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Two sufficient conditions in frequency domain for Gabor frames[☆]

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ABSTRACT

Two sufficient conditions for the Gabor system to be a frame for $L^2(\mathbb{R})$ are presented in this note. The conditions proposed are stated in terms of the Fourier transforms of the Gabor system's generating functions. It is also shown that these conditions are better than the known result.

1. Introduction

Frames were first introduced by Duffin and Schaeffer [1] in the context of nonharmonic Fourier series. Outside of this area, this idea seems to have been lost until Daubechies et al. [2] brought attention to it in 1986. They shown that Duffin and Schaeffer's definition was an abstraction of a concept introduced by Gabor [3] in 1946 for doing signal analysis. Today, the frames introduced by Gabor are called Gabor frames or Weyl–Heisenberg frames and play an important role in signal analysis. Within the past two decades, Gabor frame has established itself as a rich and fertile area of mathematical analysis [4–6]. At the same time, it has broad applications in information processing, for instance, one of the most fascinating recent applications of Gabor analysis is in the area of wireless communication [5, Chapter 12].

Gabor systems are generated by modulations and translations of several functions. We are concerned with sufficient conditions for the Gabor system to be a frame for $L^2(\mathbb{R})$. The sufficient conditions in time domain for the Gabor system to be a frame have been known [7–10]. In 2000, Czaja [11] gave characterizations of orthogonal families, tight frames and orthonormal bases of the Gabor systems via Fourier transform. In particular, Daubechies [7] mentioned a sufficient condition in frequency domain for the Gabor system to be a frame (see Theorem 2.1).

In this paper, we will present two new sufficient conditions for Gabor frame via Fourier transform. The conditions we proposed are stated in terms of the Fourier transforms of the Gabor system's generating functions, and the conditions are better than that of Daubechies. Although we consider a one-dimensional case here, our results are easily generalized to multidimensional Gabor systems.

The paper is organized as follows. In Section 2, we introduce some notations and state the main results. Section 3 gives the proofs of the results.

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2. Notations and main results

Before stating the following main results, we need some notations and known results. We use the Fourier transform of the form

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) e^{-2\pi i x\omega} dx. \tag{2.1}$$

Now, the definitions of frame and Gabor frame are listed as follows.

Definition 2.1. Let $\mathbb{H}$ be a separable Hilbert space. A sequence $\{f_i\}_{i\in N}$ of elements of $\mathbb{H}$ is a frame for $\mathbb{H}$ if there exist constants $0 < C \le D < \infty$ such that for all $h \in \mathbb{H}$,

$$C\|h\|^2 \le \sum_{i\in N} |\langle h, f_i\rangle|^2 \le D\|h\|^2. \tag{2.2}$$

The numbers C, D are called the frame lower and upper bounds, respectively (the largest C and smallest D for which (2.2) holds are the optimal frame bounds). The sequence which satisfies only the upper inequality in (2.2) is called a Bessel sequence. A frame is tight if $C = D$. If $C = D = 1$, it is called a Parseval frame.

Definition 2.2. Let $a, b > 0$ and $g^l(x) \in L^2(\mathbb{R})$ $(l = 1, 2, \ldots, L)$. We call the system

$$g^l_{j,k}(x) = e^{2i\pi jbx} g^l(x - ka), \quad j, k \in \mathbb{Z},\ l = 1, 2, \ldots, L \tag{2.3}$$

as a Gabor system generated by the functions g^l. The system $\{g^l_{j,k}(x)\}_{l,j,k}$ is called a Gabor frame if it constitutes a frame for $L^2(\mathbb{R})$.

Let $\{g^l(x) \mid l = 1, 2, \ldots, L\} \subset L^2(\mathbb{R})$ and $a, b > 0$. Set

$$T := \frac{1}{a}, \tag{2.4}$$

$$\Phi_s(\omega) := \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} |\hat{g}^l(\omega - jb)\hat{g}^l(\omega - jb + sT)|, \tag{2.5}$$

$$\alpha_s := \operatorname{ess\,sup}_{\omega} \Phi_s(\omega), \quad s \in \mathbb{Z}, \tag{2.6}$$

$$\lambda := \sum_{s\neq 0} \alpha_s, \tag{2.7}$$

$$\gamma := \operatorname{ess\,inf}_{\omega} \Phi_0(\omega), \tag{2.8}$$

$$I_s(\omega) := \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \hat{g}^l(\omega - jb)\hat{g}^l(\omega - jb + sT), \tag{2.9}$$

$$\beta_s := \operatorname{ess\,sup}_{\omega} |I_s(\omega)|, \quad s \in \mathbb{Z}, \tag{2.10}$$

$$\mu := \sum_{s\neq 0} \beta_s, \tag{2.11}$$

then the following result holds [7].

Theorem 2.1. *Let $\{g^l(x) \mid l = 1, 2, \ldots, L\} \subset L^2(\mathbb{R})$ and $a, b > 0$. If α_0, λ and γ are defined by (2.6)–(2.8), respectively, and they satisfy*

$$\lambda < \gamma \le \alpha_0 < +\infty,$$

then system (2.3) is a Gabor frame of $L^2(\mathbb{R})$ with lower bound C_1 and upper bound C_2 defined by

$$C_1 = \frac{\gamma - \lambda}{a}$$

and

$$C_2 = \frac{\alpha_0 + \lambda}{a},$$

respectively.

Remark 2.1. In [12], we established Theorem 2.1 from the study of shift-invariant spaces.

Now, the first result of the paper is stated as follows.

Theorem 2.2. *Let* $\{g^l(x) \mid l = 1, 2, \ldots, L\} \subset L^2(\mathbb{R})$ *and* $a, b > 0$. *If* α_0, γ *and* μ *satisfy*

$$\mu < \gamma \le \alpha_0 < +\infty, \tag{2.12}$$

then system (2.3) constitutes a Gabor frame for $L^2(\mathbb{R})$ *with the lower bound* C_1 *and upper bound* C_2, *where*

$$C_1 = \frac{\gamma - \mu}{a} \tag{2.13}$$

and

$$C_2 = \frac{\alpha_0 + \mu}{a}. \tag{2.14}$$

Remark 2.2. Obviously, $\mu \le \lambda$, so the frame bounds in Theorem 2.2 are better than ones in Theorem 2.1.

Set

$$\theta := \operatorname{ess\,sup}_{\omega} \sum_{s\in\mathbb{Z}\setminus\{0\}} |I_s(\omega)|, \tag{2.15}$$

then we have the following theorem.

Theorem 2.3. *Let* $\{g^l(x) \mid l = 1, 2, \ldots, L\} \subset L^2(\mathbb{R})$ *and* $a, b > 0$. *If* α_0, λ *and* θ *satisfy the following,*

$$\theta < \gamma \le \alpha_0 < +\infty, \tag{2.16}$$

then system (2.3) constitutes a Gabor frame with the frame lower bound C_1 *and upper bound* C_2 *defined by*

$$C_1 = \frac{\gamma - \theta}{a} \tag{2.17}$$

and

$$C_2 = \frac{\alpha_0 + \theta}{a}, \tag{2.18}$$

respectively.

Remark 2.3. Since $\theta \le \mu$, the frame bounds in Theorem 2.3 are better than ones in Theorem 2.2.

3. The proofs of main results

In order to prove Theorems 2.2 and 2.3, we need two lemmas. First, let Γ be the set of all functions in $L^2(\mathbb{R})$ satisfying

(i) $\|\hat{f}\|_\infty < \infty$;
(ii) there exists a constant K such that $\mathrm{supp}(\hat{f}) \subset [-K, K]$ and $0\bar{\in}\mathrm{supp}(\hat{f})$.

Then we have

Lemma 3.1. Γ *is a dense subset of* $L^2(\mathbb{R})$.

The proof of this result is well known.

Lemma 3.2. *Let* $h \in L^2(\mathbb{R})$ *and* g *be a bounded and compactly supported function. Then the series*

$$\sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} g(\xi)h(\xi)\mathrm{e}^{\mathrm{i}2\pi ka(\xi-\omega)}\mathrm{d}\xi \tag{3.1}$$

is convergent to a periodic function $G(\omega) \in L^2[0, T]$, *where* $G(\omega) := T\sum_{s\in\mathbb{Z}} g(\omega + sT)h(\omega + sT)$ *and* T *is defined by (2.4).*

Proof. By periodization, we have

$$\frac{1}{T}\int_{\mathbb{R}} g(\xi)h(\xi)\mathrm{e}^{\mathrm{i}2\pi ka(\xi-\omega)}\mathrm{d}\xi = \frac{1}{T}\int_0^T \sum_{s\in\mathbb{Z}} g(\xi + sT)h(\xi + sT)\mathrm{e}^{\mathrm{i}2\pi ka(\xi-\omega)}\mathrm{d}\xi. \tag{3.2}$$

Since g is a bounded and compactly supported function, the number of s in the above sum is finite. Thus, the left side of (3.2) is a Fourier coefficient of the function $\frac{1}{T}G(\omega)$ in $L^2([0, T])$. □

Proof of Theorem 2.2. Let $f \in L^2(\mathbb{R})$, then by Lemma 3.1, there exists a function sequence $\{f_m\}_{m=1}^{+\infty} \subset \Gamma$, such that

$$\|\hat{f}_m - \hat{f}\| \to 0, \quad m \to \infty, \tag{3.3}$$

$$\mathrm{supp}\hat{f}_m \subset [-K_m, K_m], \quad 0 \bar{\in} \mathrm{supp}\hat{f}_m. \tag{3.4}$$

For fixed $j \in \mathbb{Z}$, define the functional

$$P_j(h) := \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |\langle h, g_{j,k}^l\rangle|^2 = \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |\langle \hat{h}, \hat{g}_{j,k}^l\rangle|^2, \quad h \in L^2(\mathbb{R}). \tag{3.5}$$

Note that

$$\widehat{g_{j,k}^l}(\omega) = \mathrm{e}^{-2\pi ika(\omega - jb)} \hat{g}^l(\omega - jb), \tag{3.6}$$

hence we have

$$\begin{aligned} P_j(f_m) &= \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} |\langle \hat{f}_m, \hat{g}_{j,k}^l\rangle|^2 = \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} \langle \hat{f}_m, \hat{g}_{j,k}^l\rangle \langle \hat{g}_{j,k}^l, \hat{f}_m\rangle \\ &= \sum_{l=1}^{L} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_m(\xi) \mathrm{e}^{2\pi ika(\xi - jb)} \overline{\hat{g}^l(\xi - jb)} \mathrm{d}\xi \int_{\mathbb{R}} \overline{\hat{f}_m(\omega)} \mathrm{e}^{-2\pi ik(\omega - jb)} \hat{g}^l(\omega - jb) \mathrm{d}\omega. \end{aligned} \tag{3.7}$$

From Lemma 3.2, we obtain

$$P_j(f_m) = \sum_{l=1}^{L} \sum_{s\in\mathbb{Z}} T \int_{\mathbb{R}} \hat{f}_m(\omega + sT) \overline{\hat{g}^l(\omega - jb + sT)} \overline{\hat{f}_m(\omega)} \hat{g}^l(\omega - jb) \mathrm{d}\omega. \tag{3.8}$$

Let

$$P(f) = \sum_{l=1}^{L} \sum_{j,k\in\mathbb{Z}} |\langle f, g_{j,k}^l\rangle|^2 = \sum_{j\in\mathbb{Z}} P_j(f),$$

then

$$P(f_m) = \frac{1}{a} \sum_{l=1}^{L} \sum_{j,s\in\mathbb{Z}} \int_{\mathbb{R}} \hat{f}_m(\omega + sT) \overline{\hat{g}^l(\omega - jb + sT)} \overline{\hat{f}_m(\omega)} \hat{g}^l(\omega - jb) \mathrm{d}\omega. \tag{3.9}$$

Dividing $s \in \mathbb{Z}$ into $s = 0$ and $s \neq 0$, we can rewrite $P(f_m)$ as

$$P(f_m) = Q_1(f_m) + Q_2(f_m), \tag{3.10}$$

where

$$Q_1(f_m) := \frac{1}{a} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \int_{\mathbb{R}} |\hat{f}_m(\omega) \hat{g}^l(\omega - jb)|^2 \mathrm{d}\omega, \tag{3.11}$$

$$Q_2(f_m) := \frac{1}{a} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{s\in\mathbb{Z}\setminus\{0\}} \int_{\mathbb{R}} \hat{f}_m(\omega + sT) \overline{\hat{g}^l(\omega - jb + sT)} \overline{\hat{f}_m(\omega)} \hat{g}^l(\omega - jb) \mathrm{d}\omega. \tag{3.12}$$

Since $\alpha_0 < \infty$, the series $Q_1(f_m)$ is convergent and

$$\frac{\gamma}{a} \|\hat{f}_m\|^2 \le Q_1(f_m) \le \frac{\alpha_0}{a} \|\hat{f}_m\|^2, \tag{3.13}$$

that is

$$\frac{\gamma}{a} \|f_m\|^2 \le Q_1(f_m) \le \frac{\alpha_0}{a} \|f_m\|^2. \tag{3.14}$$

Let

$$Q_2^*(f_m) := \frac{1}{a} \sum_{l=1}^{L} \sum_{j\in\mathbb{Z}} \sum_{s\in\mathbb{Z}\setminus\{0\}} \left| \int_{\mathbb{R}} \hat{f}_m(\omega + sT) \overline{\hat{g}^l(\omega - jb + sT)} \overline{\hat{f}_m(\omega)} \hat{g}^l(\omega - jb) \mathrm{d}\omega \right|, \tag{3.15}$$

then from the fact that

$$|\hat{g}^l(\omega - jb + sT)\hat{g}^l(\omega - jb)| \le \frac{1}{2}(|\hat{g}^l(\omega - jb + sT)|^2 + |\hat{g}^l(\omega - jb)|^2), \tag{3.16}$$

we have

$$Q_2^*(f_m) \le \frac{1}{a}\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{s\neq 0}\int_{\mathbb{R}} |\hat{f}_m(\omega + jb + sT)\overline{\hat{f}_m(\omega + jb)}|\; |\hat{g}^l(\omega)|^2 \mathrm{d}\omega. \tag{3.17}$$

By the definition of Γ, we get

$$Q_2^*(f_m) \le C\|\hat{f}_m\|_\infty^2 \sum_{l=1}^{L}\int_{E_m} |\hat{g}^l(\omega)|^2 \mathrm{d}\omega, \tag{3.18}$$

where $E_m = [-K_m, K_m]$, $0\bar{\in}E_m$ and C is a constant. Thus, we obtain

$$Q_2^*(f_m) \le C\|\hat{f}_m\|_\infty^2 \sum_{l=1}^{L} \|g^l\|^2 < \infty. \tag{3.19}$$

This shows that $Q_2^*(f_m)$ is convergent, and consequently $Q_2(f_m)$ is absolutely convergent. Then by the Cauchy–Schwarz inequality,

$$\begin{aligned} |Q_2(f_m)| &= \left|\frac{1}{a}\sum_{s\neq 0}\int_{\mathbb{R}} \hat{f}_m(\omega + sT)\overline{\hat{f}_m(\omega)} I_s(\omega)\mathrm{d}\omega\right| \\ &\le \frac{1}{a}\sum_{s\neq 0}\int_{\mathbb{R}} (|\hat{f}_m(\omega + sT)||I_s(\omega)|^{\frac{1}{2}})(|\hat{f}_m(\omega)||I_s(\omega)|^{\frac{1}{2}})\mathrm{d}\omega \\ &\le \frac{1}{a}\sum_{s\neq 0}\left(\int_{\mathbb{R}} |\hat{f}_m(\omega + sT)|^2|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}} |\hat{f}_m(\omega)|^2|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}}. \end{aligned} \tag{3.20}$$

Note that

$$\int_{\mathbb{R}} |\hat{f}_m(\omega + sT)|^2|I_s(\omega)|\mathrm{d}\omega = \int_{\mathbb{R}} |\hat{f}_m(\omega)|^2|I_s(\omega - sT)|\mathrm{d}\omega$$

and

$$\begin{aligned} I_s(\omega - sT) &= \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \hat{g}^l(\omega - jb + sT)\overline{\hat{g}^l}(\omega - jb) \\ &= \overline{\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}} \overline{\hat{g}^l(\omega - jb + sT)}\hat{g}^l(\omega - jb)} \\ &= I_{-s}(\omega), \end{aligned} \tag{3.21}$$

hence, by $\beta_s = \beta_{-s}$, we have

$$|Q_2(f_m)| \le \frac{1}{a}\int_{\mathbb{R}} |\hat{f}_m(\omega)|^2 \sum_{s\neq 0}\beta_s \mathrm{d}\omega = \frac{\mu}{a}\|f_m\|^2,$$

that is to say,

$$\frac{-\mu}{a}\|f_m\|^2 \le Q_2(f_m) \le \frac{\mu}{a}\|f_m\|^2. \tag{3.22}$$

It follows from (3.14) and (3.22) that

$$\frac{\gamma - \mu}{a}\|f_m\|^2 \le P(f_m) \le \frac{\alpha_0 + \mu}{a}\|f_m\|^2. \tag{3.23}$$

Let $m \to +\infty$ in (3.23), then

$$\frac{\gamma - \mu}{a}\|f\|^2 \le P(f) \le \frac{\alpha_0 + \mu}{a}\|f\|^2.$$

Therefore, system (2.3) is a Gabor frame of $L^2(\mathbb{R})$ with the frame lower bound C_1 defined by (2.13) and the upper bound C_2 defined by (2.14), respectively. This completes the proof. □

Proof of Theorem 2.3. Similar to the proof of Theorem 2.2, (3.11)–(3.14) and (3.20) hold. It follows from (3.20), the Cauchy–Schwarz inequality and (3.21) that

$$\begin{aligned} |Q_2(f_m)| &\leq \frac{1}{a}\left(\sum_{s\neq 0}\int_{\mathbb{R}}|\hat{f}_m(\omega+sT)|^2|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\sum_{s\neq 0}\int_{\mathbb{R}}|\hat{f}_m(\omega)|^2|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\sum_{s\neq 0}\int_{\mathbb{R}}|\hat{f}_m(\omega)|^2|I_s(\omega-sT)|\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\sum_{s\neq 0}\int_{\mathbb{R}}|\hat{f}_m(\omega)|^2|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &= \frac{1}{a}\left(\int_{\mathbb{R}}|\hat{f}_m(\omega)|^2\sum_{s\neq 0}|I_{-s}(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}|\hat{f}_m(\omega)|^2\sum_{s\neq 0}|I_s(\omega)|\mathrm{d}\omega\right)^{\frac{1}{2}} \\ &\leq \frac{\theta}{a}\|f_m\|^2, \end{aligned} \tag{3.24}$$

which implies that

$$\frac{-\theta}{a}\|f_m\|^2 \leq Q_2(f_m) \leq \frac{\theta}{a}\|f_m\|^2. \tag{3.25}$$

Combining (3.14) with (3.25), we get

$$\frac{\gamma-\theta}{a}\|f_m\|^2 \leq P(f_m) \leq \frac{\alpha_0+\theta}{a}\|f_m\|^2. \tag{3.26}$$

Let $m \to +\infty$ in (3.26), then we have

$$\frac{\gamma-\theta}{a}\|f\|^2 \leq P(f) \leq \frac{\alpha_0+\theta}{a}\|f\|^2.$$

The proof is completed. □

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