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Note

Some New Conjugate Orthogonal Latin Squares

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We present some new conjugate orthogonal Latin squares which are obtained from a direct method of construction of the starter-adder type. Combining these new constructions with earlier results of K. T. Phelps and the first author, it is shown that a (3, 2, 1)- (or (1, 3, 2)-) conjugate orthogonal Latin square of order v exists for all positive integers $v \neq 2, 6$. It is also shown that a (3, 2, 1)- (or (1, 3, 2)-) conjugate orthogonal *idempotent* Latin square of order v exists for all positive integers $v \neq 2, 3, 6$ with one possible exception $v = 12$, and this result can be used to enlarge the spectrum of a certain class of Mendelsohn designs and provide better results for problems on embedding. © 1987 Academic Press, Inc.

1. INTRODUCTION

A *Latin square* of order v is a $v \times v$ matrix array (a_{ij}) based on the elements of a v-set, say $S = \{1, 2, ..., v\}$ such that in each row and in each column every element occurs exactly once. The Latin square (a_{ij}) is called *idempotent* if $a_{ii} = i$ for all i, $1 \leqslant i \leqslant v$. Two Latin squares (a_{ij}) and (b_{ij}), each based on the same set S, are said to be *orthogonal* if for each $(s, t) \in S \times S$ there exists a unique ordered pair (i, j) such that $a_{ij} = s$ and $b_{ij} = t$.

A *quasigroup* is an ordered pair $(Q, \otimes)$, where Q is a set and $\otimes$ is a binary operation on Q such that the equations $a \otimes x = b$ and $y \otimes a = b$ are uniquely solvable for every pair of elements a, b in Q. It is fairly well

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known that the multiplication table of a quasigroup defines a Latin square, that is, a Latin square can be considered as the multiplication table for a quasigroup with the headline and sideline removed. We are concerned mainly with finite Latin squares (quasigroups) in this article, and the interested reader may wish to refer to the book of Dénes and Keedwell [3] for more general information on Latin squares. The notion of conjugate of a Latin square is described below for the benefit of the reader.

If $(Q, \otimes)$ is a quasigroup, we may define on the set Q six binary operations $\otimes(1,2,3)$, $\otimes(1,3,2)$, $\otimes(2,1,3)$, $\otimes(2,3,1)$, $\otimes(3,1,2)$ and $\otimes(3,2,1)$ as follows: $a \otimes b = c$ if and only if

$$a \otimes (1,2,3)\, b = c, \qquad a \otimes (1,3,2)\, c = b, \qquad b \otimes (2,1,3)\, a = c,$$

$$b \otimes (2,3,1)\, c = a, \qquad c \otimes (3,1,2)\, a = b, \qquad c \otimes (3,2,1)\, b = a.$$

These six (not necessarily distinct) quasigroups $(Q, \otimes(i,j,k))$, where $\{i,j,k\} = \{1,2,3\}$, are called the *conjugates* of $(Q, \otimes)$ (see Stein [9]). If the multiplication table of a quasigroup $(Q, \otimes)$ defines a Latin square L, then the six Latin squares defined by the multiplication tables of its conjugates $(Q, \otimes(i,j,k))$ are called the conjugates of L. A Latin square which is orthogonal to its (i,j,k)-conjugate will be called (i,j,k)-*conjugate orthogonal*. In this article, we shall focus our attention on the problem of existence of $(3,2,1)$- (or $(1,3,2)$-) conjugate orthogonal Latin squares, which was investigated earlier by Phelps [8], Keedwell [7], and Bennett [1]. We shall present a direct method of construction of the starter-adder type to produce new and inequivalent $(3,2,1)$- (or $(1,3,2)$-) conjugate orthogonal Latin squares of orders $v = 14$, 15, 18, and 26. These constructions combined with earlier results will establish the existence of $(3,2,1)$- (or $(1,3,2)$-) conjugate orthogonal Latin squares of order v (COLS(v)) for all positive integers $v \neq 2, 6$. The constructions also guarantee the existence of $(3,2,1)$- (or $(1,3,2)$-) conjugate orthogonal idempotent Latin squares of order v (COILS(v)) for all positive integers $v \neq 2, 3, 6$ with one possible exception $v = 12$, and we thus obtain new Latin squares, which can be associated with a certain class of Mendelsohn designs described in [1, 7]. Moreover, the constructions directly provide improvements to the earlier results on incomplete conjugate orthogonal Latin squares contained in [2].

2. Notation and Construction

In most of what follows, we shall need the concept of incomplete conjugate orthogonal Latin squares described in [2]. Let $S = \{0, 1, \ldots, v-n-1\} \cup X$, where $X = \{x_1, x_2, \ldots, x_n\}$. For a Latin square L of order v based on the v-set S, we shall denote by $e_L(i,j)$ the entry in the cell (i,j) of the array. An incomplete Latin square L of order v based on S and missing a subsquare of order n based on X will be called idempotent provided $e_L(i, i) = i$ for all i, $0 \leqslant i \leqslant v-n-1$. We shall denote by ICOILS(v, n) an incomplete (3, 2, 1)-conjugate orthogonal idempotent Latin square of order v based on S and missing a subsquare of order n based on X. Our method of construction will first involve the starter–adder technique to obtain ICOILS(v, n). We then fill in the missing subsquare of order n with a COILS(n) (or COLS(n)) to get a COILS(v) (or COLS(v)). The starter–adder technique has been used extensively by various authors (see, e.g., [4–6, 10]). The interested reader may wish to refer to [2] for other techniques used in the construction of ICOILS(v, n). In this article the ICOILS(v, n) L, based on the set S described above, will be generated cyclically from its first row given by the vectors $\mathbf{e} = (e_L(0, 0), ..., e_L(0, v-n-1))$ and $\mathbf{f} = (e_L(0, v-n), ..., e_L(0, v-1))$, and the last n elements of its first column given by the vector $\mathbf{g} = (e_L(v-n, 0), ..., e_L(v-1, 0))$. The ICOILS($v, n$) is constructed modulo $v-n$ in the range $\{0, 1, ..., v-n-1\}$, where the x_i's act as "infinity" elements as follows:

(a) $e_L(s+1, t+1) = e_L(s, t)$ if $e_L(s, t) = x_i$, and $e_L(s+1, t+1) \equiv e_L(s, t) + 1 \pmod{v-n}$ otherwise, where $0 \leqslant s, t \leqslant v-n-1$.

(b) $e_L(s+1, v-n-1+t) \equiv e_L(s, v-n-1+t) + 1 \pmod{v-n}$, where $1 \leqslant t \leqslant n$, $0 \leqslant s < v-n-1$.

(c) $e_L(v-n-1+t, s+1) \equiv e_L(v-n-1+t, s) + 1 \pmod{v-n}$, where $1 \leqslant t \leqslant n$, $0 \leqslant s < v-n-1$.

We remark that there are obviously conditions which the vectors **e**, **f**, and **g** must satisfy in order to produce the ICOILS(v, n) and this becomes the major task in our constructions. For $v = 14$, 15, 18, and 26, we shall present the vectors **e**, **f**, and **g** for a specified n in Table I. It is a straightforward matter to check that a (3, 2, 1)-ICOILS(v, n) results in each case. From the constructions given in Table I, we may obtain new or inequivalent (3, 2, 1)-COILS(v) (or COLS(v)) for $v = 14$, 15, 18, and 26 by

TABLE I

(v, n)	**e**	**f**	**g**
(14, 3)	$(0, 8, 10, 5, x_1, 6, x_2, 1, x_3, 4, 2)$	(7, 3, 9)	(9, 10, 4)
(14, 4)	$(0, 6, x_1, 5, x_2, 9, x_3, 3, x_4, 7)$	(8, 4, 2, 1)	(1, 7, 3, 9)
(15, 4)	$(0, x_1, x_2, 7, 3, 8, 4, 9, x_3, 10, x_4)$	(6, 1, 2, 5)	(8, 5, 6, 7)
(18, 4)	$(0, x_1, x_2, 10, 7, 13, 2, x_3, 6, 4, 9, 12, x_4, 1)$	(3, 8, 11, 5)	(11, 5, 4, 6)
(18, 5)	$(0, 5, x_1, x_2, x_3, 3, 8, 4, 2, x_4, 6, x_5, 11)$	(1, 7, 9, 10, 12)	(6, 1, 5, 3, 8)
(26, 7)	$(0, 10, 9, x_1, 12, x_2, 3, 2, x_3, 14, x_4, x_5, x_6, 16, x_7, 7, 17, 4, 1)$	(5, 6, 8, 11, 13, 15, 18)	(12, 4, 15, 17, 10, 13, 18).

(3, 2, 1)-COILS(14)

0	6	A	5	B	9	C	3	D	7	8	4	2	1
8	1	7	A	6	B	0	C	4	D	9	5	3	2
D	9	2	8	A	7	B	1	C	5	0	6	4	3
6	D	0	3	9	A	8	B	2	C	1	7	5	4
C	7	D	1	4	0	A	9	B	3	2	8	6	5
4	C	8	D	2	5	1	A	0	B	3	9	7	6
B	5	C	9	D	3	6	2	A	1	4	0	8	7
2	B	6	C	0	D	4	7	3	A	5	1	9	8
A	3	B	7	C	1	D	5	8	4	6	2	0	9
5	A	4	B	8	C	2	D	6	9	7	3	1	0
1	2	3	4	5	6	7	8	9	0	A	D	B	C
7	8	9	0	1	2	3	4	5	6	C	B	D	A
3	4	5	6	7	8	9	0	1	2	D	A	C	B
9	0	1	2	3	4	5	6	7	8	B	C	A	D

FIGURE 1

appropriately filling in the missing subsquares of the ICOILS(v, n). In particular, we illustrate the construction for $v = 14$, $n = 4$ to obtain a (3, 2, 1)-COILS(14) in Fig. 1. The existence of such a square has been in doubt for quite some time (see [1, 8]). Note that by using $v = 14$ and $n = 3$ with a (3, 2, 1)-COLS(3), we obtain a (3, 2, 1)-COLS(14) which is inequivalent to that of Fig. 1. For convenience we have put $x_1 = A$, $x_2 = B$, $x_3 = C$ and $x_4 = D$ in Fig. 1.

Since there exist (3, 2, 1)-COILS(n) for $n = 4, 5$, and 7, we may obtain from Table I a (3, 2, 1)-COILS(15) which is inequivalent to that arising from a construction of Keedwell (see [7]). We also obtain two new and inequivalent (3, 2, 1)-COILS(18) (see [1]) and a (3, 2, 1)-COILS(26), which is inequivalent to that presented by the first author in [1]. In addition, we observe that the constructions provide for Latin squares with aligned subsquares and hence lead to improvements of the results in [2]. Noting that the existence of a (3, 2, 1)-COILS(v) (or COLS(v)) is equivalent to the existence of a (1, 3, 2)-COILS(v) (or COLS(v)) (see [8]), we may combine our constructions with the previous results in [1, 7, 8] to obtain the following:

THEOREM 2.1. *A (3, 2, 1) (or (1, 3, 2))-COLS(v) exists for every positive integer $v \neq 2, 6$.*

THEOREM 2.2. *A (3, 2, 1) (or (1, 3, 2))-COILS(v) exists for every positive integer $v \neq 2, 3, 6$ with the possible exception of $v = 12$.*

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