed Point Theorem, the authors proved that problems (1.4)–(1.5) have at least three positive pseudosymmetric solutions.

In this paper, motivated by [1] and [2], we shall show that problem (1.1) and (1.2) or (1.3) has at least three positive solutions by means of the Five Functionals Fixed Point Theorem.

Throughout, it is assumed that

(1) \( f \in C([0, \infty), [0, \infty)) \),

(2) \( a(t) \) is nonnegative measurable function defined in \( (0, 1) \), and \( a(t) \) does not identically vanish on any subinterval of \( (0, 1) \) and
\[
0 < \int_0^1 a(t) dt < \infty.
\]

2. Preliminaries

In this section, we give some definitions and results that we shall use in the rest of the paper.

**Definition 2.1.** Let \( E \) be a real Banach space. A nonempty, closed, convex set \( P \subset E \) is said to be a cone provided the following conditions are satisfied:

(i) if \( x \in P \) and \( \lambda \geq 0 \), then \( \lambda x \in P \);

(ii) if \( x \in P \) and \( -x \in P \), then \( x = 0 \).

Every cone \( P \subset E \) induces an ordering in \( E \) given by
\[ x \leq y \quad \text{if and only if} \quad y - x \in P. \]
**Definition 2.2.** A map \( \alpha \) is said to be a nonnegative, continuous, concave functional on a cone \( P \) of a real Banach space \( E \), if

\[
\alpha : P \to [0, \infty)
\]

is continuous, and

\[
\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y)
\]

for all \( x, y \in P \) and \( t \in [0, 1] \). Similarly, we say the map \( \beta \) is a nonnegative, continuous, convex functional on a cone \( P \) of a real Banach space \( E \), if

\[
\beta : P \to [0, \infty)
\]

is continuous, and

\[
\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y)
\]

for all \( x, y \in P \) and \( t \in [0, 1] \).

Let \( \gamma, \beta, \theta \) be nonnegative, continuous, convex functionals on \( P \) and \( \alpha, \psi \) be nonnegative, continuous, concave functionals on \( P \). Then, for nonnegative real numbers \( h, a, b, d \) and \( c \), we define the convex sets

\[
P(\gamma, c) = \{x \in P : \gamma(x) < c\},
\]

\[
P(\gamma, \alpha, a, c) = \{x \in P : a \leq \alpha(x), \gamma(x) \leq c\},
\]

\[
Q(\gamma, \beta, d, c) = \{x \in P : \beta(x) \leq d, \gamma(x) \leq c\},
\]

\[
P(\gamma, \theta, \alpha, a, b, c) = \{x \in P : a \leq \alpha(x), \theta(x) \leq b, \gamma(x) \leq c\}
\]

and

\[
Q(\gamma, \beta, \psi, h, a, c) = \{x \in P : h \leq \psi(x), \beta(x) \leq d, \gamma(x) \leq c\}.
\]

To prove our main results, we need the following theorem, which is the Five Functionals Fixed Point Theorem [2].

**Theorem 2.1.** Let \( P \) be a cone in a real Banach space \( E \). Suppose there exist positive numbers \( c \) and \( M \), nonnegative, continuous, concave functionals \( \alpha \) and \( \psi \) on \( P \), and nonnegative, continuous, convex functionals \( \gamma \) and \( \beta \) on \( P \), with

\[
\alpha(x) \leq \beta(x) \quad \text{and} \quad \|x\| \leq M\gamma(x)
\]

for all \( x \in \overline{P(\gamma, c)} \). Suppose

\[
\Phi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)}
\]

is completely continuous and there exist nonnegative numbers \( h, a, k, b \), with \( 0 < a < b \) such that:

1. \( \{x \in P(\gamma, \theta, \alpha, b, k, c) : \alpha(x) > b\} \neq \emptyset \) and \( \alpha(\Phi x) > b \) for \( x \in P(\gamma, \theta, \alpha, b, k, c) \);
2. \( \{x \in Q(\gamma, \beta, \psi, h, a, c) : \beta(x) < a\} \neq \emptyset \) and \( \beta(\Phi x) < a \) for \( x \in Q(\gamma, \beta, \psi, h, a, c) \);
3. \( \alpha(\Phi x) > b \) for \( x \in P(\gamma, \alpha, b, c) \) with \( \theta(\Phi x) > k \);
4. \( \beta(\Phi x) < a \) for \( x \in Q(\gamma, \beta, a, c) \) with \( \psi(\Phi x) < h \).
Then $\Phi$ has at least three fixed points $x_1, x_2, x_3 \in P(\gamma, c)$ such that

$$\beta(x_1) < a, \quad b < \alpha(x_2), \quad \text{and} \quad a < \beta(x_3) \quad \text{with} \quad \alpha(x_3) < b.$$ 

3. Existence results

In this section, we shall obtain existence results for the problem (1.1) and (1.2) or (1.3), by using the Five Functionals Fixed Point Theorem.

We first consider the problem (1.1)–(1.2).

Let $E = C[0, 1]$ be the Banach space with the sup-norm $\|u\| = \sup\{|u(x)|: 0 \leq x \leq 1\}$, and define the cone $P \subset E$ by

$$P = \{u \in E: u(t) \geq 0, \ u(t) \text{ is concave on } [0, 1], \ u'(1) = 0\}.$$ 

Now, fix $l$ such that

$$0 < \eta < l < 1,$$

and

$$\int_{l}^{1} a(t) \, dt > 0.$$ 

Define the nonnegative, continuous, concave functionals $\alpha, \psi$ and the nonnegative, continuous, convex functionals $\beta, \theta, \gamma$ on the cone $P$ by

$$\gamma(u) = \theta(u) := \max_{0 \leq t \leq \eta} u(t) = u(\eta),$$

$$\alpha(u) := \min_{l \leq t \leq 1} u(t) = u(l), \quad \beta(u) := \max_{0 \leq t \leq l} u(t) = u(l),$$

$$\psi(u) := \min_{\eta \leq t \leq 1} u(t) = u(\eta).$$

It is clear that $\alpha(u) = \beta(u)$ for all $u \in P$.

For notational convenience, we denote by

$$M = AG \left( \int_{\eta}^{1} a(r) \, dr \right) + \eta G \left( \int_{0}^{1} a(r) \, dr \right),$$

$$m = (B + l)G \left( \int_{l}^{1} a(r) \, dr \right),$$

$$\lambda_l = (A + l)G \left( \int_{0}^{1} a(r) \, dr \right),$$

where $G(w) = |w|^{1/(p-1)} \text{sgn}(w)$ is the inverse of $g$. 
Lemma 3.1. Let $u \in P$, then

1. $u(t) \geq t\|u\|$ for $t \in [0, 1]$, and
2. $\eta u(l) \leq lu(\eta)$.

Proof. (1) is Lemma 2.1 of [1]. Now, we prove (2). Since $u(t)$ is concave and $0 < \eta < l < 1$, we have

$$\frac{u(l) - u(0)}{l} \leq \frac{u(\eta) - u(0)}{\eta},$$

thus

$$lu(\eta) \geq \eta u(l) + (l - \eta)u(0) \geq \eta u(l),$$

which completes this proof.

By Lemma 3.1, we have $\gamma(u) = u(\eta) \geq \eta\|u\|$. Hence $\|u\| \leq \frac{1}{\eta}\gamma(u)$ for all $u \in P$. Now, we state and prove our main results.

Theorem 3.1. Let $0 < a < lb < l\eta c$, $Mb < mc$ and suppose that $f$ satisfies the following conditions:

(H1) $f(x) < g\left(\frac{c}{M}\right)$ for all $0 \leq x \leq \frac{c}{\eta}$,
(H2) $f(x) > g\left(\frac{b}{m}\right)$ for all $b \leq x \leq \frac{b}{\eta^2}$,
(H3) $f(x) < g\left(\frac{a}{\lambda l}\right)$ for all $0 \leq x \leq \frac{a}{\lambda}$.

Then the boundary value problem (1.1)–(1.2) has at least three positive solutions $u_1, u_2$ and $u_3$ such that

$$\beta(u_1) < a, \quad b < \alpha(u_2), \quad a < \beta(u_3)$$

with $\alpha(u_3) < b$.

Proof. Define a completely continuous operator $\Phi : P \rightarrow E$ by

$$(\Phi u)(t) = B_0 \left(G\left(\int_0^1 a(r) f(u(r)) \, dr\right)\right)$$

$$+ \int_0^t G\left(\int_0^1 a(r) f(u(r)) \, dr\right) \, ds, \quad 0 \leq t \leq 1.$$ 

By [1], it is known that $\Phi(u) \in P$ and every fixed point of $\Phi$ is a solution of problem (1.1)–(1.2).

Let $u \in P(\gamma, c)$, then $\gamma(u) = \max_{0 \leq t \leq \eta} u(t) = u(\eta) \leq c$, consequently, $0 \leq u(t) \leq c$ for $t \in [0, \eta]$. Since $u(\eta) \geq \eta u(1)$, so $\|u\| = u(1) \leq \frac{u(\eta)}{\eta} \leq \frac{c}{\eta}$, this implies $0 \leq u(t) \leq \frac{c}{\eta}$ for $t \in [0, 1]$. Thus
\[ \gamma(\Phi u) = (\Phi u)(\eta) \]
\[ = B_0 \left( G \left( \int_{\eta}^{1} a(r) f(u(r)) \, dr \right) \right) + \int_{0}^{\eta} G \left( \int_{s}^{1} a(r) f(u(r)) \, dr \right) \, ds \]
\[ \leq AG \left( \int_{\eta}^{1} a(r) f(u(r)) \, dr \right) + \eta G \left( \int_{0}^{1} a(r) f(u(r)) \, dr \right) \]
\[ < \frac{c}{M} \left[ AG \left( \int_{\eta}^{1} a(r) \, dr \right) + \eta G \left( \int_{0}^{1} a(r) \, dr \right) \right] = c. \]

Therefore
\[ \Phi u \in \bar{P}(\gamma, c). \]

Now, we show that (i)–(iv) of Theorem 2.1 are satisfied.

Firstly, let \( u \equiv \frac{b}{\eta}, \; k = \frac{b}{\eta}, \) it follows that
\[
\alpha(u) = u(l) = \frac{b}{\eta} > b, \quad \theta(u) = u(\eta) = \frac{b}{\eta} = k, \quad \gamma(u) = \frac{b}{\eta} < c,
\]
which shows that \( \{ u \in P(\gamma, \theta, \alpha, b, k, c), \alpha(u) > b \} \neq \emptyset, \) and for \( u \in P(\gamma, \theta, \alpha, b, \frac{b}{\eta}, c), \) we have
\[ b \leq u(t) \leq \frac{b}{\eta^2} \quad \text{for all} \; t \in [l, 1]. \]

By the condition (H2) of Theorem 3.1, we can obtain
\[ \alpha(\Phi u) = (\Phi u)(l) \]
\[ = B_0 \left( G \left( \int_{\eta}^{1} a(r) f(u(r)) \, dr \right) \right) + \int_{0}^{l} G \left( \int_{s}^{1} a(r) f(u(r)) \, dr \right) \, ds \]
\[ \geq BG \left( \int_{l}^{1} a(r) f(u(r)) \, dr \right) + lG \left( \int_{l}^{1} a(r) f(u(r)) \, dr \right) \]
\[ > (B + l)G \left( \int_{l}^{1} a(r) \, dr \right) \times \frac{b}{m} = b. \]

So the condition (i) of Theorem 2.1 is satisfied.

Secondly, we prove that (ii) of Theorem 2.1 is fulfilled. We take \( u \equiv \eta a, \; h = \eta a, \) then
\[ \gamma(u) = \eta a < c, \quad \psi(u) = \eta a = h, \quad \beta(u) = \eta a < a. \]

From this we know that \( \{ u \in Q \in (\gamma, \beta, \psi, h, a, c): \beta(u) < a \} \neq \emptyset. \) If \( u \in Q(\gamma, \beta, \psi, \eta a, a, c), \) then
\[ 0 \leq u \leq \frac{a}{l} \quad \text{for} \; t \in [0, 1]. \]
In view of (H3) of Theorem 3.1, we have
\[ \beta(\Phi u) = (\Phi u)(l) = B_0 \left( \int_{\eta}^{1} a(r) f(u(r)) \, dr \right) + \int_{0}^{l} \left( \int_{s}^{1} a(r) f(u(r)) \, dr \right) ds \leq AG \left( \int_{\eta}^{1} a(r) f(u(r)) \, dr \right) + lG \left( \int_{0}^{1} a(r) f(u(r)) \, dr \right) < (A + l)G \left( \int_{0}^{1} a(r) \, dr \right) \times \frac{a}{\lambda l} = a. \]

Thirdly, we show that (iii) of Theorem 2.1 is satisfied. If \( u \in P(\gamma, \alpha, b, c) \) and \( \theta(\Phi u) = \Phi u(\eta) > b \eta \), then
\[ \alpha(\Phi u) = (\Phi u)(l) \geq l\Phi u(l) \geq l\Phi u(\eta) > \frac{l}{\eta} b > b. \]

Finally, if \( u \in Q(\gamma, \beta, a, c) \) and \( \psi(\Phi u) = \Phi u(\eta) < \eta a \), then
\[ \beta(\Phi u) = \Phi u(l) \leq \frac{\Phi u(l)}{l} \leq \frac{\Phi u(\eta)}{\eta} < a, \]

which shows that the condition (iv) of Theorem 2.1 is fulfilled.

Thus, all the conditions in Theorem 2.1 are met. So the boundary value problem (1.1)–(1.2) has at least three solutions \( u_1, u_2, u_3 \) and
\[ \beta(u_1) < a, \quad b < \alpha(u_2) \quad \text{and} \quad a < \beta(u_3) \quad \text{with} \quad \alpha(u_3) < b. \]

This proof is complete. \( \square \)

Now, we deal with the problem (1.1) and (1.3). Fix \( \xi \) such that
\[ 0 < \xi < \eta < 1, \]
and
\[ \int_{0}^{\xi} a(t) \, dt > 0. \]

Define a cone \( P \subset E \) by
\[ P = \{ u \in E: u(t) \geq 0 \, \text{for} \, t \in [0, 1], \, u(t) \text{ is concave on} \, [0, 1], \, u'(0) = 0 \}, \]
and define the nonnegative, continuous, concave functionals \( \alpha, \psi \) and the nonnegative, continuous, convex functionals \( \beta, \theta, \gamma \) on \( P \) respectively as
\( \gamma(u) = \theta(u) := \max_{\eta \leq t \leq 1} u(t) = u(\eta), \)
\( \alpha(u) := \min_{0 \leq t \leq \xi} u(t) = u(\xi), \quad \beta(u) := \max_{\xi \leq t \leq 1} u(t) = u(\xi), \)
\( \psi(u) := \min_{0 \leq t \leq \eta} u(t) = u(\eta). \)

Similar to Lemma 3.1, we have

**Lemma 3.2.** If \( u \in P \), then

1. \( u(t) \geq (1 - t)\|u\|, \)
2. \((1 - \eta)u(\xi) \leq (1 - \xi)u(\eta).\)

Let

\[
M_1 = AG\left( \int_0^\eta a(r) \, dr \right) + (1 - \eta)G\left( \int_0^1 a(r) \, dr \right),
\]
\[
m_1 = (B + 1 - \xi)G\left( \int_0^\xi a(r) \, dr \right),
\]
\[
\lambda_\xi = (A + 1 - \xi)G\left( \int_0^1 a(r) \, dr \right).
\]

We have the following result.

**Theorem 3.2.** Let \( 0 < a < (1 - \xi)b < (1 - \xi)(1 - \eta)c, M_1b < m_1c \), and assume that:

(H4) \( f(x) < g(\frac{c}{M_1}) \) for all \( 0 \leq x \leq \frac{c}{1-\eta} \),

(H5) \( f(x) > g(\frac{b}{m_1}) \) for all \( b \leq x \leq \frac{b}{(1-\eta)^2} \),

(H6) \( f(x) < g(\frac{a}{\lambda_\xi}) \) for all \( 0 \leq x \leq \frac{a}{1-\eta} \).

Then the problem (1.1) and (1.3) has at least three positive solutions \( u_1, u_2 \) and \( u_3 \) such that

\( \beta(u_1) < a, \quad b < \alpha(u_2) \quad \text{and} \quad a < \beta(u_3) \quad \text{with} \quad \alpha(u_3) < b. \)

Define a completely continuous operator \( \Phi : \overline{P(\gamma, c)} \to \overline{P(\gamma, c)} \) by

\[
\Phi u(t) = B_1\left( G\left( \int_0^\eta a(r)f(u(r)) \, dr \right) \right)
+ \int_t^1 G\left( \int_0^s a(r)f(u(r)) \, dr \right) \, ds, \quad 0 \leq t \leq 1.
\]
The proof of Theorem 3.2 is similar to that of Theorem 3.1, we omit it here.

References


