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Multidimensional generalized automatic sequences and shape-symmetric morphic words

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ABSTRACT

An infinite word is S-automatic if, for all $n \geq 0$, its $(n+1)$th letter is the output of a deterministic automaton fed with the representation of n in the numeration system S. In this paper, we consider an analogous definition in a multidimensional setting and study its relation to the shape-symmetric infinite words introduced by Arnaud Maes. More precisely, for $d \geq 1$, we show that a multidimensional infinite word $x: \mathbb{N}^d \to \Sigma$ over a finite alphabet Σ is S-automatic for some abstract numeration system S built on a regular language containing the empty word if and only if x is the image by a coding of a shape-symmetric infinite word.

1. Introduction

Let $k \geq 2$. An infinite word $x = (x_n)_{n\geq 0}$ is *k-automatic*if for all $n \geq 0$, x_n is obtained by feeding a deterministic finite automaton with output (DFAO for short) with the k-ary representation of n. In his seminal paper [4], Cobham showed that an infinite word is *k-automatic* if and only if it is the image by a coding of a fixed point of a uniform morphism of constant length k.

If we relax the assumption on the uniformity of the morphism, Cobham's result still holds but k-ary systems are replaced by a wider class of numeration systems, the so-called *abstract numeration systems* [6,13,12]. If an abstract numeration system is denoted by S, the corresponding sequences that can be generated are said to be *S-automatic*. That is, the $(n+1)$th element of such a sequence is obtained by feeding a DFAO with the representation of n in the particular abstract numeration system S.

In the vein of Arnaud Maes' thesis, this paper studies the relationship between sequences generated by automata and sequences generated by morphisms, but extended to the framework of multidimensional infinite words, i.e., maps from $\mathbb{N}^d$ to some finite alphabet Σ. For instance, k-automatic sequences have been generalized either by considering d-tuples of k-ary representations given to a suitable DFAO or by iterating morphisms for which images of letters are d-dimensional cubes of constant size; see [14,11] for questions related to frequencies of letters. In [13] multidimensional S-automatic sequences have been introduced mimicking O. Salon's construction. Let us also mention [2] where a different notion of bidimensional morphism is introduced in connection with problems arising in discrete geometry. In [5] bidimensional S-automatic sequences turn out to be useful in the context of combinatorial game theory. They play a central role to get new characterizations of P-positions for the famous Wythoff game and some of its variations. Another motivation for studying the set of multidimensional S-automatic words w over $\{0, 1\}$ is to consider them as characteristic words of subsets P_w of $\mathbb{N}^d$, to extend the structure $\langle \mathbb{N}; < \rangle$ by the corresponding predicates P_w and to study the decidability of the corresponding first-order theory. Also see [3] for the relationship with second-order monadic theory.

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Our main result in this paper can be precisely stated as follows.

Theorem. *Let $d \geq 1$. The d-dimensional infinite word x is S-automatic for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ if and only if x is the image by a coding of a shape-symmetric d-dimensional infinite word.*

Our first task is to define the different concepts occurring in this statement.

1.1. Abstract numeration systems

If Σ is a finite alphabet, Σ^* denotes the free monoid generated by Σ having concatenation of words as product and the empty word ε as neutral element. If $w = w_0 \cdots w_{\ell-1}$ is a word, $\ell \geq 0$, where the w_j's are letters, then $|w|$ denotes its length ℓ. Let $(\Sigma, <)$ be a totally ordered alphabet and u, v be two words over Σ. We say that u is *genealogically less* than v, and we write $u \prec v$, if either $|u| < |v|$ (i.e., u is shorter than v) or $|u| = |v|$ and there exist $p, s, t \in \Sigma^*$, $a, b \in \Sigma$ such that $u = pas$, $v = pbt$ and $a < b$ (i.e., u is lexicographically less than v). Note that in the literature, genealogical order is also called radix order. Also let us mention that we have adopted the convention that all words and arrays have indices starting from 0.

Definition 1. An *abstract numeration system* [6] is a triple $S = (L, \Sigma, <)$ where L is an infinite regular language over a totally ordered finite alphabet $(\Sigma, <)$. Enumerating the words of L using the genealogical ordering $\prec$ induced by the ordering $<$ of Σ gives a one-to-one correspondence $\mathrm{rep}_S\colon \mathbb{N} \to L$ mapping the non-negative integer n onto the $(n + 1)$th word in L. In particular, 0 is sent onto the first word in the genealogically ordered language L. The inverse map of rep_S is the S-value map denoted by $\mathrm{val}_S\colon L \to \mathbb{N}$.

Example 2. Consider the alphabet $\Sigma = \{a, b\}$ with $a < b$. The first few words in the regular language $L = \{a, ba\}^*\{\varepsilon, b\}$ are $\varepsilon, a, b, aa, ab, ba, aaa, aab$. For $S = (L, \Sigma, <)$, we have, for instance, $\mathrm{val}_S(b) = 2$ and $\mathrm{rep}_S(5) = ba$.

Remark 3. Any positional numeration system built on a strictly increasing sequence $(U_n)_{n\geq 0}$ of integers such that $U_0 = 1$ gives an abstract numeration system whenever $\mathbb{N}$ is U-recognizable, i.e., whenever the set of greedy representations of the non-negative integers in terms of the sequence $(U_n)_{n\geq 0}$ is regular.

Any regular language is accepted by a deterministic finite automaton, which is defined as follows. A *deterministic finite automaton* $\mathcal{A}$ (DFA for short) is given by $\mathcal{A} = (Q, q_0, \Sigma, \delta, F)$ where Q is the finite *set of states*, $q_0 \in Q$ is the *initial state*, $\delta\colon Q \times \Sigma \to Q$ is the *transition function* and $F \subseteq Q$ is the *set of final states*. The function δ can be extended to $Q \times \Sigma^*$ by $\delta(q, \varepsilon) = q$ for all $q \in Q$ and $\delta(q, aw) = \delta(\delta(q, a), w)$ for all $q \in Q$, $a \in \Sigma$ and $w \in \Sigma^*$. A word $w \in \Sigma^*$ is *accepted by* $\mathcal{A}$ if $\delta(q_0, w) \in F$. The *language accepted by* $\mathcal{A}$ is the set of the accepted words. A *deterministic finite automaton with output* (DFAO for short) $\mathcal{B} = (Q, q_0, \Sigma, \delta, \Gamma, \tau)$ is defined analogously where Γ is the *output alphabet* and $\tau\colon Q \to \Gamma$ is the *output function*. The *output corresponding to the input* $w \in \Sigma^*$ is $\tau(\delta(q_0, w))$.

1.2. S-automatic multidimensional infinite words

Let $d \geq 1$. To work with d-tuples of words of the same length, we introduce the following map.

Definition 4. If $w_1, \ldots, w_d$ are finite words over the alphabet Σ, the *padding map* $(\cdot)^\#\colon (\Sigma^*)^d \to ((\Sigma \cup \{\#\})^d)^*$ is defined as

$$(w_1, \ldots, w_d)^\# := (\#^{m-|w_1|}w_1, \ldots, \#^{m-|w_d|}w_d)$$

where $m = \max\{|w_1|, \ldots, |w_d|\}$.

As an example, $(ab, bbaa)^\# = (\#\#ab, bbaa)$. In what follows, we use the notation $\Sigma_\#$ as a shorthand for $\Sigma \cup \{\#\}$.

Definition 5. A *d-dimensional infinite word* over the alphabet Γ is a map $x\colon \mathbb{N}^d \to \Gamma$. We use notation like $x_{n_1,\ldots,n_d}$ or $x(n_1, \ldots, n_d)$ to denote the value of x at $(n_1, \ldots, n_d)$. Such a word is said to be *S-automatic* for an abstract numeration system $S = (L, \Sigma, <)$ if there exists a deterministic finite automaton with output $\mathcal{A} = (Q, q_0, (\Sigma_\#)^d, \delta, \Gamma, \tau)$ such that, for all $n_1, \ldots, n_d \geq 0$,

$$\tau(\delta(q_0, (\mathrm{rep}_S(n_1), \ldots, \mathrm{rep}_S(n_d))^\#)) = x_{n_1,\ldots,n_d}.$$

In this case, we say that the DFAO $\mathcal{A}$ generates the infinite word x. The notion of an S-automatic infinite word was introduced in [13] (also see [10]) as a natural generalization of the multidimensional k-automatic sequences introduced in [14].

Example 6. Consider the abstract numeration system introduced in Example 2, $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO depicted in Fig. 1. Since this automaton is fed with entries of the form $(\mathrm{rep}_S(n_1), \mathrm{rep}_S(n_2))^\#$, we do not consider the transitions on input $(\#, \#)$. If the outputs of the DFAO are considered to be the states themselves, then the DFAO generates the bidimensional infinite S-automatic word given in Fig. 2.

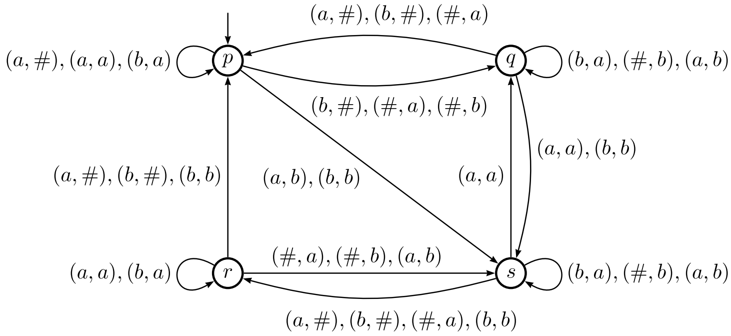

Fig. 1. A deterministic finite automaton with output.

	ε	a	b	aa	ab	ba	aaa	aab	$\cdots$
ε	p	q	q	p	q	p	q	q	$\cdots$
a	p	p	s	s	q	s	p	s	
b	q	p	s	q	s	q	p	s	
aa	p	p	s	p	s	q	q	s	
ab	q	p	s	p	s	s	s	r	
ba	p	s	q	p	s	q	s	q	
aaa	p	p	s	p	s	q	p	s	
aab	q	p	s	p	s	s	p	s	
$\vdots$	$\vdots$								$\ddots$

Fig. 2. A bidimensional infinite S-automatic word.

1.3. Multidimensional morphism

This section is given for the sake of completeness and is intended to present the notion of a multidimensional morphism.

Let d be a positive integer. It is fixed for the whole section. If $i \leq j$ are integers, then $[\![i, j]\!]$ denotes the interval of integers $\{i, i+1, \dots, j\}$. For all $\mathbf{n} \in \mathbb{N}^d$ and $i \in [\![1, d]\!]$, we let n_i denote the ith component of $\mathbf{n}$ and $\mathbf{n}_{\hat{i}}$ denote the $(d-1)$-tuple $(n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_d)$ in $\mathbb{N}^{d-1}$. Let $\mathbf{m}$ and $\mathbf{n}$ be two d-tuples in $\mathbb{N}^d$. We write $\mathbf{m} \leq \mathbf{n}$ (resp. $\mathbf{m} < \mathbf{n}$), if $m_i \leq n_i$ (resp. $m_i < n_i$) for all $i = 1, \dots, d$. In particular, we set $\mathbf{0} := (0, \dots, 0)$ and $\mathbf{1} := (1, \dots, 1)$.

Definition 7. Let $s_1, \dots, s_d$ be positive integers or ∞. A *d-dimensional array* over the alphabet Σ is a map x with domain $[\![0, s_1-1]\!] \times \cdots \times [\![0, s_d-1]\!]$ taking values in Σ. By convention, if we have $s_i = \infty$ for some i, then we set $[\![0, s_i-1]\!] = \mathbb{N}$. If x is such an array, we write $|x|$ for the d-tuple $(s_1, \dots, s_d) \in (\mathbb{N} \cup \{\infty\})^d$, which is called the *shape* of x. We let ε_d denote the d-dimensional array of shape $\mathbf{0}$. Note that we have $\varepsilon_1 = \varepsilon$. If $\mathbf{n} = (n_1, \dots, n_d)$ belongs to the domain of x, we indifferently use the notation $x_{n_1,\dots,n_d}$, $x_{\mathbf{n}}$, $x(n_1, \dots, n_d)$ or $x(\mathbf{n})$. A d-dimensional array x is said to be *bounded* if we have $|x|_i < \infty$ for all $i \in [\![1, d]\!]$. The set of d-dimensional bounded arrays over Σ is denoted by $B_d(\Sigma)$. A bounded array x is a *square* of *size* $c \in \mathbb{N}$ if $|x| = (c, \dots, c)$.

Definition 8. Let x be a d-dimensional array. If we have $\mathbf{0} \leq \mathbf{s} \leq \mathbf{t} \leq |x| - \mathbf{1}$, then $x[\mathbf{s}, \mathbf{t}]$ is said to be a *factor* of x and is defined as the array y of shape $\mathbf{t} - \mathbf{s} + \mathbf{1}$ given by $y(\mathbf{n}) = x(\mathbf{n} + \mathbf{s})$ for all $\mathbf{n} \in \mathbb{N}^d$ such that $\mathbf{n} \leq \mathbf{t} - \mathbf{s}$. For any $\mathbf{u} \in \mathbb{N}^d$, the set of factors of x of shape $\mathbf{u}$ is denoted by $\mathrm{Fact}_{\mathbf{u}}(x)$.

Example 9. Consider the bidimensional (bounded) array of shape $(2, 5)$,

$$x = \begin{array}{|c|c|c|c|c|}\hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline\end{array}\,.$$

We have

$$x[(0,0),(1,1)] = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad \text{and} \quad x[(0,2),(1,4)] = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline\end{array}\,.$$

For instance, $\mathrm{Fact}_{\mathbf{1}}(x) = \{a, b, c, d\}$ and

$$\mathrm{Fact}_{(2,3)}(x) = \left\{ \begin{array}{|c|c|c|}\hline a & b & a \\ \hline c & d & b \\ \hline\end{array}\,, \begin{array}{|c|c|c|}\hline b & a & a \\ \hline d & b & c \\ \hline\end{array}\,, \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline\end{array} \right\}.$$

Definition 10. Let x be a d-dimensional array of shape $\mathbf{s} = (s_1, \ldots, s_d)$. For all $i \in [\![1, d]\!]$ and $k < s_i$, we let $x_{|i,k}$ denote the $(d-1)$-dimensional array of shape

$$|x|_{\widehat{i}} = \mathbf{s}_{\widehat{i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$$

defined by setting the ith coordinate equal to k in x, that is,

$$x_{|i,k}(n_1, \ldots, n_{i-1}, n_{i+1}, \ldots, n_d) = x(n_1, \ldots, n_{i-1}, k, n_{i+1}, \ldots, n_d)$$

for all $n_j \in [\![0, s_j - 1]\!]$ with $j \in [\![1, d]\!] \setminus \{i\}$.

Definition 11. Let x, y be two d-dimensional arrays. If for some $i \in [\![1, d]\!]$, $|x|_{\widehat{i}} = |y|_{\widehat{i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_d)$, then we define the *concatenation* of x and y *in the direction* i to be the d-dimensional array $x \odot^i y$ of shape

$$(s_1, \ldots, s_{i-1}, |x|_i + |y|_i, s_{i+1}, \ldots, s_d)$$

satisfying

(i) $x = (x \odot^i y)[\mathbf{0}, |x| - \mathbf{1}]$
(ii) $y = (x \odot^i y)[(0, \ldots, 0, |x|_i, 0, \ldots, 0), (0, \ldots, 0, |x|_i, 0, \ldots, 0) + |y| - \mathbf{1}]$.

The d-dimensional empty word ε_d is a word of shape $\mathbf{0}$. We extend the definition to the concatenation of ε_d and any d-dimensional word x in the direction $i \in [\![1, d]\!]$ by

$$\varepsilon_d \odot^i x = x \odot^i \varepsilon_d = x.$$

In particular, $\varepsilon_d \odot^i \varepsilon_d = \varepsilon_d$.

Example 12. Consider the two bidimensional arrays

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array} \quad \text{and} \quad y = \begin{array}{|c|c|c|}\hline a & a & b \\ \hline b & c & d \\ \hline\end{array}$$

of shape respectively $|x| = (2, 2)$ and $|y| = (2, 3)$. Since $|x|_{\widehat{2}} = |y|_{\widehat{2}} = 2$, we get

$$x \odot^2 y = \begin{array}{|c|c|c|c|c|}\hline a & b & a & a & b \\ \hline c & d & b & c & d \\ \hline\end{array}.$$

However $x \odot^1 y$ is not defined because $2 = |x|_{\widehat{1}} \neq |y|_{\widehat{1}} = 3$.

Let x be a d-dimensional array and $\mu\colon \Sigma \to B_d(\Sigma)$ be a map. Note that μ cannot necessarily be extended to a morphism on Σ^*. Indeed, if x is an array over Σ, $\mu(x)$ is not always well defined. Depending on the shapes of the images by μ of the letters in Σ, when trying to build $\mu(x)$ by concatenating the images $\mu(x_{\mathbf{i}})$ we can obtain "holes" or "overlaps". Therefore, we introduce some restrictions on μ.

Definition 13. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a map and x be a d-dimensional array such that

$$\forall i \in [\![1, d]\!], \forall k < |x|_i, \forall a, b \in \mathrm{Fact}_{\mathbf{1}}(x_{|i,k}) : |\mu(a)|_i = |\mu(b)|_i. \tag{1}$$

Then the image of x by μ is the d-dimensional array defined as

$$\mu(x) = \odot^1_{0 \le n_1 < |x|_1} \left(\cdots \left(\odot^d_{0 \le n_d < |x|_d} \mu(x(n_1, \ldots, n_d)) \right) \cdots \right).$$

Note that the ordering of the products in the different directions is unimportant.

Example 14. Consider the map μ given by

$$a \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline b & d \\ \hline\end{array}, b \mapsto \begin{array}{|c|}\hline c \\ \hline b \\ \hline\end{array}, c \mapsto \begin{array}{|c|c|}\hline a & a \\ \hline\end{array}, d \mapsto \begin{array}{|c|}\hline d \\ \hline\end{array}.$$

Let

$$x = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array}.$$

Since $|\mu(a)|_1 = |\mu(b)|_1 = 2$, $|\mu(c)|_1 = |\mu(d)|_1 = 1$, $|\mu(a)|_2 = |\mu(c)|_2 = 2$ and $|\mu(b)|_2 = |\mu(d)|_2 = 1$, $\mu(x)$ is well defined and given by

$$\mu(x) = \begin{array}{|c|c|c|}\hline a & a & c \\ \hline b & d & b \\ \hline a & a & d \\ \hline\end{array}.$$

However, $\mu^2(x)$ is not well defined.

Definition 15. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a map. If for all $a \in \Sigma$ and all $n \geq 1$, $\mu^n(a)$ is inductively well defined from $\mu^{n-1}(a)$, that is, $\mu^{n-1}(a)$ satisfies (1), then μ is said to be a *d-dimensional morphism*.

The usual notion of a *prolongable morphism* can also be given in this multidimensional setting.

Definition 16. Let μ be a d-dimensional morphism and a be a letter such that $(\mu(a))_{\mathbf{0}} = a$. We say that μ is *prolongable on a*. Then the limit

$$w = \mu^\omega(a) := \lim_{n\to+\infty} \mu^n(a)$$

is well defined and $w = \mu(w)$ is a *fixed point* of μ. A d-dimensional infinite word x over Σ is said to be *pure morphic* if it is a fixed point of a d-dimensional morphism. It is said to be *morphic* if there exists a coding $\nu\colon \Gamma^* \to \Sigma^*$ (i.e., a letter-to-letter morphism) such that $x = \nu(y)$ for some pure morphic word y over Γ.

Note that if a d-dimensional infinite word w is a fixed point of a d-dimensional morphism μ, then (1) implies

$$\forall i \in [\![1, d]\!], \forall k \in \mathbb{N}, \forall a, b \in \mathrm{Fact}_{\mathbf{1}}(w_{|i,k}) : |\mu(a)|_i = |\mu(b)|_i.$$

1.4. Shape-symmetric morphic words

The property of shape-symmetry was first introduced by A. Maes and was used mainly in connection with logical questions about the decidability of first-order theories where $\langle \mathbb{N}; <\rangle$ is extended by some morphic predicate [7,9,8]. This property is a natural generalization of uniform morphisms where all images are squares of the same size [14]. Let $d \geq 2$ be an integer.

Definition 17. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. If the images $\mu(x_{n,\ldots,n})$, for all $n \in \mathbb{N}$, of the letters on the diagonal of x are squares, then x is said to be *shape-symmetric* (with respect to μ).

Remark 18. Two equivalent formulations of shape-symmetry are given as follows. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. This word is shape-symmetric if and only if

$$\forall i, j \leq d, \forall k \in \mathbb{N}, \forall a \in \mathrm{Fact}_{\mathbf{1}}(x_{|i,k}), \forall b \in \mathrm{Fact}_{\mathbf{1}}(x_{|j,k}) : |\mu(a)|_i = |\mu(b)|_j$$

or, if and only if for any permutation f of $[\![1, d]\!]$, we have, for all $n_1, \ldots, n_d \geq 0$,

$$|\mu(x(n_1, \ldots, n_d))| = (s_1, \ldots, s_d) \Rightarrow |\mu(x(n_{f(1)}, \ldots, n_{f(d)}))| = (s_{f(1)}, \ldots, s_{f(d)}).$$

Remark 19. A. Maes showed that determining whether or not a map $\mu\colon \Sigma \to B_d(\Sigma)$ is a d-dimensional morphism is a decidable problem. Moreover he showed that if μ is prolongable on a letter a, then it is decidable whether or not the fixed point $\mu^\omega(a)$ is shape-symmetric [7–9].

Example 20. The following morphism has a fixed point $\mu^\omega(a)$ which is shape-symmetric.

$$\mu(a) = \mu(f) = \begin{array}{|c|c|}\hline a & b \\ \hline c & d \\ \hline\end{array},\ \mu(b) = \begin{array}{|c|}\hline e \\ \hline c \\ \hline\end{array},\ \mu(c) = \begin{array}{|c|c|}\hline e & b \\ \hline\end{array},\ \mu(d) = \begin{array}{|c|}\hline f \\ \hline\end{array},\ \mu(e) = \begin{array}{|c|c|}\hline e & b \\ \hline g & d \\ \hline\end{array},$$

$$\mu(g) = \begin{array}{|c|c|}\hline h & b \\ \hline\end{array},\ \mu(h) = \begin{array}{|c|c|}\hline h & b \\ \hline c & d \\ \hline\end{array}.$$

In Fig. 3 we have represented the beginning of the array. Some elements are underlined for the use of Example 34.

Definition 21. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. The *shape sequence* of x with respect to μ *in the direction* $i \in [\![1, d]\!]$ is the sequence

$$\mathrm{Shape}_{\mu,i}(x) = (|\mu(x_{|i,k})|_i)_{k\geq 0}.$$

For a unidimensional morphism μ having the infinite word $x = x_0x_1x_2\cdots$ as a fixed point, the *shape sequence* of x with respect to μ is $\mathrm{Shape}_\mu(x) = (|\mu(x_k)|)_{k\geq 0}$.

Remark 22. Let $\mu\colon \Sigma \to B_d(\Sigma)$ be a d-dimensional morphism having the d-dimensional infinite word x as a fixed point. Note that x is shape-symmetric if and only if

$$\mathrm{Shape}_{\mu,1}(x) = \cdots = \mathrm{Shape}_{\mu,d}(x).$$

$\underline{a}$	$\underline{b}$	e	e	b	e	b	e	e	b	$\cdots$
c	d	$\underline{c}$	g	d	g	d	c	g	d	
e	b	f	e	$\underline{b}$	h	b	f	h	b	
e	b	e	a	b	e	b	e	h	b	
g	d	c	c	d	g	d	$\underline{c}$	c	d	
e	b	e	e	b	a	b	e	e	b	
g	d	c	g	d	c	d	c	g	d	
h	b	f	e	b	e	b	f	h	b	
e	b	e	e	b	e	b	e	a	b	
g	d	c	g	d	g	d	c	c	d	
$\vdots$										$\ddots$

Fig. 3. A fixed point of μ.

$$x' = y \odot^2 z = \begin{array}{|c|c|c|c|c|}\hline a & b & a & e & a \\ \hline e & e & e & e & e \\ \hline a & a & e & e & b \\ \hline e & e & a & b & b \\ \hline b & a & b & e & a \\ \hline \end{array} \quad \text{and} \quad x'' = y' \odot^1 z' = \begin{array}{|c|c|c|c|c|}\hline a & b & a & e & a \\ \hline a & a & e & e & b \\ \hline e & e & a & b & b \\ \hline b & a & b & e & a \\ \hline \end{array}.$$

Fig. 4. The successive e-erased arrays from x.

1.5. Erasing hyperplanes from multidimensional arrays

This short section is quite technical and it is only needed in the second part of the proof of Theorem 25. Here we define how to erase hyperplanes from a multidimensional array.

Definition 23. Let $d \geq 2$ be an integer. Let x be a d-dimensional array of shape $(s_1, s_2, \ldots, s_d)$ over $\Sigma \cup \{e\}$, where e does not belong to Σ. A $(d-1)$-dimensional array $x_{|i,k}$ is called an *e-hyperplane* of x if each letter in $x_{|i,k}$ is equal to e. *Erasing* an e-hyperplane $x_{|i,k}$ of x means replacing x with a d-dimensional array $x' = y \odot^i z$, where

$$y = \begin{cases} x[\mathbf{0}, (s_1, \ldots, s_{i-1}, k, s_{i+1}, \ldots, s_d) - \mathbf{1}], & \text{if } k \geq 1; \\ \varepsilon_d, & \text{otherwise} \end{cases}$$

and

$$z = \begin{cases} x[(0, \ldots, 0, k+1, 0, \ldots, 0), |x| - \mathbf{1}], & \text{if } k < s_i - 1; \\ \varepsilon_d, & \text{otherwise.} \end{cases}$$

We let ρ_e denote the map which associates with any d-dimensional array x over $\Sigma \cup \{e\}$ the array $\rho_e(x)$ obtained by erasing iteratively every e-hyperplane of x. Moreover, we say that x is *e-erasable* if the array $\rho_e(x)$ does not contain the letter e as a factor. In other words, for each position $\mathbf{n}$ such that $x_{\mathbf{n}} = e$, there exists an integer $i \in [\![1, d]\!]$ such that $x_{|i,n_i}$ is an e-hyperplane.

Example 24. Consider the bidimensional array

$$x = \begin{array}{|c|c|c|c|c|c|}\hline a & b & a & e & e & a \\ \hline e & e & e & e & e & e \\ \hline a & a & e & e & e & b \\ \hline e & e & a & e & b & b \\ \hline b & a & b & e & e & a \\ \hline \end{array}$$

of shape $(5, 6)$. Clearly, $x_{|2,3}$ is a e-hyperplane. By erasing $x_{|2,3}$ from x, we obtain the bidimensional array $x' = y \odot^2 z$ of shape $(5, 5)$, where $y = x[(0, 0), (4, 2)]$ and $z = x[(0, 4), (4, 5)]$. Then $x'_{|1,1}$ is a e-hyperplane of x'. By erasing $x'_{|1,1}$ from x', we obtain the bidimensional array $x'' = y' \odot^1 z'$ of shape $(4, 5)$, where $y' = x'[(0, 0), (0, 4)]$ and $z' = x'[(2, 0), (4, 4)]$. The erased arrays x' and x'' are depicted in Fig. 4. Moreover, we have $\rho_e(x) = x''$ since there is no e-hyperplane in x''. Because the letter e still occurs in x'', the bidimensional array x is not e-erasable.

2. Main result

Let us recall that our goal is to prove the following result.

Theorem 25. *Let $d \geq 1$. The d-dimensional infinite word x is S-automatic for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ if and only if x is the image by a coding of a shape-symmetric infinite d-dimensional word.*

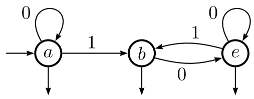

Fig. 5. The automaton $\mathcal{A}_{\mu_1,a}$.

The case $d = 1$ is proved in [13]. It is a natural generalization of the classical theorem of Cobham from 1972 [4]. For the sake of clarity, we present the proof in the case $d = 2$. We split the proof into two parts.

Part 1. Assume that $x = \nu(\mu^\omega(a))$ where $\nu\colon \Sigma^* \to \Gamma^*$ is a coding and $\mu\colon \Sigma^* \to B_2(\Sigma)$ is a two-dimensional morphism prolongable on a such that $y = \mu^\omega(a)$ is shape-symmetric. We show in this part that x is S-automatic for some $S = (L, \Sigma, <)$ where $\varepsilon \in L$.

Let $Y_1 = (y_{0,n})_{n\geq 0}$ be the first row of y. This word Y_1 is a unidimensional infinite word over a subset Σ_1 of Σ. It is clear that Y_1 is generated by a unidimensional morphism μ_1 derived from μ (one only has to consider the first row occurring in the images by μ of the letters in Σ).

Definition 26. With each (unidimensional) morphism $\mu\colon \Sigma \to \Sigma^*$ and with each letter $a \in \Sigma$ we can canonically associate a DFA denoted by $\mathcal{A}_{\mu,a}$ and defined as follows. Let $r_\mu := \max_{b\in\Sigma} |\mu(b)|$. The alphabet of $\mathcal{A}_{\mu,a}$ is $\{0, \ldots, r_\mu - 1\}$. The set of states is Σ. The initial state is a and every state is final. The (partial) transition function δ_μ is defined by $\delta_\mu(b, i) = (\mu(b))_i$, for all $b \in \Sigma$ and $i \in \{0, \ldots, |\mu(b)| - 1\}$. By removing from the language accepted by $\mathcal{A}_{\mu,a}$ the words having 0 as a prefix, we obtain the *directive language* of (μ, a). We let $L_{\mu,a}$ denote this directive language. Note that $L_{\mu,a}$ is a prefix language since all states in $\mathcal{A}_{\mu,a}$ are final. In particular, we have $\varepsilon \in L_{\mu,a}$. The reason why we call it *directive* will be clear; see Lemma 29 and Corollary 30.

Example 27. Considering the morphism μ of Example 20, we get $\Sigma_1 = \{a, b, e\}$, $\mu_1\colon a \mapsto ab,\ b \mapsto e,\ e \mapsto eb$ and $Y_1 = abeebebeebeebebeebebeeb\cdots$. The DFA associated with (μ_1, a) is depicted in Fig. 5. The first few words in the directive language of (μ_1, a) are ε, 1, 10, 100, 101, 1000, 1001, 1010, 10000.

Lemma 28. *Let $\mu\colon \Sigma \to \Sigma^*$ be a morphism prolongable on $a \in \Sigma$. We have*

$$\#(L_{\mu,a} \cap \{0, \ldots, r_\mu - 1\}^s) = \begin{cases} |\mu^s(a)| - |\mu^{s-1}(a)|, & \text{if } s > 0; \\ 1, & \text{if } s = 0. \end{cases}$$

Proof. The adjacency matrix $M \in \mathbb{N}^{\Sigma\times\Sigma}$ of $\mathcal{A}_{\mu,a}$ is defined for all $b, c \in \Sigma$ by $M_{b,c} = \#\{i \mid \delta_\mu(b, i) = c\}$. For all $s \geq 0$, $[M^s]_{b,c}$ is the number of paths of length s from b to c in $\mathcal{A}_{\mu,a}$. Since all states are final, the number N_s of words of length s accepted by $\mathcal{A}_{\mu,a}$ is obtained by summing up all the entries of M^s in the row corresponding to a. In particular, we have $N_0 = 1$. Because $\mathcal{A}_{\mu,a}$ has a loop of label 0 in a, the number of words of length s accepted by $\mathcal{A}_{\mu,a}$ and starting with 0 is equal to the number N_{s-1} of words of length $s - 1$ accepted by $\mathcal{A}_{\mu,a}$. Consequently, the number of words of length $s > 0$ in the directive language $L_{\mu,a}$ is exactly $N_s - N_{s-1}$. Of course, the matrix M can also be related to the morphism μ and $M_{b,c}$ is also the number of occurrences of c in $\mu(b)$. In particular, summing up all entries in the row of M^s corresponding to a gives $|\mu^s(a)|$. Therefore, the number of words of length $s > 0$ in the directive language $L_{\mu,a}$ is $|\mu^s(a)| - |\mu^{s-1}(a)|$. $\square$

Lemma 29. *Let $\mu\colon \Sigma \to \Sigma^*$ be a morphism prolongable on $a \in \Sigma$. Let S be the abstract numeration system built on the directive language $L_{\mu,a}$ of (μ, a) with the ordered alphabet $(\{0, \ldots, r_\mu - 1\}, 0 < \cdots < r_\mu - 1)$. Then, for the infinite word $\mu^\omega(a) = y_0y_1y_2\cdots$ and for all $n \geq 0$, we have*

$$y_n = \delta_\mu(a, \mathrm{rep}_S(n))$$

and, by setting $\mathrm{val}_S(0) = 0$,

$$\mu(y_n) = \mu^\omega(a)[\mathrm{val}_S(\mathrm{rep}_S(n)0), \mathrm{val}_S(\mathrm{rep}_S(n)(|\mu(y_n)| - 1))].$$

This latter formula is equivalent to

$$\forall n \in \mathbb{N},\ \forall i \in [\![0, |\mu(y_n)| - 1]\!],\ (\mu(y_n))_i = y_{\mathrm{val}_S(\mathrm{rep}_S(n)i)}.$$

Proof. Proceed by induction on the length s of the words in $L_{\mu,a}$. The only word of length 0 in $L_{\mu,a}$ is $\mathrm{rep}_S(0) = \varepsilon$. Since μ is prolongable on a, we have $y_0 = a = \delta_\mu(a, \mathrm{rep}_S(0))$. Moreover, for any $i \in [\![0, |\mu(a)| - 1]\!]$, we have $\mathrm{val}_S(i) = i$. So for any $i \in [\![0, |\mu(a)| - 1]\!]$, we get $(\mu(y_0))_i = y_i$.

Now take $s \in \mathbb{N}\setminus\{0\}$ and assume that the lemma holds for all integers m such that $|\mathrm{rep}_S(m)| \in [\![0, s-1]\!]$. Take $n \in \mathbb{N}\setminus\{0\}$ such that $|\mathrm{rep}_S(n)| = s$. Write $\mathrm{rep}_S(n) = wk$ where $|w| = s - 1$ and $k \in [\![0, |\mu(\delta_\mu(a, w))| - 1]\!]$. Since $L_{\mu,a}$ is prefix-closed, there exists an integer m such that $w = \mathrm{rep}_S(m)$. Hence, we have

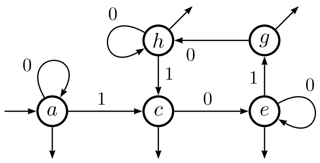

Fig. 6. The automaton $\mathcal{A}_{\mu_2,a}$.

$$\begin{aligned}\delta_\mu(a, \mathrm{rep}_S(n)) &= \delta_\mu(a, wk)\\ &= \delta_\mu(\delta_\mu(a, \mathrm{rep}_S(m)), k)\\ &= \delta_\mu(y_m, k) \text{ (by the induction hypothesis)}\\ &= (\mu(y_m))_k \text{ (by the definition of } \delta_\mu)\\ &= y_{\mathrm{val}_S(\mathrm{rep}_S(m)k)} \text{ (by the induction hypothesis)}\\ &= y_n.\end{aligned}$$

Thus, we have shown $y_\ell = \delta_\mu(a, \mathrm{rep}_S(\ell))$ for all $\ell \in [\![0, |\mu^s(a)| - 1]\!]$. Moreover, from the lemma above, we get

$$|\mathrm{rep}_S(n)| = t \Leftrightarrow n \in [\![|\mu^{t-1}(a)|, |\mu^t(a)| - 1]\!]. \tag{2}$$

Therefore we can write

$$\mu^{s+1}(a) = \underbrace{\mu^{s-1}(a)uy_nv}_{\mu^s(a)}\,\mu(u)\mu(y_n)\mu(v)$$

for some finite words u, v. Hence, we get

$$\forall i \in [\![0, |\mu(y_n)| - 1]\!],\ (\mu(y_n))_i = y_{|\mu^s(a)|+|\mu(u)|+i}.$$

By the definition of $L_{\mu,a}$, we have

$$\forall i \in [\![0, |\mu(y_n)| - 1]\!],\ \mathrm{val}_S(\mathrm{rep}_S(n)i) = \mathrm{val}_S(\mathrm{rep}_S(n)0) + i.$$

Hence, to conclude the proof, it suffices to show $\mathrm{val}_S(\mathrm{rep}_S(n)0) = |\mu^s(a)| + |\mu(u)|$.

From (2) we know that $|\mu^s(a)|$ is the S-value of the first word of length $s+1$ in $L_{\mu,a}$ with respect to the genealogical order. Next, from the definition of $L_{\mu,a}$ and from the first part of the proof, it follows that $L_{\mu,a}$ contains exactly $|\mu(\delta_\mu(a, \mathrm{rep}_S(\ell)))| = |\mu(y_\ell)|$ words of the form $\mathrm{rep}_S(\ell)j$, where ℓ belongs to $[\![0, |\mu^s(a)|-1]\!]$ and j is a letter. Since $\mathrm{rep}_S(|\mu^{s-1}(a)|)$ is the first word of length s in $L_{\mu,a}$ with respect to the genealogical order, we get that $|\mu(u)| = \sum_{\ell=|\mu^{s-1}(a)|}^{n-1} |\mu(y_\ell)|$ is exactly the number of words of length $s+1$ in $L_{\mu,a}$ of the form $\mathrm{rep}_S(\ell)j$ with $|\mathrm{rep}_S(\ell)| = s$ and $\ell < n$, i.e., the number of words in $L_{\mu,a}$ of length $s+1$ less than $\mathrm{rep}_S(n)0$ with respect to the genealogical order. The result follows. □

Corollary 30. *Let $\mu: \Sigma \to \Sigma^*$ be a morphism prolongable on $a \in \Sigma$ and let $\mu^\omega(a) = y_0y_1y_2\cdots$. Let S be the abstract numeration system built on the directive language $L_{\mu,a}$ of (μ, a) with the ordered alphabet $(\{0, \ldots, r_\mu - 1\}, 0 < \cdots < r_\mu - 1)$. Let $n \geq 0$ and $\mathrm{rep}_S(n) = w_0 \cdots w_\ell$, where the w_j's are letters. Define $z_0 := \mu(a)$ and for $j = 0, \ldots, \ell - 1$, set $z_{j+1} := \mu((z_j)_{w_j})$. Then, $y_n = (z_\ell)_{w_\ell}$.*

Example 31. Continue Example 27. The fixed point Y_1 of μ_1 starts with

$$abeebebe = y_0 \cdots y_7$$

and $\mathrm{rep}_S(7) = 1010$. From Lemma 29, $y_7 = e$ has been generated by applying μ_1 to the letter in the position $\mathrm{val}_S(101) = 4$, i.e., $y_4 = b$. We have $y_7 = (\mu_1(b))_0$. In turn, y_4 occurs in the image by μ_1 of the letter in the position $\mathrm{val}_S(10) = 2$, $y_2 = e$ and we have $y_4 = (\mu_1(e))_1$. Now y_2 appears in the image of the letter in the position $\mathrm{val}_S(1) = 1$ and we have $y_2 = (\mu_1(b))_0$.

The following result is obvious.

Lemma 32. *Let x, y be two infinite (unidimensional) words and λ, μ be two morphisms such that there exist letters a, b such that $x = \lambda^\omega(a)$ and $y = \mu^\omega(b)$. The languages $L_{\lambda,a}$ and $L_{\mu,b}$ are equal if and only if* $\mathrm{Shape}_\lambda(x) = \mathrm{Shape}_\mu(y)$.

Example 33. If one considers the morphism μ_2 defined by $a \mapsto ac$, $c \mapsto e$, $e \mapsto eg$, $g \mapsto h$ and $h \mapsto hc$ (which is derived from the first column of the bidimensional morphism in Example 20), we have the DFA $\mathcal{A}_{\mu_2,a}$ depicted in Fig. 6. The automata in Figs. 5 and 6 clearly accept the same language (the first one being minimal).

Let $Y_2 = (y_{n,0})_{n\geq 0}$ be the first column of y. This word Y_2 is a unidimensional infinite word over a subset Σ_2 of Σ. It is clear that Y_2 is generated by a morphism μ_2 derived from μ. Since y is shape-symmetric, thanks to Remark 22 and to Lemma 32,

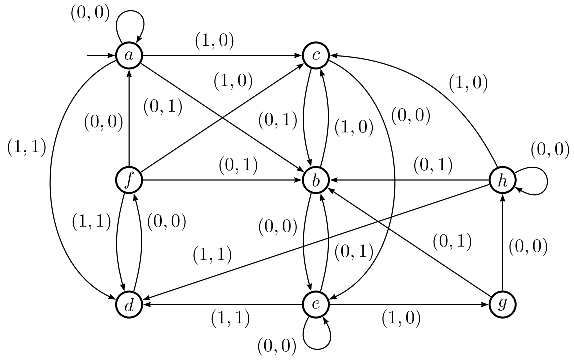

Fig. 7. DFAO generating $\mu^\omega(a)$ as an S-automatic word.

we have

$$L_{\mu_1,a} = L_{\mu_2,a} =: L.$$

We consider the abstract numeration system built upon this language L (with the natural ordering of digits). With all the above discussion and in particular in view of Corollary 30, it is clear that if $\mathrm{rep}_S(m) = ub$ and $\mathrm{rep}_S(n) = vc$ where b and c are letters, then

$$(\mu(y_{\mathrm{val}_S(u),\mathrm{val}_S(v)}))_{b,c} = y_{m,n}. \tag{3}$$

Example 34. Consider the letter c occurring in the position $(4, 7)$ in the fixed point y of μ underlined in Fig. 3. We have $(4, 7) = (\mathrm{val}_S(101), \mathrm{val}_S(1010))$. If we consider the pair $(\mathrm{val}_S(10), \mathrm{val}_S(101)) = (2, 4)$, we get $(\mu(y_{2,4}))_{1,0} = (\mu(b))_{1,0} = c = y_{4,7}$. In other words, $y_{4,7}$ comes from $y_{2,4}$. We can continue in this way. We have $b = y_{2,4} = (\mu(y_{1,2}))_{0,1}$ because $(\mathrm{val}_S(1), \mathrm{val}_S(10)) = (1, 2)$. Now $y_{1,2} = c = (\mu(y_{0,1}))_{1,0}$ because $(\mathrm{val}_S(\varepsilon), \mathrm{val}_S(1)) = (0, 1)$. Finally $y_{0,1} = b = (\mu(y_{0,0}))_{0,1} = (\mu(a))_{0,1}$ because $(\mathrm{val}_S(\varepsilon), \mathrm{val}_S(\varepsilon)) = (0, 0)$.

Now we extend Definition 26 to the multidimensional case.

Definition 35. For each d-dimensional morphism $\mu: \Sigma \to B_d(\Sigma)$ and for each letter $a \in \Sigma$, define a DFA $\mathcal{A}_{\mu,a}$ over the alphabet $\{0, \ldots, r_\mu - 1\}^d$ where $r_\mu = \max\{|\mu(b)|_i \mid b \in \Sigma, i = 1, \ldots, d\}$. The set of states is Σ, the initial state is a and all states are final. The (partial) transition function is defined by

$$\delta_\mu(b, \mathbf{n}) = (\mu(b))_{\mathbf{n}},$$

for all $b \in \Sigma$ and $\mathbf{n} \le |\mu(b)|$.

Thanks to (3), the automaton $\mathcal{A}_{\mu,a}$ is such that, for all $m, n \ge 0$,

$$y_{m,n} = \delta_\mu(a, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^0),$$

where we have padded the shortest word with enough 0's to make two words of the same length as in Definition 4. If we consider the coding ν as the output function, the corresponding DFAO generates x as an S-automatic sequence. Note that padding with 0's correctly works since 0 is the lexicographically smallest letter and the directive language L does not contain any words starting with 0. This concludes the first part.

Example 36. Consider the two-dimensional morphism μ of Example 20 and its fixed point $\mu^\omega(a)$ depicted in Fig. 3. If $S = (L, \{0, 1\}, 0 < 1)$ is the abstract numeration system constructed on $L = \{\varepsilon, 1, 10, 100, 101, 1000, 1001, 1010, \ldots\}$, then the corresponding DFAO depicted in Fig. 7, where the output function is the identity, generates $\mu^\omega(a)$ as an S-automatic word. For instance, if we continue Example 34, by reading $(\mathrm{rep}_S(4), \mathrm{rep}_S(7))^0 = (0101, 1010)$, we get

$$y_{0,0} = a \overset{(0,1)}{\rightarrow} y_{0,1} = b \overset{(1,0)}{\rightarrow} y_{1,2} = c \overset{(0,1)}{\rightarrow} y_{2,4} = b \overset{(1,0)}{\rightarrow} y_{4,7} = c,$$

and the letters appearing in this sequence of transitions are exactly the underlined ones in Fig. 3.

Part 2. Assume that $x = (x_{m,n})_{m,n\ge 0}$ is a two-dimensional S-automatic infinite word over an alphabet Γ for some abstract numeration system $S = (L, \Sigma, <)$ where $\varepsilon \in L$ and $\Sigma = \{a_1, \ldots, a_r\}$ with $a_1 < \cdots < a_r$. Let $\mathcal{A} = (Q_\mathcal{A}, q_0, (\Sigma_\#)^2, \delta_\mathcal{A}, \Gamma, \tau_\mathcal{A})$ be a deterministic finite automaton with output generating x. We may assume that $\# =: a_0$ is a symbol not belonging to Σ and that $a_0 < a_1$. Recall that this means that $x_{m,n} = \tau_\mathcal{A}(\delta_\mathcal{A}(q_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^\#))$ for all $m, n \ge 0$. Without loss of generality, we suppose that $\delta_\mathcal{A}(q, (\#, \#)) = q$, for all $q \in Q_\mathcal{A}$. In this part we prove that x can be represented as the image by a coding of a morphic shape-symmetric two-dimensional infinite word. We do the proof in

	ε	a	b	$a\#$	aa	ab	$b\#$	ba	bb	$a\#\#$	$a\#a$	
ε	p	q	q	q	p	q	q	p	q	q	p	$\cdots$
a	p	p	s	p	s	q	p	s	q	p	s	
b	q	p	s	p	q	s	p	q	s	p	q	
$a\#$	p	q	q	p	q	q	s	r	s	p	q	
aa	p	p	s	p	p	s	r	q	s	p	p	
ab	q	p	s	q	p	s	r	s	r	q	p	
$b\#$	q	p	q	p	q	q	s	r	s	p	q	
ba	p	s	q	p	p	s	r	q	s	p	p	
bb	p	q	s	q	p	s	r	s	r	q	p	
$a\#\#$	p	q	q	q	p	q	q	p	q	p	q	
$a\#a$	p	p	s	p	s	q	p	s	q	p	p	
$\vdots$												$\ddots$

Fig. 8. The fixed point $\mu_{\mathcal{A}}{}^{\omega}(p)$.

three steps. First, we show that x can be obtained by applying an erasing map to a fixed point of a uniform two-dimensional morphism. In the second step we prove that x is morphic. The generating morphism μ and the coding ν are obtained using a construction based on a unidimensional construction from [1]. Finally, we show that the considered fixed point of μ is shape-symmetric.

Definition 37. Let $d \geq 1$. Any DFA of the form $\mathcal{A} = (Q, q_0, \Sigma^d, \delta, F)$, where $\Sigma = \{a_0, a_1, \ldots, a_r\}$ with the ordering $a_i < a_{i+1}$ for all $0 \leq i \leq r-1$, can be canonically associated with a d-dimensional morphism denoted by $\mu_{\mathcal{A}}: Q \to B_d(Q)$ and defined as follows. The image of a letter $q \in Q$ is a d-dimensional square x of size $r+1$ defined by $x_{\mathbf{n}} = \delta(q, (a_{n_1}, \ldots, a_{n_d}))$ for all $\mathbf{0} \leq \mathbf{n} = (n_1, \ldots, n_d) \leq (r, \ldots, r)$.

Example 38. Consider the alphabet $\Sigma = \{\#, a, b\}$ with $\# < a < b$ and the automaton $\mathcal{A}$ depicted in Fig. 1 with added loops of label $(\#, \#)$ on all states. Then we get

$$\mu_{\mathcal{A}}(p) = \begin{array}{|c|c|c|}\hline p & q & q \\ \hline p & p & s \\ \hline q & p & s \\ \hline\end{array}\,, \mu_{\mathcal{A}}(q) = \begin{array}{|c|c|c|}\hline q & p & q \\ \hline p & s & q \\ \hline p & q & s \\ \hline\end{array}\,, \mu_{\mathcal{A}}(r) = \begin{array}{|c|c|c|}\hline r & s & s \\ \hline p & r & s \\ \hline p & r & p \\ \hline\end{array}\,, \mu_{\mathcal{A}}(s) = \begin{array}{|c|c|c|}\hline s & r & s \\ \hline r & q & s \\ \hline r & s & r \\ \hline\end{array}$$

and $\mu_{\mathcal{A}}{}^{\omega}(p)$ is the two-dimensional infinite word depicted in Fig. 8. Notice that $\mu_{\mathcal{A}}{}^{\omega}(p)$ is different from the S-automatic word given in Fig. 2. However, by erasing some rows and columns in Fig. 8, namely the ones corresponding to the words not belonging to $L = \{a, ba\}^*\{\varepsilon, b\}$, we obtain exactly the word in Fig. 2.

By assumption, L is a regular language over Σ. Hence, there exists a DFA accepting L and we may easily modify it to obtain a DFA $\mathcal{L} = (Q_{\mathcal{L}}, \ell_0, \Sigma_{\#}, \delta_{\mathcal{L}}, F_{\mathcal{L}})$ accepting $\{\#\}^*L$ and satisfying $\delta_{\mathcal{L}}(\ell_0, \#) = \ell_0$. Note that ℓ_0 is a final state since $\varepsilon \in L$. Next, let us define a "product" automaton $\mathcal{P} = (Q, p_0, (\Sigma_{\#})^2, \delta, F)$ imitating the behavior of $\mathcal{A}$ and two copies of the automaton $\mathcal{L}$, one for each dimension. The set of states of $\mathcal{P}$ is the Cartesian product $Q = Q_{\mathcal{A}} \times Q_{\mathcal{L}} \times Q_{\mathcal{L}}$, where the initial state p_0 is (q_0, ℓ_0, ℓ_0). The transition function $\delta: Q \times (\Sigma_{\#})^2 \to Q$ is defined by

$$\delta((q, k, \ell), (a, b)) = (\delta_{\mathcal{A}}(q, (a, b)), \delta_{\mathcal{L}}(k, a), \delta_{\mathcal{L}}(\ell, b)),$$

where (q, k, ℓ) belongs to Q and (a, b) is a pair of letters in $(\Sigma_{\#})^2$. The set of final states is $F = Q_{\mathcal{A}} \times F_{\mathcal{L}} \times F_{\mathcal{L}}$. Let $y = (y_{m,n})_{m,n \geq 0}$ be the infinite word satisfying

$$y_{m,n} = \delta(p_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^{\#}).$$

Note that both the first and the second component of $(\mathrm{rep}_S(m), \mathrm{rep}_S(n))^{\#}$ belong to the language $\{\#\}^*L$ and, therefore, $\delta(p_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^{\#})$ is a final state. Define $\tau: F \to \Gamma$ to be the coding satisfying $\tau((q, k, \ell)) = \tau_{\mathcal{A}}(q)$ for all $(q, k, \ell) \in F$. By construction, it is clear that $\tau(y) = (x_{m,n})_{m,n \geq 0}$. We consider the canonically associated morphism $\mu_{\mathcal{P}}: Q \to B_2(Q)$ given in Definition 37. Note that $\mu_{\mathcal{P}}$ is prolongable on p_0, since $\delta(p_0, (a_0, a_0)) = (\delta_{\mathcal{A}}(q_0, (\#, \#)), \delta_{\mathcal{L}}(\ell_0, \#), \delta_{\mathcal{L}}(\ell_0, \#)) = (q_0, \ell_0, \ell_0) = p_0$.

Example 39. Let us continue Example 6 and consider again the abstract numeration system $S = (\{a, ba\}^*\{\varepsilon, b\}, \{a, b\}, a < b)$ and the DFAO depicted in Fig. 1, with additional loops of label $(\#, \#)$ on all states. The minimal automaton of $\{\#\}^*\{a, ba\}^*\{\varepsilon, b\}$ is depicted in Fig. 9. If $\mathcal{P}$ is the corresponding product automaton, then the fixed point $\mu_{\mathcal{P}}{}^{\omega}((p, g, g))$ of $\mu_{\mathcal{P}}$ is the two-dimensional infinite word depicted in Fig. 10.

Let e be a new symbol. Recall that ρ_e is the erasing map given in Definition 23. Let ρ denote $\rho_e \circ \lambda$, where λ is a morphism on $Q \cup \{e\}$ defined by

$$\lambda(p) = \begin{cases} e, & \text{if } p \notin F; \\ p, & \text{otherwise.} \end{cases}$$

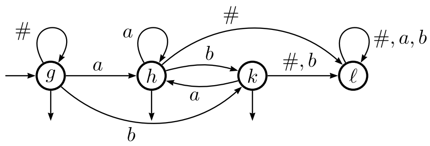

Fig. 9. The minimal automaton accepting $\{\#\}^*\{a, ba\}^*\{\varepsilon, b\}$.

(p,g,g)	(q,g,h)	(q,g,k)	(q,g,ℓ)	(p,g,h)	(q,g,k)	(q,g,ℓ)	(p,g,h)	$\cdots$
(p,h,g)	(p,h,h)	(s,h,k)	(p,h,ℓ)	(s,h,h)	(q,h,k)	(p,h,ℓ)	(s,h,h)	
(q,k,g)	(p,k,h)	(s,k,k)	(p,k,ℓ)	(q,k,h)	(s,k,k)	(p,k,ℓ)	(q,k,h)	
(p,ℓ,g)	(q,ℓ,h)	(q,ℓ,k)	(p,ℓ,ℓ)	(q,ℓ,h)	(q,ℓ,k)	(s,ℓ,ℓ)	(r,ℓ,h)	
(p,h,g)	(p,h,h)	(s,h,k)	(p,h,ℓ)	(p,h,h)	(s,h,k)	(r,h,ℓ)	(q,h,h)	
(q,k,g)	(p,k,h)	(s,k,k)	(q,k,ℓ)	(p,k,h)	(s,k,k)	(r,k,ℓ)	(s,k,h)	
(q,ℓ,g)	(p,ℓ,h)	(q,ℓ,k)	(p,ℓ,ℓ)	(q,ℓ,h)	(q,ℓ,k)	(s,ℓ,ℓ)	(r,ℓ,h)	
(p,h,g)	(s,h,h)	(q,h,k)	(p,h,ℓ)	(p,h,h)	(s,h,k)	(r,h,ℓ)	(q,h,h)	
(p,ℓ,g)	(p,ℓ,h)	(s,ℓ,k)	(p,ℓ,ℓ)	(p,ℓ,h)	(s,ℓ,k)	(r,ℓ,ℓ)	(s,ℓ,h)	
(p,ℓ,g)	(q,ℓ,h)	(q,ℓ,k)	(q,ℓ,ℓ)	(p,ℓ,h)	(q,ℓ,k)	(q,ℓ,ℓ)	(p,ℓ,h)	
(p,ℓ,g)	(p,ℓ,h)	(s,ℓ,k)	(p,ℓ,ℓ)	(s,ℓ,h)	(q,ℓ,k)	(p,ℓ,ℓ)	(s,ℓ,h)	
$\vdots$								$\ddots$

Fig. 10. The fixed point ${\mu_{\mathcal{P}}}^{\omega}((p, g, g))$.

We first claim that $y = \rho({\mu_{\mathcal{P}}}^{\omega}(p_0))$. Observe that the infinite word $\lambda({\mu_{\mathcal{P}}}^{\omega}(p_0))$ is e-erasable. Namely, all letters in a fixed column C of the infinite bidimensional word ${\mu_{\mathcal{P}}}^{\omega}(p_0)$ are of the form (q, k, ℓ) where the third component ℓ is fixed. If ℓ does not belong to $F_{\mathcal{L}}$, the word $\lambda(C)$ is a unidimensional e-hyperplane of $\lambda({\mu_{\mathcal{P}}}^{\omega}(p_0))$. Thus, the map ρ erases all columns where the third component ℓ does not belong to $F_{\mathcal{L}}$. The same holds for rows and second components k of the letters in Q. Hence, the two-dimensional infinite word $\rho({\mu_{\mathcal{P}}}^{\omega}(p_0))$ contains only letters belonging to F. By the construction of the morphism $\mu_{\mathcal{P}}$, those letters are coming from the automaton $\mathcal{P}$ by feeding it with words belonging to $((\Sigma_{\#})^2)^* \cap (\{\#\}^*L)^2$. More precisely, all rows and columns corresponding to words not belonging to L are erased and $(\rho({\mu_{\mathcal{P}}}^{\omega}(p_0)))_{m,n}$ is equal to $\delta(p_0, (\mathrm{rep}_S(m), \mathrm{rep}_S(n))^{\#}) = y_{m,n}$. Hence, defining $\vartheta = \tau \circ \rho$, we get a map from Σ to Γ such that $x = \vartheta({\mu_{\mathcal{P}}}^{\omega}(p_0))$.

Example 40. We continue Example 39 and this time we consider the bidimensional infinite S-automatic word depicted in Fig. 2. This word is exactly the bidimensional infinite word obtained by erasing all rows and columns corresponding to words not belonging to L from the bidimensional word $\mu_{\mathcal{A}}^{\omega}(p)$ depicted in Fig. 8. By the previous construction, we obtain that this word is also the two-dimensional infinite word obtained by first erasing all columns with ℓ as the third component and all rows with ℓ as the second component from the two-dimensional infinite word ${\mu_{\mathcal{P}}}^{\omega}((p, g, g))$ depicted in Fig. 10 and then mapping the infinite word by τ.

Next we show that x is morphic by getting rid of the erasing map ρ. We construct a morphism μ prolongable on some letter α and a coding ν such that $x = \nu(\mu^{\omega}(\alpha))$. We follow the guidelines of [1, Theorem 7.7.4]. First we need the following definitions.

Definition 41. Let μ be a morphism on some finite alphabet Σ and let $\Psi \subseteq \Sigma$. We say that a letter $a \in \Sigma$ is

(i) (μ, Ψ)-*dead* if the word $\mu^n(a) \in \Psi^*$ for every $n \geq 0$.
(ii) (μ, Ψ)-*moribund* if there exists $m \geq 0$ such that the word $\mu^m(a)$ contains at least one letter in $\Sigma \setminus \Psi$, and for every $n > m$, $\mu^n(a) \in \Psi^*$.
(iii) (μ, Ψ)-*robust* if there exist infinitely many $n \geq 0$ such that the word $\mu^n(a)$ contains at least one letter in $\Sigma \setminus \Psi$.

The following lemma from [1, Lemma 7.7.3] is also valid for multidimensional morphisms, since the proof is only based on the finiteness of the alphabet Σ.

Lemma 42. *Let μ be a morphism on some finite alphabet Σ and let $\Psi \subseteq \Sigma$. Then there exists an integer $T \geq 1$ such that the morphism $\varphi = \mu^T$ satisfies:*

(a) *If a is (φ, Ψ)-moribund, then $\varphi^n(a) \in \Psi^*$ for all $n > 0$ and $a \in \Sigma \setminus \Psi$.*
(b) *If a is (φ, Ψ)-robust, then the word $\varphi^n(a)$ contains at least one letter in $\Sigma \setminus \Psi$ for all $n > 0$.*

Remark 43. Note that, by Lemma 42, a letter in Ψ is either (φ, Ψ)-dead or (φ, Ψ)-robust and a letter in $\Sigma \setminus \Psi$ is either (φ, Ψ)-moribund or (φ, Ψ)-robust.

We may assume, by taking a power of $\mu_{\mathcal{P}}$ if necessary, that $\mu_{\mathcal{P}}$ satisfies the properties (a) and (b) listed for φ in Lemma 42 with $\Psi = F^c := Q \setminus F$. For the sake of simplicity, we use the words *dead*, *moribund* and *robust* instead of $(\mu_{\mathcal{P}}, F^c)$-dead, $(\mu_{\mathcal{P}}, F^c)$-moribund and $(\mu_{\mathcal{P}}, F^c)$-robust from now on.

$T_k \backslash T_\ell$	Δ	M	R_{F^c}	R_F
Δ	Δ	Δ	Δ	Δ
M	Δ	M	Δ	M
R_{F^c}	Δ	Δ	R_{F^c}	R_{F^c}
R_F	Δ	M	R_{F^c}	R_F

Fig. 11. Type T_p of a letter $p = (q, k, \ell) \in Q$.

Next we classify the states of $Q_\mathcal{L}$ and Q into four categories. The *type of a state* $k \in Q_\mathcal{L}$ is

$$T_k = \begin{cases} \Delta, & \text{if } k \notin F_\mathcal{L} \text{ and } \forall a \in \Sigma_\#, \delta_\mathcal{L}(k, a) \notin F_\mathcal{L}; \\ M, & \text{if } k \in F_\mathcal{L} \text{ and } \forall a \in \Sigma_\#, \delta_\mathcal{L}(k, a) \notin F_\mathcal{L}; \\ R_{F^c}, & \text{if } k \notin F_\mathcal{L} \text{ and } \exists a \in \Sigma_\#, \delta_\mathcal{L}(k, a) \in F_\mathcal{L}; \\ R_F, & \text{if } k \in F_\mathcal{L} \text{ and } \exists a \in \Sigma_\#, \delta_\mathcal{L}(k, a) \in F_\mathcal{L}. \end{cases}$$

The *type of a state* $p = (q, k, \ell) \in Q$ is

$$T_p = \begin{cases} \Delta, & \text{if } p \text{ is dead;} \\ M, & \text{if } p \text{ is moribund;} \\ R_{F^c}, & \text{if } p \in F^c \text{ and } p \text{ is robust;} \\ R_F, & \text{if } p \in F \text{ and } p \text{ is robust.} \end{cases}$$

By these definitions, it is clear that the type of $(q, k, \ell) \in Q$ only depends on the types of k and $\ell \in Q_\mathcal{L}$ according to Fig. 11. Note that by the properties (a) and (b) of Lemma 42, it suffices to consider transitions $\delta_\mathcal{L}(k, a)$ for each letter $a \in \Sigma_\#$ instead of transitions $\delta_\mathcal{L}(k, w)$ for all words w in $(\Sigma_\#)^*$. For instance, if the type of k is R_{F^c} and the type of ℓ is R_F, then $k \notin F_\mathcal{L}$ and (q, k, ℓ) belongs to F^c. Moreover, there exist $m, n \in [\![0, r]\!]$ such that $\delta_\mathcal{L}(k, a_m) \in F_\mathcal{L}$ and $\delta_\mathcal{L}(\ell, a_n) \in F_\mathcal{L}$. This means that $(\mu_\mathcal{P}((q, k, \ell)))_{m,n}$ belongs to F. Hence, by Lemma 42 and Remark 43, (q, k, ℓ) is robust.

Let us define two morphisms λ_Δ and λ_M on $Q \cup \{e\}$ in a similar way as λ was defined above:

$$\lambda_\Delta(p) = \begin{cases} e, & \text{if } p \text{ is dead;} \\ p, & \text{otherwise} \end{cases} \quad \text{and} \quad \lambda_M(p) = \begin{cases} e, & \text{if } p \text{ is moribund;} \\ p, & \text{otherwise.} \end{cases}$$

By the property (b) of Lemma 42, we know that if p is robust, then $\mu_\mathcal{P}(p)$ contains at least one letter in F and since every dead letter must belong to F^c, the word $\lambda_\Delta(\mu_\mathcal{P}(p))$ contains at least one letter in F. For any $\ell \in Q_\mathcal{L}$, let us define a sequence $(d_\ell(i))_{0 \le i \le h_\ell}$ such that $d_\ell(0) = 0$, $d_\ell(h_\ell) = r + 1$ and for all $i \in [\![0, h_\ell - 1]\!]$, $d_\ell(i) < d_\ell(i + 1)$ and there exists exactly one index $n \in [\![d_\ell(i), d_\ell(i + 1) - 1]\!]$ satisfying

$$\delta_\mathcal{L}(\ell, a_n) \in F_\mathcal{L}. \tag{4}$$

Note that h_ℓ is the number of letters $a_n \in \Sigma_\#$ satisfying condition (4). Hence, for each robust letter $p = (q, k, \ell)$, we get $h_k, h_\ell \ge 1$ and we may define the factorization

$$\lambda_\Delta(\mu_\mathcal{P}(p)) = \boxed{\begin{array}{cccc} w_p(0, 0) & w_p(0, 1) & \cdots & w_p(0, h_\ell - 1) \\ w_p(1, 0) & w_p(1, 1) & \cdots & w_p(1, h_\ell - 1) \\ \vdots & \vdots & \ddots & \vdots \\ w_p(h_k - 1, 0) & w_p(h_k - 1, 1) & \cdots & w_p(h_k - 1, h_\ell - 1) \end{array}},$$

where each bidimensional array

$$w_p(i, j) = \lambda_\Delta(\mu_\mathcal{P}(p))[(d_k(i), d_\ell(j)), (d_k(i + 1) - 1, d_\ell(j + 1) - 1)]$$

contains exactly one letter in F.

Example 44. Let us continue Example 40. Recall that the product automaton $\mathcal{P}$ is produced from the automaton $\mathcal{A}$ depicted in Fig. 1 and the automaton $\mathcal{L}$ depicted in Fig. 9. Note that the type of the state ℓ in $\mathcal{L}$ is $T_\ell = \Delta$ and all other states have type R_F. By Fig. 10, we see that

$$\mu_\mathcal{P}(p, g, g) = \boxed{\begin{array}{ccc} (p, g, g) & (q, g, h) & (q, g, k) \\ (p, h, g) & (p, h, h) & (s, h, k) \\ (q, k, g) & (p, k, h) & (s, k, k) \end{array}}$$

and

$$\mu_\mathcal{P}(q, g, h) = \boxed{\begin{array}{ccc} (q, g, \ell) & (p, g, h) & (q, g, k) \\ (p, h, \ell) & (s, h, h) & (q, h, k) \\ (p, k, \ell) & (q, k, h) & (s, k, k) \end{array}}.$$

Since h_ℓ is the number of letters $a_n \in \Sigma_\#$ such that $\delta_{\mathcal{L}}(\ell, a_n) \in F_{\mathcal{L}}$, we notice that $h_g = 3$ and $h_h = 2$. By Fig. 11, we have $\lambda_\Delta(\mu_{\mathcal{P}}(p, g, g)) = \mu_{\mathcal{P}}(p, g, g)$ and

$$\lambda_\Delta(\mu_{\mathcal{P}}(q, g, h)) = \boxed{\begin{array}{ccc} e & (p, g, h) & (q, g, k) \\ e & (s, h, h) & (q, h, k) \\ e & (q, k, h) & (s, k, k) \end{array}}.$$

Since all letters in $\mu_{\mathcal{P}}(p, g, g)$ belong to F, the array $w_{(p,g,g)}(i, j)$ is a square of size 1 for $(i, j) \in [\![0, h_g - 1]\!] \times [\![0, h_g - 1]\!]$. We also obtain

$$\begin{array}{rclrcl} w_{(q,g,h)}(0, 0) & = & \boxed{e \quad (p, g, h)}, & w_{(q,g,h)}(0, 1) & = & \boxed{(q, g, k)}, \\ w_{(q,g,h)}(1, 0) & = & \boxed{e \quad (s, h, h)}, & w_{(q,g,h)}(1, 1) & = & \boxed{(q, h, k)}, \\ w_{(q,g,h)}(2, 0) & = & \boxed{e \quad (q, k, h)}, & w_{(q,g,h)}(2, 1) & = & \boxed{(s, h, k)} \end{array}$$

from the image $\lambda_\Delta(\mu_{\mathcal{P}}(q, g, h))$.

Now we show that if p is a robust state, the bidimensional array $\lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(p)))$ is e-erasable. If $v := \lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(p)))$ is not e-erasable, then there must exist $m, n \geq 0$ such that $v_{m,n} = e$, $v_{m,n'} \neq e$ for some n' and $v_{m',n} \neq e$ for some m'. By construction, the letter $p' = (\mu_{\mathcal{P}}(p))_{m,n} = (q, k, \ell)$ is mapped to e either if $T_{p'} = \Delta$ or if $T_{p'} = M$. By the same reason, the letters $v_{m,n'} = (q', k, \ell')$ and $v_{m',n} = (q'', k', \ell)$ must be robust. Thus, there exist letters $a_{m''}, a_{n''} \in \Sigma_\#$ such that $\delta_{\mathcal{L}}(k, a_{m''}) \in F_{\mathcal{L}}$ and $\delta_{\mathcal{L}}(\ell, a_{n''}) \in F_{\mathcal{L}}$. Hence, it follows that $p' = (q, k, \ell)$ is robust, since the letter $(\mu_{\mathcal{P}}(p'))_{m'',n''}$ belongs to F, which is a contradiction. Then, for each robust letter $p = (q, k, \ell)$, for each i with $0 \leq i < h_k$ and for each j with $0 \leq j < h_\ell$, write

$$(\rho_e(\lambda_M(w_p(i, j))))_{m,n} =: v_{p,i,j}(m, n)$$

where $(m, n) < \mathbf{s}_{p,i,j} := |\rho_e(\lambda_M(w_p(i, j)))|$. Note that the array $\lambda_M(w_p(i, j))$ is e-erasable as a factor of the e-erasable array $\lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(p)))$.

Now we are ready to introduce a two-dimensional morphism μ on a new alphabet Ξ and a coding $\nu' : \Xi \to Q$ such that $y = \nu'(\mu^\omega(\alpha))$ for a letter $\alpha \in \Xi$. The alphabet of the new symbols is

$$\Xi = \{\alpha(p, i, j) \mid p = (q, k, \ell) \text{ is robust}, 0 \leq i < h_k \text{ and } 0 \leq j < h_\ell\}.$$

We define the bidimensional arrays $u_{p,i,j}(m, n)$ for each robust letter $p = (q, k, \ell) \in Q$, $(i, j) \in [\![0, h_k - 1]\!] \times [\![0, h_\ell - 1]\!]$ and $(m, n) < \mathbf{s}_{p,i,j}$ as follows. If $v_{p,i,j}(m, n) = (q', k', \ell')$, then $u_{p,i,j}(m, n)$ is an array of shape $(h_{k'}, h_{\ell'})$ such that

$$(u_{p,i,j}(m, n))_{i',j'} = \alpha(v_{p,i,j}(m, n), i', j')$$

for $(i', j') \in [\![0, h_{k'} - 1]\!] \times [\![0, h_{\ell'} - 1]\!]$. The image of $\alpha(p, i, j)$ by the morphism $\mu : \Xi \to B_2(\Xi)$ is defined as the array

$$\boxed{\begin{array}{cccc} u_{p,i,j}(0, 0) & u_{p,i,j}(0, 1) & \cdots & u_{p,i,j}(0, s_2 - 1) \\ u_{p,i,j}(1, 0) & u_{p,i,j}(1, 1) & \cdots & u_{p,i,j}(1, s_2 - 1) \\ \vdots & \vdots & \ddots & \vdots \\ u_{p,i,j}(s_1 - 1, 0) & u_{p,i,j}(s_1 - 1, 1) & \cdots & u_{p,i,j}(s_1 - 1, s_2 - 1) \end{array}},$$

where $(s_1, s_2) = \mathbf{s}_{p,i,j}$. Note that the above concatenation of the arrays $u_{p,i,j}(m, n)$ is well defined. Since all letters occurring on a row of $w_p(i, j)$ are of the form (q', k', ℓ') where the second component k' is fixed, it also means that the letters $v_{p,i,j}(m, n)$ and $v_{p,i,j}(m, n')$ occurring on the same row of $\rho_e(\lambda_M(w_p(i, j)))$ have the same second component k'. Hence, $|u_{p,i,j}(m, n)|_{\hat{2}} = |u_{p,i,j}(m, n')|_{\hat{2}} = h_{k'}$ and the words $u_{p,i,j}(m, n)$ and $u_{p,i,j}(m, n')$ can be concatenated in the direction 2. The same holds for $u_{p,i,j}(m, n)$ and $u_{p,i,j}(m', n)$ in the direction 1. The coding $\nu' : \Xi \to Q$ is defined by

$$\nu'(\alpha(p, i, j)) = \rho(w_p(i, j)). \tag{5}$$

Note that by the definition of $w_p(i, j)$, there is only one letter belonging to F and the array $\lambda(w_p(i, j))$ is e-erasable, since only one letter is different from e. This shows that ν' is a coding.

Following the proof of [1, Theorem 7.7.4], we may prove by induction that

$$\nu' \circ \mu^n \left(\boxed{\begin{array}{cccc} \alpha(p, 0, 0) & \alpha(p, 0, 1) & \cdots & \alpha(p, 0, h_\ell - 1) \\ \alpha(p, 1, 0) & \alpha(p, 1, 1) & \cdots & \alpha(p, 1, h_\ell - 1) \\ \vdots & \vdots & \ddots & \vdots \\ \alpha(p, h_k - 1, 0) & \alpha(p, h_k - 1, 1) & \cdots & \alpha(p, h_k - 1, h_\ell - 1) \end{array}} \right) = \rho \circ \mu_{\mathcal{P}}^{n+1}(p) \tag{6}$$

for all robust letters $p = (q, k, \ell)$ and for all $n \geq 0$. Since $\mu_{\mathcal{P}}$ is prolongable on p_0 and $x = \vartheta(\mu_{\mathcal{P}}^\omega(p_0))$ is a two-dimensional infinite word, the letter p_0 must be robust. Therefore, we have $(w_{p_0}(0, 0))_{0,0} = v_{p_0,0,0}(0, 0) = p_0$. Thus,

$(u_{p_0,0,0}(0,0))_{0,0} = \alpha(p_0, 0, 0)$ and, consequently, the morphism μ is prolongable on $\alpha := \alpha(p_0, 0, 0)$. For all $n \geq 0$, it follows from (6) that we have

$$\nu'(\mu^{n+1}(\alpha)) = \begin{bmatrix} \nu'(\mu^n(u_{p_0,0,0}(0,0))) & U \\ V & W \end{bmatrix} = \begin{bmatrix} \rho(\mu_{\mathcal{P}}^{n+1}(p_0)) & U \\ V & W \end{bmatrix},$$

where U, V and W are bidimensional arrays. Since $\rho(\mu_{\mathcal{P}}^{n+1}(p_0))$ tends to y as n tends to infinity, we have

$$\nu'(\mu^{\omega}(\alpha)) = \rho(\mu_{\mathcal{P}}^{\omega}(p_0)) = y.$$

Hence, defining the coding $\nu: \Xi \to \Gamma$ as $\nu = \tau \circ \nu'$ we obtain

$$\nu(\mu^{\omega}(\alpha)) = \tau(y) = x.$$

Example 45. Let us continue Example 44. We obtain

$$\rho_e(\lambda_M(\lambda_\Delta(\mu_{\mathcal{P}}(q, g, h)))) = \boxed{\begin{matrix} (p, g, h) & (q, g, k) \\ (s, h, h) & (q, h, k) \\ (q, k, h) & (s, k, k) \end{matrix}}.$$

Since $w_{(p,g,g)}(i, j)$ is a square of size 1 for every $(i, j) \in [\![0, 2]\!] \times [\![0, 2]\!]$, we have

$$\mathbf{s}_{(p,g,g),i,j} = |\rho_e(\lambda_M(w_{(p,g,g)}(i, j)))| = (1, 1)$$

and

$$v_{(p,g,g),i,j}(0, 0) = w_{(p,g,g)}(i, j) = (\mu_{\mathcal{P}}(p, g, g))_{i,j}.$$

In particular, we have $v_{(p,g,g),0,0}(0, 0) = (p, g, g)$ and $v_{(p,g,g),0,1}(0, 0) = (q, g, h)$. Hence, $u_{(p,g,g),0,0}(0, 0)$ is an array of shape $(h_g, h_g) = (3, 3)$ such that

$$(u_{(p,g,g),0,0}(0, 0))_{i',j'} = \alpha(v_{(p,g,g),0,0}(0, 0), i', j') = \alpha((p, g, g), i', j')$$

for $(i', j') \in [\![0, 2]\!] \times [\![0, 2]\!]$ and the image $\mu(\alpha((p, g, g), 0, 0)) = u_{(p,g,g),0,0}(0, 0)$ is

$$\boxed{\begin{matrix} \alpha((p, g, g), 0, 0) & \alpha((p, g, g), 0, 1) & \alpha((p, g, g), 0, 2) \\ \alpha((p, g, g), 1, 0) & \alpha((p, g, g), 1, 1) & \alpha((p, g, g), 1, 2) \\ \alpha((p, g, g), 2, 0) & \alpha((p, g, g), 2, 1) & \alpha((p, g, g), 2, 2) \end{matrix}}.$$

Similarly, $|u_{(p,g,g),0,1}(0, 0)| = (h_g, h_h) = (3, 2)$ and

$$(u_{(p,g,g),0,1}(0, 0))_{i',j'} = \alpha(v_{(p,g,g),0,1}(0, 0), i', j') = \alpha((q, g, h), i', j')$$

for $(i', j') \in [\![0, 1]\!] \times [\![0, 2]\!]$. Thus, the image $\mu(\alpha((p, g, g), 0, 1)) = u_{(p,g,g),0,1}(0, 0)$ is

$$\boxed{\begin{matrix} \alpha((q, g, h), 0, 0) & \alpha((q, g, h), 0, 1) \\ \alpha((q, g, h), 1, 0) & \alpha((q, g, h), 1, 1) \\ \alpha((q, g, h), 2, 0) & \alpha((q, g, h), 2, 1) \end{matrix}}.$$

Next we apply the coding ν to the images above. Hence, by (5), we have

$$\nu'(\mu(\alpha((p, g, g), 0, 0))) = \mu_{\mathcal{P}}(p, g, g)$$

and

$$\nu'(\mu(\alpha((p, g, g), 0, 1))) = \boxed{\begin{matrix} (p, g, h) & (q, g, k) \\ (s, h, h) & (q, h, k) \\ (q, k, h) & (s, k, k) \end{matrix}}.$$

Since $\nu = \tau \circ \nu'$, the infinite word $\nu(\mu^{\omega}(\alpha((p, g, g), 0, 0)))$ begins with

$$\nu\left(\mu(\alpha((p, g, g), 0, 0)) \odot^2 \mu(\alpha((p, g, g), 0, 1))\right) = \boxed{\begin{matrix} p & q & q & p & q \\ p & p & s & s & q \\ q & p & s & q & s \end{matrix}},$$

which is exactly the left upper corner of the infinite word depicted in Fig. 2.

Finally, we have to show that $w = \mu^{\omega}(\alpha)$ is shape-symmetric, that is, $|\mu(w_{n,n})|$ is a square for all $n \in \mathbb{N}$. First, observe that since we have $\alpha = \alpha(p_0, 0, 0)$, where the second and the third component of $p_0 = (q_0, \ell_0, \ell_0)$ are equal, the letter $\mu^{\omega}(\alpha)_{n,n}$ must be of the form $\alpha((q, k, k), i, i)$, where (q, k, k) is a robust letter in Q and i belongs to $[\![0, h_k - 1]\!]$. Second, if $p = (q, k, k)$ is a robust letter in Q, then $\mu(\alpha(p, i, i))$ is a square for all $i \in [\![0, h_k - 1]\!]$. Hence, the result follows.

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References

[1] J.-P. Allouche, J. Shallit, Automatic Sequences: Theory, Applications, Generalizations, Cambridge University Press, 2003.
[2] P. Arnoux, V. Berthé, A. Siegel, Two-dimensional iterated morphisms and discrete planes, Theoret. Comput. Sci. 319 (2004) 145–176.
[3] O. Carton, W. Thomas, The monadic theory of morphic infinite words and generalizations, Inform. and Comput. 176 (2002) 51–76.
[4] A. Cobham, Uniform tag sequences, Math. Systems Theory 6 (1972) 164–192.
[5] E. Duchêne, A.S. Fraenkel, R. Nowakowski, M. Rigo, Extensions and restrictions of Wythoff's game preserving Wythoff's sequence as set of P positions, J. Combin. Theory Ser. A, in press (doi:10.1016/j.jcta.2009.07.010).
[6] P.B.A. Lecomte, M. Rigo, Numeration systems on a regular language, Theory Comput. Syst. 34 (2001) 27–44.
[7] A. Maes, Decidability of the first-order theory of $\langle \mathbb{N}; <, P\rangle$ for morphic predicates P, Preprint 9806, Inst. für Informatik und Praktische Math., Christian-Albrechts-Univ. Kiel (1998).
[8] A. Maes, An automata-theoretic decidability proof for first-order theory of $\langle \mathbf{N}, <, P\rangle$ with morphic predicate P, J. Autom. Lang. Comb. 4 (1999) 229–245.
[9] A. Maes, Morphic predicates and applications to the decidability of arithmetic theories, Ph.D. Thesis, Univ. Mons-Hainaut, (1999).
[10] S. Nicolay, M. Rigo, About the frequency of letters in generalized automatic sequences, Theoret. Comp. Sci. 374 (2007) 25–40.
[11] J. Peyrière, Fréquence des motifs dans les suites doubles invariantes par une substitution, Ann. Sci. Math. Québec 11 (1987) 133–138.
[12] M. Rigo, Generalization of automatic sequences for numeration systems on a regular language, Theoret. Comp. Sci. 244 (2000) 271–281.
[13] M. Rigo, A. Maes, More on generalized automatic sequences, J. Autom. Lang. Comb. 7 (2002) 351–376.
[14] O. Salon, Suites automatiques à multi-indices, Séminaire de théorie des nombres de Bordeaux, Exp. 4 (1986–1987), 4.01–4.27; followed by an Appendix by J. Shallit, 4-29A–4-36A.