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New fixed point theorems and applications of mixed monotone operator ☆

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Abstract

In this paper, we study the uniqueness and existence of fixed point of mixed monotone operators in the partially ordered Banach space. The results extend and improve recent related results.

Keywords: Mixed monotone operators; Cone and semiorder; Fixed point

1. Introduction and preliminaries

It is well known that mixed monotone operator equation is important for applications due to there is a quite extensive class of integro-differential equations as well as boundary value problems in nonlinear analysis which are related to the solvability of this kind of equation. There are many useful results about mixed monotone operator (see [1–8]). In this paper,without assuming operators to be continuous or compact, we study mixed monotone operators and give a several of new existence and uniqueness theorems. The results extend and improve recent related results (see [5–8, 13]).

Let the real Banach space E (or, more general case, topological linear space) be partially ordered by a cone P of E, i.e. $x \leqslant y$ (or denoted $y \geqslant x$) if and only if $y - x \in P$. By θ we denote the zero element of E. Recall that a nonempty closed convex set $P \subset E$ is a cone if it satisfies

(i) $x \in P, \lambda \geqslant 0 \Rightarrow \lambda x \in P$, and
(ii) $x, -x \in P \Rightarrow x = \theta$.

We denote by P^0 the interior set of P. A cone P is said to be solid cone if $P^0 \neq \phi$. P is said to be normal if there exists a positive constant N such that $\theta \leqslant x \leqslant y$ implies $\|x\| \leqslant N\|y\|$; N is called the normal constant of P. We write $x \ll y$ if and only if $y - x \in P^0$.

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Definition 1.1. (See [1,13].) Let $D \subset E$. Operator $A : D \times D \to E$ is said to be mixed monotone if $A(x, y)$ is nondecreasing in x and nonincreasing in y, i.e. $u_1 \leqslant u_2$, $v_2 \leqslant v_1$, $u_i, v_i \in D$ $(i = 1, 2)$ implies $A(u_1, v_1) \leqslant A(u_2, v_2)$. Point $(x^*, y^*) \in D \times D$, $x^* \leqslant y^*$ is called a coupled lower–upper fixed point of A if $x^* \leqslant A(x^*, y^*)$ and $A(y^*, x^*) \leqslant y^*$. Point $(x^*, y^*) \in D \times D$ is called a coupled fixed point of A if $A(x^*, y^*) = x^*$ and $A(y^*, x^*) = y^*$. Element $x^* \in D$ is called a fixed point of A if $A(x^*, x^*) = x^*$.

For all $x, y \in E$, the notation $x \sim y$ means that there exists $0 < \lambda \leqslant \mu$ such that $\lambda x \leqslant y \leqslant \mu x$. Clearly, $\sim$ is an equivalence relation. Given $h > \theta$ (i.e. $h \geqslant \theta$ and $h \neq \theta$), we denote by P_h the set

$$P_h = \{x \in E, \text{ there exists } \lambda(x), \mu(x) > 0 \text{ such that } \lambda(x)h \leqslant x \leqslant \mu(x)h\}.$$

It is easy to see that $P_h \subset P$.

All the concepts discussed above can be found in [1,2,12,13].

The following result on an operator in ordered spaces can be similarly proved as in [6] and therefore its proof is omitted.

Lemma 1.1. *Let E be a topological linear space, partially ordered by a cone P of E. $h > \theta$, and $A : P_h \times P_h \to E$. Then the following two statements are equivalent*:

(a) *For all $0 < t < 1$ and $u, v \in P_h$, there exists $0 < \alpha = \alpha(t, u, v) < 1$ such that $A(tu, \frac{1}{t}v) \geqslant t^{\alpha(t,u,v)} A(u, v)$.*
(b) *For all $0 < t < 1$ and $u, v \in P_h$, there exists $\eta = \eta(t, u, v) > 0$ such that $A(tu, \frac{1}{t}v) \geqslant t[1 + \eta(t, u, v)]A(u, v)$, where $t[1 + \eta(t, u, v)] < 1$.*

Remark 1.1. If operator A is a mixed monotone, and satisfies condition (a) or (b) of Lemma 1.1, then we call A a $t - \alpha(t, u, v)$ or $t - \eta(t, u, v)$ mixed monotone model operator.

We also need the following result (see [5, Theorem 2.3]).

Lemma 1.2. *(See [5].) Let E be a real ordered Banach space, P a normal cone in E, $h > \theta$, and $A : P_h \times P_h \to P_h$ a mixed monotone operator. There exists a function $\eta : (0, 1) \times P_h \times P_h \to (0, +\infty)$ such that for all $x, y \in P_h$, $t \in (0, 1)$, we have*

$$A\left(tx, \frac{1}{t}y\right) \geqslant t\big[1 + \eta(t, x, y)\big]A(x, y). \tag{1}$$

If $(u_0, v_0) \in P_h \times P_h$ is coupled lower–upper fixed point of A, and

$$\xi(t) = \inf_{x,y \in [u_0, v_0]} \eta(t, x, y) > 0, \quad \forall t \in (0, 1), \tag{2}$$

then A has exactly one fixed point x^* in P_h. Moreover, constructing successively the sequences $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, for any initial value $x_0 \in P_h$, we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

For more facts about mixed monotone operators and other related concepts the reader is referred to [1–15] and some of the references therein.

2. $t - \eta(t, u, v)$ mixed monotone model operator's main results

Firstly, we give a new uniqueness and existence theorem of fixed point of $t - \eta(t, u, v)$ mixed monotone model operators.

Theorem 2.1. *Let P be a normal and solid cone of a real Banach space E. For a class of operators $A = B + D$, we assume that*

(H_1) *$B : P^0 \times P^0 \to P^0$ is a mixed monotone operator, and there exists a function $\varphi : (0, 1) \times P^0 \times P^0 \to (0, +\infty)$ and $u_0, v_0 \in P^0$, $u_0 \leqslant v_0$ such that*

(i) *for all* $x, y \in P^0$, $t \in (0,1)$,

$$B\big(tx, t^{-1}y\big) \geqslant \varphi(t,x,y)B(x,y), \tag{3}$$

$\varphi(t,x,y) > t$, $\frac{\varphi(t,x,y)}{t}$ *is decreasing in* t;

(ii) $u_0 \leqslant B(u_0, v_0) + Du_0$ *and* $B(v_0, u_0) + Dv_0 \leqslant v_0$.

(H_2) $D : P \to P$ *satisfies the following conditions*:

(i) *for all* $x, y \in P$, $x \geqslant y$: $D(x-y) = Dx - Dy$;

(ii) *for all* $x \in P$, $t \geqslant 0$: $D(tx) = tD(x)$.

Then there exists a unique fixed point $x^* \in [u_0, v_0]$ *such that* $A(x^*, x^*) = x^*$. *Moreover, for any initial point* $x_0 \in [u_0, v_0]$, *constructing successively the sequences* $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, *we have* $\|x_n - x^*\| \to 0$.

Proof. According to the condition (H_2), we have that the operator $D : P \to P$ is nondecreasing. From the monotonicity of the operator B, we can obtain that the operator A is mixed monotone.

For all $x, y \in [u_0, v_0]$, there exists constant $c > 0$ such that

$$A(x,y) \leqslant A(v_0, u_0) \leqslant cB(u_0, v_0) \leqslant cB(x,y). \tag{4}$$

Let

$$\eta(t,x,y) = \frac{\varphi(t,x,y)}{t} - 1, \quad x, y \in [u_0, v_0],\ t \in (0,1). \tag{5}$$

We have

$$\begin{aligned} A\big(tx, t^{-1}y\big) &= B\big(tx, t^{-1}y\big) + D(tx) \geqslant t\big[1 + \eta(t,x,y)\big]B(x,y) + tD(x) \geqslant tA(x,y) + t\eta(t,x,y)\frac{1}{c}A(x,y) \\ &= t\left[1 + \frac{1}{c}\eta(t,x,y)\right]A(x,y). \end{aligned}$$

Set

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots,$$

$$t_n = \sup\{t > 0 \mid u_n \geqslant tv_n\}, \quad n = 0, 1, \ldots.$$

It is clear that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0,$$

$$u_n \geqslant t_n v_n, \quad n = 0, 1, \ldots.$$

Notice that $u_{n+1} \geqslant u_n \geqslant t_n v_n \geqslant t_n v_{n+1}$, $n = 0, 1, \ldots$, thus we have that

$$0 < t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n \leqslant t_{n+1} \leqslant \cdots < 1.$$

So there exists $\lim_{n\to\infty} t_n = t$ for some $0 < t \leqslant 1$.

We can show that $t = 1$. Otherwise, if $0 < t < 1$, it follows from $\eta(t, u_n, v_n) > 0$. We consider two cases:

(i) There exists some natural number N such that $t_N = t$. In this case, we can obtain that $t_n = t$, $u_n \geqslant tv_n$, as $n \geqslant N$ and

$$u_{n+1} = A(u_n, v_n) \geqslant A\big(tv_n, t^{-1}u_n\big) \geqslant t\left[1 + \frac{1}{c}\eta(t, v_n, u_n)\right]A(v_n, u_n), \quad n \geqslant N.$$

Thus

$$t_{n+1} \geqslant t\left[1 + \frac{1}{c}\eta(t, v_n, u_n)\right] > t, \quad n \geqslant N,$$

which is a contradiction.

(ii) For all $n = 1, 2, \ldots$, $t_n < t$. In this case, we have that

$$u_{n+1} = A(u_n, v_n) \geqslant A\big(t_n v_n, t_n^{-1} u_n\big) \geqslant t_n\left[1 + \frac{1}{c}\eta(t_n, v_n, u_n)\right] A(v_n, u_n) = t_n\left[1 + \frac{1}{c}\eta(t_n, v_n, u_n)\right] v_{n+1}.$$

Thus, by the definition of t_{n+1}, we have

$$t_{n+1} \geqslant t_n\left[1 + \frac{1}{c}\eta(t_n, v_n, u_n)\right] = t_n\left[1 + \frac{1}{c}\left(\frac{\varphi(t_n, v_n, u_n)}{t_n} - 1\right)\right] \geqslant t_n\left[1 + \frac{1}{c}\left(\frac{\varphi(t, v_n, u_n)}{t} - 1\right)\right].$$

Furthermore, we have

$$t \geqslant t\left[1 + \frac{1}{c}\left(\frac{\varphi(t, v_n, u_n)}{t} - 1\right)\right] > t.$$

This is a contradiction. Hence $t = 1$.

Thus for any natural number p,

$$\theta \leqslant u_{n+p} - u_n \leqslant v_n - u_n \leqslant (1 - t_n)v_0, \quad n = 1, 2, \ldots,$$

$$\theta \leqslant v_n - v_{n+p} \leqslant v_n - u_n \leqslant (1 - t_n)v_0, \quad n = 1, 2, \ldots.$$

From the normality of P it follows that $\|u_{n+p} - u_n\| \to 0$, $\|v_{n+p} - v_n\| \to 0$ as $n \to \infty$. Hence there exists $u^*, v^* \in [u_0, v_0]$, such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$, and $\theta \leqslant v^* - u^* \leqslant v_n - u_n \leqslant (1 - t_n)v_0$, $n = 1, 2, \ldots$, thus $u^* = v^*$.

Let $x^* \triangleq u^* = v^*$. Notice that $u_n \leqslant u^* \leqslant v_n$, hence

$$u_{n+1} = A(u_n, v_n) \leqslant A(u^*, u^*) \leqslant A(v_n, u_n) = v_{n+1}.$$

Letting $n \to \infty$, we have $A(u^*, u^*) = u^*$, $u^* = x^*$ is a fixed point of A.

If the operator A has other fixed point $w^* \in [u_0, v_0]$, then

$$u_0 \leqslant A(w^*, w^*) = w^* \leqslant v_0.$$

Repeating the above iterative procedure, we have $u_n \leqslant w^* \leqslant v_n$.Taking into account that P is normal cone we conclude that $u^* = w^* = x^*$.

Finally, for all $x_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, by using the mixed monotone of A, we have $u_n \leqslant x_n \leqslant v_n$, which implies that $\|x_n - x^*\| \to 0$ as $n \to \infty$. The proof is over. □

Further, we have the following result.

Theorem 2.2. *Let P be a normal cone of a real Banach space E, $h > \theta$. For a class of operators $A = B + \lambda C + D$, where $\lambda \geqslant 0$ is a constant, we assume that*

(H$_3$) *$B : P_h \times P_h \to P_h$ is a mixed monotone operator, and there exists a function $\alpha : P_h \times P_h \times (0, 1) \to (0, 1)$ and $u_0, v_0 \in P_h$, $u_0 \leqslant v_0$ such that*
  (i) *for all $x, y \in P_h$, $t \in (0, 1)$:*

$$B\big(tx, t^{-1}y\big) \geqslant t^{\alpha(t,x,y)} B(x, y); \tag{6}$$

  (ii) *$u_0 \leqslant B(u_0, v_0) + \lambda C(u_0, v_0) + Du_0$ and $B(v_0, u_0) + \lambda C(v_0, u_0) + Dv_0 \leqslant v_0$.*

(H$_4$) *$C : P_h \times P_h \to P_h$ is a mixed monotone operator, and there exists a function $\beta : (0, +\infty) \to (1, +\infty)$ such that for all $x, y \in P_h$, $t > 0$:*

$$C\big(tx, t^{-1}y\big) \geqslant t^{\beta(t)} C(x, y). \tag{7}$$

(H$_5$) *$D : P \to P$ satisfies the following conditions:*
  (i) *$D(x - y) = Dx - Dy$, for all $x, y \in P$, $x \geqslant y$;*
  (ii) *$D(tx) = tD(x)$, for all $x \in P$, $t \geqslant 0$.*

Suppose that

$$\gamma(t) = \inf_{x,y\in[u_0,v_0]} t^{\alpha(t,x,y)} > t\big[1+\lambda c\big(1-t^{\beta(t)-1}\big)\big], \quad t\in(0,1), \tag{8}$$

where

$$c = \inf\big\{r \;\big|\; C(x,y)\leqslant rB(x,y),\ x,y\in[u_0,v_0]\big\}. \tag{9}$$

Then there exists a unique fixed point $x^*\in[u_0,v_0]$ such that $A(x^*,x^*)=x^*$. Moreover, for any initial point $x_0\in[u_0,v_0]$, constructing successively the sequences $x_n=A(x_{n-1},x_{n-1})$, $n=1,2,\ldots$, we have $\|x_n-x^*\|\to 0$ as $n\to\infty$.

Proof. Firstly, for $\forall x,y\in[u_0,v_0]$, since $B(u_0,v_0)$ and $C(v_0,u_0)\in P_h$, there exists constant $r>0$ such that

$$C(x,y)\leqslant C(v_0,y)\leqslant C(v_0,u_0)\leqslant rB(u_0,v_0)\leqslant rB(x,v_0)\leqslant rB(x,y).$$

Thus $c=\inf\{r \mid C(x,y)\leqslant rB(x,y),\ x,y\in[u_0,v_0]\}\geqslant 0$.

Notice that if $c=0$, then for $\forall x,y\in[u_0,v_0]$, we have $C(x,y)\leqslant cB(x,y)=\theta$, this is a contradiction. Hence $c>0$.

By using the mixed monotone properties of operators B and C, and the condition of (H_5), we can obtain that the operator A is mixed monotone.

Similarly to the proof of $c>0$, we can have that

$$\delta = \sup\big\{r \;\big|\; B(x,y)\geqslant rA(x,y),\ x,y\in[u_0,v_0]\big\} > 0.$$

According to Lemma 1.1, there exists $\eta(t,x,y)=t^{\alpha(t,x,y)-1}-1>0$, such that for all $x,y\in P_h$, $t\in(0,1)$: $B(tx,t^{-1}y)\geqslant t[1+\eta(t,x,y)]B(x,y)$.

Let $\xi(t)\triangleq\inf_{x,y\in[u_0,v_0]}\eta(t,x,y)$, then $\xi(t)>\lambda c(1-t^{\beta(t)-1})$.

Hence for all $x,y\in[u_0,v_0]$, $t\in(0,1)$, we have

$$\begin{aligned}
A\big(tx,t^{-1}y\big) &= B\big(tx,t^{-1}y\big)+\lambda C\big(tx,t^{-1}y\big)+D(tx)\geqslant t\big[1+\eta(t,x,y)\big]B(x,y)+\lambda t^{\beta(t)}C(x,y)+tD(x)\\
&\geqslant t\big[1+\xi(t)\big]B(x,y)+\lambda t^{\beta(t)}C(x,y)+tD(x)\\
&= tA(x,y)+t\xi(t)B(x,y)+\lambda t^{\beta(t)}C(x,y)-\lambda tC(x,y)\\
&\geqslant tA(x,y)+t\xi(t)B(x,y)-\lambda c\big(t-t^{\beta(t)}\big)B(x,y)=tA(x,y)+t\big[\xi(t)-\lambda c\big(1-t^{\beta(t)-1}\big)\big]B(x,y)\\
&\geqslant tA(x,y)+t\big[\xi(t)-\lambda c\big(1-t^{\beta(t)-1}\big)\big]\delta A(x,y)=t\big[1+\delta\big(\xi(t)-c\lambda\big(1-t^{\beta(t)-1}\big)\big)\big]A(x,y).
\end{aligned}$$

Let $\omega(t)\triangleq\delta(\xi(t)-c\lambda(1-t^{\beta(t)-1}))$, thus $\omega(t)>0$, and $A(tx,t^{-1}y)\geqslant t[1+\omega(t)]A(x,y)$. By using Lemma 1.2, there exists a unique fixed point $x^*\in[u_0,v_0]$ such that $A(x^*,x^*)=x^*$. Moreover, for any initial point $x_0\in[u_0,v_0]$, constructing successively the sequences $x_n=A(x_{n-1},x_{n-1})$, $n=1,2,\ldots$, we have $\|x_n-x^*\|\to 0$. This completes the proof. □

Remark 2.1. If the function $\beta:(0,+\infty)\to(1,+\infty)$ in Theorem 2.2 reduces to a constant $\beta>1$, and for all $x,y\in P_h$, $t>0$: $C(tx,t^{-1}y)=t^{\beta}C(x,y)$, $D:E\to E$ is a positive linear operator such that $D(P_h)\subset P_h\cup\{\theta\}$, then Theorem 2.2 will coincide with [7, Theorem 2.1]. This means that Theorem 2.2 improves the corresponding results in [7].

Theorem 2.3. *Let P be a normal cone of a real Banach space E, $u_0,v_0\in E$, and $u_0<v_0$. $A:[u_0,v_0]\times[u_0,v_0]\to E$ is a mixed monotone operator. Assume that*

(i) $u_0\leqslant A(u_0,v_0)$, $A(v_0,u_0)\leqslant v_0$;
(ii) *there exists a constant $0\leqslant k<1$ such that*

$$A(v,u)-A(u,v)\leqslant k(v-u)\quad(u_0\leqslant u\leqslant v\leqslant v_0). \tag{10}$$

Then there exists a unique fixed point $x^* \in [u_0, v_0]$ such that $A(x^*, x^*) = x^*$. Moreover, for any initial point $x_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = A(x_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ as $n \to \infty$.

Proof. Set

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots.$$

Thus it is easy to obtain that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0.$$

Hence

$$v_n - u_n = A(v_{n-1}, u_{n-1}) - A(u_{n-1}, v_{n-1}) \leqslant k(v_{n-1} - u_{n-1}) \leqslant k^2(v_{n-2} - u_{n-2}) \leqslant \cdots \leqslant k^n(v_0 - u_0).$$

Thus for any natural number m and n,

$$\theta \leqslant u_{n+m} - u_n \leqslant v_n - u_n \leqslant k^n(v_0 - u_0), \qquad \theta \leqslant v_n - v_{n+m} \leqslant v_n - u_n \leqslant k^n(v_0 - u_0).$$

Taking into account that P is normal cone we conclude that

$$\|u_{n+m} - u_n\| \leqslant N\|v_n - u_n\| \leqslant N^2 k^n \|v_0 - u_0\| \to 0,$$

$$\|v_{n+m} - v_n\| \leqslant N\|v_n - u_n\| \leqslant N^2 k^n \|v_0 - u_0\| \to 0,$$

as $n \to \infty$, where N is the normal constant of P. Hence there exists $u^*, v^* \in [u_0, v_0]$, such that $u_n \to u^*$, $v_n \to v^*$ as $n \to \infty$, and $u^* = v^*$. The rest of the proof is similar to that of Theorem 2.1. □

Remark 2.2. Compared with the corresponding results in the literature [13, Theorem 3.3.1, Corollary 3.3.1], it shows that our hypotheses are weaker than that conditions in the following sense: delete the condition that A is condensing mapping; the condition that $\|A(v, u) - A(u, v)\| \leqslant k\|u - v\|$, $0 \leqslant k < 1$, is reduced to that $A(v, u) - A(u, v) \leqslant k(v - u)$, $0 \leqslant k < 1$.

3. About mixed monotone operator with convexity and concavity properties

In this section, we first give both necessary and sufficient conditions for the existence and uniqueness of fixed point of mixed monotone operator. It not only generalizes some known results but also offers a general method to cope with a class of mixed monotone operators with convexity and concavity.

Theorem 3.1. *Let P be a normal cone of a real Banach space E and $A : P \times P \to E$ a mixed monotone operator. Assume that there exists function $w : (0, 1) \times P \times P \to (0, +\infty)$ such that*

(i) *for all $t \in (0, 1)$, $x, y \in P$:*

$$A(tx, y) \geqslant w(t, x, y)A(x, ty); \tag{11}$$

(ii) *for all $t \in (0, 1)$, $x \in P$: $t < w(t, x, x) < 1$, and $w(t, x, x)$ is nonincreasing or nondecreasing in x, and is continuous from left in t.*

Then A has exactly one fixed point $x^* \in P^0$ if and only if for some $u_0, v_0 \in P^0$ with $u_0 \leqslant v_0$ such that $u_0 \leqslant A(u_0, v_0)$ and $A(v_0, u_0) \leqslant v_0$.

Moreover, if sufficient conditions hold, then for any initial point $x_0, y_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Proof. *Necessity*. Assume that x^* is a fixed point of A in P^0. Let $u_0 = v_0 = x^*$, then $u_0 \leqslant A(u_0, v_0)$ and $A(v_0, u_0) \leqslant v_0$.

Sufficiency. Let

$$u_n = A(u_{n-1}, v_{n-1}), \quad v_n = A(v_{n-1}, u_{n-1}), \quad n = 1, 2, \ldots.$$

Thus it is easy to obtain that

$$u_0 \leqslant u_1 \leqslant \cdots \leqslant u_n \leqslant \cdots \leqslant v_n \leqslant \cdots \leqslant v_1 \leqslant v_0.$$

Since $u_0, v_0 \in P^0$, there exists $0 < r_0 \leqslant 1$ such that $u_0 \geqslant r_0 v_0$. If $r_0 = 1$, then u_0 is fixed point of A. Without loss of generality, suppose $0 < r_0 < 1$. Then by condition (i) and the fact that A is a mixed monotone operator, we have

$$u_1 = A(u_0, v_0) \geqslant A(r_0 v_0, v_0) \geqslant w(r_0, v_0, v_0) A(v_0, r_0 v_0) \geqslant w(r_0, v_0, v_0) A(v_0, u_0) = w(r_0, v_0, v_0) v_1.$$

Let $r_1 = w(r_0, v_0, v_0)$, then $1 > r_1 > r_0$ for condition (ii) and $u_1 \geqslant r_1 v_1$.

It is not hard to obtain by induction that

$$u_n \geqslant r_n v_n, \quad n = 1, 2, \ldots,$$

where $r_n = w(r_{n-1}, v_{n-1}, v_{n-1})$.

By condition (ii), we have $0 < r_{n-1} < r_n < 1$, i.e., the sequence $\{r_n\}$ is strictly increasing. Suppose $\lim_{n\to\infty} r_n = r$, then $r = 1$.

Otherwise, if $0 < r < 1$, then we have

$$r_n = w(r_{n-1}, v_{n-1}, v_{n-1}) \geqslant w(r_{n-1}, v_0, v_0)$$

when $w(t, x, x)$ is nonincreasing in x; or

$$r_n = w(r_{n-1}, v_{n-1}, v_{n-1}) \geqslant w(r_{n-1}, u_0, u_0)$$

when $w(t, x, x)$ is nondecreasing in x.

Let $n \to \infty$ in above we get

$$r \geqslant \lim_{s\to r^-} w(s, v_0, v_0) = w(r, v_0, v_0) > r,$$

or

$$r \geqslant \lim_{s\to r^-} w(s, u_0, u_0) = w(r, u_0, u_0) > r,$$

which is a contradiction.

Hence we have $r = 1$.

Therefore, for all $n, p \geqslant 1$, we get

$$\theta \leqslant v_n - u_n \leqslant v_n - r_n v_n = (1 - r_n) v_n \leqslant (1 - r_n) v_0,$$

$$\theta \leqslant u_{n+p} - u_n \leqslant v_n - u_n, \qquad \theta \leqslant v_n - v_{n+p} \leqslant v_n - u_n.$$

By the normality of P, it is easy to see that $v_n - u_n \to \theta$ as $n \to \infty$.

Thus there exists $u^*, v^* \in E$ such that $\|u_n - u^*\| \to 0$ and $\|v_n - v^*\| \to 0$ as $n \to \infty$, and $u^* = v^*$. Let $x^* \triangleq u^* = v^*$, then $u_n \leqslant x^* \leqslant v_n$, $n = 0, 1, 2, \ldots$.

Now we will show that $A(x^*, x^*) = x^*$, i.e., x^* is fixed point of A.

On one hand,

$$A(x^*, x^*) \geqslant A(u_n, v_n) = u_{n+1} \to x^*.$$

On the other hand,

$$A(x^*, x^*) \leqslant A(v_n, u_n) = v_{n+1} \to x^*.$$

Hence $A(x^*, x^*) = x^*$.

Moreover, if the sufficient condition holds, set $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, where $x_0, y_0 \in [u_0, v_0]$.

Notice that A is mixed monotone operator, we can induce that

$$u_n \leqslant x_n \leqslant v_n, \quad u_n \leqslant y_n \leqslant v_n, \quad n = 0, 1, 2, \ldots.$$

Let $n \to \infty$, then $x_n \to x^*$, $y_n \to x^*$ as $n \to \infty$.

Finally, the proof of uniqueness of the fixed point is similar to that proof of Theorem 2.1 and therefore omitted. The proof is over. □

By using the above result, we can obtain a series of new fixed point theorems for mixed monotone operators with certain concavity and convexity. The following result (see [8, Theorem 2.1 and Remark 2.1]) is corollary of Theorem 3.1.

Corollary 3.2. *Let P be a normal cone of a real Banach space E. Let $u_0, v_0 \in E$ with $u_0 \leqslant v_0$ and $A : [u_0, v_0] \times [u_0, v_0] \to E$ be a ϕ concave-$(-\psi)$ convex mixed monotone operator (i.e. A is mixed monotone operator and there exist functions $\phi, \psi : (0, 1] \times [u_0, v_0] \to (0, \infty)$ such that*

(i) *$t < \phi(t, x)\psi(t, x) < 1$ for $(t, x) \in (0, 1] \times [u_0, v_0]$;*
(ii) *for all $t \in (0, 1)$, $x, y \in [u_0, v_0]$: $A(tx, y) \geqslant \phi(t, x)A(x, y)$;*
(iii) *for all $t \in (0, 1)$, $x, y \in [u_0, v_0]$: $A(x, ty) \leqslant \psi(t, y)^{-1}A(x, y)$).*

Suppose that

(h_1) *there exists a real positive number r_0, such that $u_0 \geqslant r_0 v_0$;*
(h_2) *u_0, v_0 are such that $u_0 \leqslant A(u_0, v_0)$, $A(v_0, u_0) \leqslant v_0$;*
(h_3) *$\phi(t, x)\psi(t, x)$ is nonincreasing (nondecreasing) in x, and is continuous from left in t.*

Then A has exactly one fixed point $x^* \in [u_0, v_0]$. Moreover, for any initial point $x_0, y_0 \in [u_0, v_0]$, constructing successively the sequences $x_n = A(x_{n-1}, y_{n-1})$, $y_n = A(y_{n-1}, x_{n-1})$, $n = 1, 2, \ldots$, we have $\|x_n - x^*\| \to 0$ and $\|y_n - x^*\| \to 0$ as $n \to \infty$.

Proof. Let $w(t, x, y) = \phi(t, x) \times \psi(t, y)$. Then it is easy to test all conditions of Theorem 3.1 are satisfied. Therefore, the conclusions of Corollary 3.2 hold. □

Likewise, we have

Corollary 3.3. *Let P be a normal and solid cone of a real Banach space E. $A : P \times P \to P$ is mixed monotone operator. Suppose that*

(i) *there exist $u_0, v_0 \in P^0$ with $u_0 \leqslant v_0$, such that $u_0 \leqslant A(u_0, v_0)$, $A(v_0, u_0) \leqslant v_0$;*
(ii) *for fixed y, $A(\cdot, y) : P \to P$ is concave; for fixed x, $A(x, \cdot) : P \to P$ satisfies*

$$A(x, ty) \leqslant \big[t\big(1 + \eta(t, y)\big)\big]^{-1} A(x, y),$$

where $y \in P$, $0 < t < 1$, $\eta = \eta(t, y) > 0$;
(iii) *$\eta(t, y)$ is nonincreasing (or nondecreasing) in y, and is continuous from left in t, and there exists $\epsilon > 0$, such that $A(\theta, v_0) \geqslant \epsilon A(v_0, u_0)$, for all $(t, y) \in (0, 1) \times [u_0, v_0]$, we have*

$$\big[t + \epsilon(1 - t)\big]^{-1} < \eta(t, y) + 1 < \big[t^2 + \epsilon t(1 - t)\big]^{-1}.$$

Then the conclusion of Corollary 3.2 still holds.

Proof. Let $w(t, x, y) = [t + \epsilon(1 - t)][1 + \eta(t, y)]t$, $t \in (0, 1)$, $x, y \in [u_0, v_0]$.

Then all conditions of Theorem 3.1 are satisfied. Therefore, the conclusions of Corollary 3.3 hold. □

Similarly, the following result can be obtained. The proof is omitted.

Corollary 3.4. *Let P be a normal cone of a real Banach space E. $A : P \times P \to P$ is mixed monotone operator. Suppose that*

(i) *there exist $u_0, v_0 \in P$ with $u_0 \leqslant v_0$ and a real number $r_0 > 0$, such that $u_0 \geqslant r_0 v_0$ and*

$$u_0 \leqslant A(u_0, v_0), \qquad A(v_0, u_0) \leqslant v_0;$$

(ii) *for fixed y, $A(\cdot, y) : P \to P$ is α_1-concave; for fixed x, $A(x, \cdot) : P \to P$ is $(-\alpha_2)$-convex, where $0 \leqslant \alpha_1 + \alpha_2 < 1$.*

Then the conclusion of Corollary 3.2 *still holds.*

4. Applications

In this section, we use some of our results to several examples.

Example 4.1. Consider the following boundary value problem

$$\begin{cases} Lu = -\big(p(x)u'\big)' + q(x)u = f(x,u,u) + \mu g^{\beta}(x,u,u) + \lambda u, \quad x \in [0,1], \\ au(0) - bu'(0) = 0, \quad cu(1) + du'(1) = 0, \end{cases} \tag{12}$$

where $p \in C^1[0,1]$, $q \in C[0,1]$, and for all $x \in [0,1]$, $p(x) > 0$, $q(x) \geqslant 0$, $a, b, c, d, \lambda, \mu \geqslant 0$, $\beta > 1$ and $(a+b)(c+d) > 0$, $f, g : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are continuous functions.

It is easy to known that $u(x) \in C^2[0,1]$ is a solution of (12) if and only if $u(x) \in C[0,1]$ is a solution of the following integral equation:

$$u(x) = \int_0^1 G(x,y)\big[f\big(y, u(y), u(y)\big) + \mu g^{\beta}\big(y, u(y), u(y)\big) + \lambda u(y)\big]\,dy, \quad x \in [0,1], \tag{13}$$

where $G(x,y)$ is corresponding Green function.

Conclusion 4.1. Suppose that $f(x,u,v), g(x,u,v) : [0,1] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ are increasing in u, decreasing in v, and there exists $\alpha \in (0,1)$, such that for all $t \in (0,1)$, $u, v \in [0,\infty)$ we have

$$f\big(x, tu, t^{-1}v\big) \geqslant t^{\alpha} f(x,u,v), \qquad g\big(x, tu, t^{-1}v\big) = t^{\beta} g(x,u,v).$$

Then, when λ, μ are sufficiently small, Eq. (12) has solution in $C^2[0,1]$.

Proof. Select $E = C[0,1]$, $P = \{u \in E \mid u(x) \geqslant 0,\ x \in [0,1]\}$. Then P is normal cone in E. Let $h(x) = \int_0^1 G(x,y)\,dy$, $x \in [0,1]$,

$$B(u,v)(x) = \int_0^1 G(x,y) f\big(y, u(y), v(y)\big)\,dy, \quad u, v \in P_h,$$

$$C(u,v)(x) = \int_0^1 G(x,y) g^{\beta}\big(x, u(y), v(y)\big)\,dy, \quad u, v \in P_h,$$

$$Du(x) = \lambda \int_0^1 G(x,y) u(y)\,dy, \quad u \in P_h.$$

Then

$$A(u,v) = B(u,v) + \mu C(u,v) + D(u), \quad u, v \in P_h.$$

Obviously Eq. (13) is equivalent to the following operator equation

$$A(u,u) = u.$$

It is easy to obtain that $B, C : P_h \times P_h \to P_h$ are mixed monotone operators, and there exists $\eta(t,x,y) = t^{\alpha-1} - 1$ such that

$$B\big(tx, t^{-1}y\big) \geqslant t\big[1 + \eta(t,x,y)\big]B(x,y), \qquad C\big(tx, t^{-1}y\big) = t^{\beta}C(x,y)$$

and that $\inf_{t\in[0,1]} \frac{\eta(t,x,y)}{1-t^{\beta-1}} > 0$. It is not hard to get that there exist $u_0 = t_0 h$, $v_0 = t_0^{-1}h$ is lower–upper solution of operator A, where $1 > t_0 > 0$, $h \in P_h$ is an eigenvector of D. The rest conditions of Theorem 2.2 are obvious. Therefore, Conclusion 4.1 holds by means of Theorem 2.2. □

Example 4.2. Let $E = C_B(R^N)$ denote the set of all bounded continuous functions on R^N. Equipped with the norm $\|x\| = \sup\{|x(t)|\colon t \in R^N\}$, E is a real Banach space. The set $P = C_B^+(R^N)$ of nonnegative functions in $C_B(R^N)$ is a normal and solid cone in $C_B(R^N)$.

Consider the following nonlinear integral equation:

$$x(t) = (Ax)(t) = \int_{R^N} K(t,s)\left[\sqrt{x(s)} + \frac{1}{\sqrt[4]{x(s)}}\right] ds. \tag{14}$$

Conclusion 4.2. Suppose that $K : R^N \times R^N \to R^1$ is nonnegative and continuous with

$$\frac{1}{110} \leqslant \int_{R^N} K(t,s)\,ds \leqslant \frac{1}{1+\sqrt{10}}. \tag{15}$$

Then Eq. (14) has a unique positive solution $x^*(t) \in [0.01, 1]$. Moreover, constructing successively the sequences $x_n(t)$ and $y_n(t)$ $(n = 1, 2, \ldots)$ with

$$x_n(t) = \int_{R^N} K(t,s)\left[\sqrt{x_{n-1}(s)} + \frac{1}{\sqrt[4]{y_{n-1}(s)}}\right] ds$$

and

$$y_n(t) = \int_{R^N} K(t,s)\left[\sqrt{y_{n-1}(s)} + \frac{1}{\sqrt[4]{x_{n-1}(s)}}\right] ds$$

for any initial $(x_0, y_0) \in [0.01, 1] \times [0.01, 1]$, we have $\|x_n(t) - x^*(t)\| \to 0$, $\|y_n(t) - x^*(t)\| \to 0$ as $n \to \infty$.

Proof. Obviously Eq. (14) can be written in the form $x = A(x,x)$, where

$$A(x,y) = A_1(x) + A_2(y),$$

$$A_1(x) = \int_{R^N} K(t,s)\sqrt{x(s)}\,ds, \qquad A_2(y) = \int_{R^N} K(t,s)\frac{1}{\sqrt[4]{y(s)}}\,ds.$$

Select $u_0 = 0.01$, $v_0 = 1$, $\alpha_1 = \frac{1}{2}$, $\alpha_2 = \frac{1}{4}$. It is easy to obtain that $A : P \times P \to P$ is mixed monotone operator, $A(u_0, v_0) = A(0.01, 1) \geqslant u_0$ and $A(v_0, u_0) = A(1, 0.01) \leqslant v_0$. The rest conditions of Corollary 3.4 are all satisfied. Hence, Conclusion 4.2 holds. □

Remark 4.3. In the above examples, the existence of solutions of equations are discussed by using some results of mixed monotone operators. This do produce evidence to show that these results are significant since they have applications to a class of nonlinear differential and integral equations.

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