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Monopole–antimonopole chains

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Abstract

We present new static axially symmetric solutions of SU(2) Yang–Mills–Higgs theory, representing chains of monopoles and antimonopoles in static equilibrium. They correspond to saddlepoints of the energy functional and exist both in the topologically trivial sector and in the sector with topological charge one.

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1. Introduction

Magnetic monopole solutions are a generic prediction of grand unified theories. Such solutions possess a topological charge proportional to their magnetic charge. In Yang–Mills–Higgs (YMH) theory with gauge group SU(2) the simplest solution with unit topological charge is the spherically symmetric 't Hooft–Polyakov monopole [1,2]. SU(2) monopoles with higher topological charge cannot be spherically symmetric [3] and possess at most axial symmetry [4–7].

In the Bogomol'nyi–Prasad–Sommerfield (BPS) limit of vanishing Higgs potential spherically symmetric monopole and axially symmetric multimonopole solutions, which satisfy the first order Bogomol'nyi equations [8] as well as the second order field equations, are known analytically [5–7,9]. For these solu-

tions all zeros of the Higgs field are superimposed at a single point. Multimonopole solutions of the Bogomol'nyi equations which do not possess any rotational symmetry [10], have recently been constructed numerically [11]. In these solutions the zeros of the Higgs field are no longer all superimposed at a single point but are located at several isolated points.

As shown by Taubes, in each topological sector there exist in addition smooth, finite energy solutions of the second order field equations, which do not satisfy the Bogomol'nyi equations [12]. Consequently, the energy of these solutions exceeds the Bogomol'nyi bound. The simplest such solution resides in the topologically trivial sector and forms a saddlepoint of the energy functional [13]. It possesses axial symmetry, and the two zeros of its Higgs field are located symmetrically on the positive and negative z -axis. This solution corresponds to a monopole and antimonopole in static equilibrium [14,15].

In this Letter we present new axially symmetric saddlepoint solutions, where the Higgs field vanishes at $m > 2$ isolated points on the symmetry axis. These

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configurations represent chains of monopoles and antimonopoles, in alternating order. Chains with an equal number of monopoles and antimonopoles, $m = 2k$, reside in the topologically trivial sector, whereas chains with $k + 1$ monopoles and k antimonopoles reside in the sector with topological charge one.

After briefly reviewing SU(2) Yang–Mills–Higgs theory in Section 2, we discuss the axially symmetric ansatz in Section 3 together with the boundary conditions. We present the numerical results in Section 4 and our conclusions in Section 5.

2. Yang–Mills–Higgs theory

The Lagrangian of SU(2) Yang–Mills–Higgs theory is given by

$$-L = \int \left\{ \frac{1}{2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + \frac{1}{4} \text{Tr}(D_\mu \Phi D^\mu \Phi) + \frac{\lambda}{8} \text{Tr}[(\Phi^2 - \eta^2)^2] \right\} d^3r, \quad (1)$$

with su(2) gauge potential $A_\mu = \frac{A_\mu^a \tau^a}{2}$, field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ie[A_\mu, A_\nu]$, and covariant derivative of the Higgs field $D_\mu \Phi = \partial_\mu \Phi + ie[A_\mu, \Phi]$. e denotes the gauge coupling constant, η the vacuum expectation value of the Higgs field and λ the strength of the Higgs selfcoupling.

The field equations are derived from the Lagrangian as the variational equations with respect to the gauge potential and the Higgs field,

$$D_\mu F^{\mu\nu} - \frac{1}{4} ie[\Phi, D^\nu \Phi] = 0, \quad (2)$$

$$D_\mu D^\mu \Phi - \lambda(\Phi^2 - \eta^2)\Phi = 0. \quad (3)$$

3. Ansatz and boundary conditions

We parametrize the gauge potential and the Higgs field by the ansatz

$$A_\mu dx^\mu = \left(\frac{K_1}{r} dr + (1 - K_2) d\theta \right) \frac{\tau_\varphi}{2e} - \sin\theta \left(K_3 \frac{\tau_r^{(m)}}{2e} + (1 - K_4) \frac{\tau_\theta^{(m)}}{2e} \right) d\varphi, \quad (4)$$

$$\Phi = \Phi_1 \tau_r^{(m)} + \Phi_2 \tau_\theta^{(m)}, \quad (5)$$

which generalizes the axially symmetric ansatz employed in [14,15] for the monopole–antimonopole pair solution. Here the su(2) matrices $\tau_r^{(m)}$, $\tau_\theta^{(m)}$, and τ_φ are defined as

$$\tau_r^{(m)} = \sin(m\theta)\tau_\rho + \cos(m\theta)\tau_z,$$

$$\tau_\theta^{(m)} = \cos(m\theta)\tau_\rho - \sin(m\theta)\tau_z,$$

$$\tau_\varphi = -\sin\varphi\tau_x + \cos\varphi\tau_y,$$

where $\tau_\rho = \cos\varphi\tau_x + \sin\varphi\tau_y$, and m is an integer to which we refer as θ winding number.

The profile functions K_1 – K_4 and Φ_1 , Φ_2 depend on the coordinates r and θ , only. With this ansatz the general field equations (2) and (3) reduce to six PDEs in the coordinates r and θ . The ansatz possesses a residual U(1) gauge symmetry. To fix the gauge we impose the condition $r\partial_r K_1 - \partial_\theta K_2 = 0$. For convenience we change to dimensionless coordinates by rescaling $r \rightarrow r/(e\eta)$ and $\Phi \rightarrow \eta\Phi$.

To obtain regular solutions with finite energy and energy density we have to impose appropriate boundary conditions. Regularity at the origin requires

$$K_1(0, \theta) = 0, \quad K_2(0, \theta) = 1,$$

$$K_3(0, \theta) = 0, \quad K_4(0, \theta) = 1,$$

$$\sin(m\theta)\Phi_1(0, \theta) + \cos(m\theta)\Phi_2(0, \theta) = 0,$$

$$\partial_r [\cos(m\theta)\Phi_1(r, \theta) - \sin(m\theta)\Phi_2(r, \theta)]|_{r=0} = 0.$$

To obtain the boundary conditions at infinity we require that solutions in the vacuum sector ($m = 2k$) tend to a gauge transformed trivial solution,

$$\Phi \rightarrow U \tau_z U^\dagger, \quad A_\mu \rightarrow i \partial_\mu U U^\dagger,$$

and that solutions in the sector with unit topological charge ($m = 2k + 1$) tend to

$$\Phi \rightarrow U \Phi_\infty^{(1)} U^\dagger, \quad A_\mu \rightarrow U A_{\mu\infty}^{(1)} U^\dagger + i \partial_\mu U U^\dagger,$$

where

$$\Phi_\infty^{(1)} = \tau_r^{(1)}, \quad A_{\mu\infty}^{(1)} dx^\mu = \frac{\tau_\varphi}{2} d\theta - \sin\theta \frac{\tau_\theta^{(1)}}{2} d\varphi$$

is the asymptotic solution of a monopole, and $U = \exp\{-ik\theta\tau_\varphi\}$, both for even and odd m . Consequently, solutions with even m have vanishing magnetic charge, whereas solutions with odd m possess unit magnetic charge.

In terms of the functions K_1 – K_4 , Φ_1 and Φ_2 these boundary conditions read

$$K_1 \rightarrow 0, \quad K_2 \rightarrow 1 - m, \tag{6}$$

$$K_3 \rightarrow \frac{\cos \theta - \cos(m\theta)}{\sin \theta} \quad m \text{ odd},$$

$$K_3 \rightarrow \frac{1 - \cos(m\theta)}{\sin \theta} \quad m \text{ even}, \tag{7}$$

$$K_4 \rightarrow 1 - \frac{\sin(m\theta)}{\sin \theta}, \tag{8}$$

$$\Phi_1 \rightarrow 1, \quad \Phi_2 \rightarrow 0. \tag{9}$$

Regularity on the z -axis requires

$$K_1 = K_3 = \Phi_2 = 0,$$

$$\partial_\theta K_2 = \partial_\theta K_4 = \partial_\theta \Phi_1 = 0,$$

for $\theta = 0$ and $\theta = \pi$.

4. Results

We have constructed numerically axially symmetric solutions of the Yang–Mills–Higgs theory in the BPS limit $\lambda = 0$ for θ winding number $1 \leq m \leq 6$.

These m -chains possess m nodes of the Higgs field on the z -axis. Due to reflection symmetry, each node on the negative z -axis corresponds to a node on the positive z -axis. The nodes of the Higgs field are associated with the location of the monopoles and antimonopoles. For odd m ($m = 2k + 1$) the Higgs field possesses k nodes on the positive z -axis and one node at the origin. The node at the origin corresponds to a monopole if k is even and to an antimonopole if k is odd. For even m ($m = 2k$) there is no node of the Higgs field at the origin.

The $m = 1$ solution is the well-known 't Hooft–Polyakov monopole [1,2]. The $m = 3$ (M–A–M) and $m = 5$ (M–A–M–A–M) chains represent saddlepoints with unit topological charge. The $m = 2$ chain is identical to the monopole–antimonopole (M–A) pair discussed in [14,15]. The M–A pair as well as the $m = 4$ (M–A–M–A) and $m = 6$ (M–A–M–A–M–A) chains form saddlepoints in the vacuum sector.

In Table 1 we present the dimensionless energy $E/4\pi\eta$, the dimensionless magnetic dipole moment μ and the nodes z_i of the Higgs field for the solutions with θ winding number $1 \leq m \leq 6$.

Table 1

The dimensionless energy, the dipole moment μ , the estimated dipole moment $\mu_{\text{est}}^{(m)}$, and the nodes z_i are given for the m -chains with $m = 1, \dots, 6$

m	$E[4\pi\eta]$	μ	μ_{est}	z_i
1	1.00	0.0	0.0	0.0
2	1.70	4.72	4.18	± 2.09
3	2.44	0.0	0.0	0.0 ± 4.67
4	3.12	9.86	9.21	$\pm 2.39 \pm 6.99$
5	3.78	0.0	0.0	$0.0 \pm 4.79 \pm 9.61$
6	4.40	16.06	15.40	$\pm 2.46 \pm 7.57 \pm 12.63$

We observe that the energy $E^{(m)}$ of an m -chain is always smaller than the energy of m single monopoles or antimonopoles (with infinite separation between them), i.e., $E^{(m)} < E_\infty = 4\pi\eta m$. On the other hand $E^{(m)}$ exceeds the minimal energy bound given by the Bogomol'nyi limit $E_{\text{min}}/4\pi\eta = 0$ for even m , and $E_{\text{min}}/4\pi\eta = 1$ for odd m . We observe an (almost) linear dependence of the energy $E^{(m)}$ on m . This can be modelled by taking into account only the energy of m single (infinitely separated) monopoles and the next-neighbour interaction between monopoles and antimonopoles on the chain. Defining the interaction energy as the binding energy of the monopole–antimonopole pair,

$$\Delta E = 2(4\pi\eta) - E^{(2)},$$

we obtain as energy estimate for the m -chain

$$E_{\text{est}}^{(m)}/4\pi\eta = m + (m - 1)\Delta E/4\pi\eta.$$

In Fig. 1 we show this estimate for the energy, $E_{\text{est}}^{(m)}/4\pi\eta$, together with the exact energy for chains with θ winding number $m = 1, \dots, 6$. We note that the deviation of the estimated energy from the exact energy is indeed very small.

In Table 2 we list the possible decompositions of the m -chains into subchains of smaller length together with the total energy of the subchains (at infinite separation). We observe that the energetically most favourable state is the m -chain. This supports our interpretation of the m -chain as an equilibrium state of m monopoles and antimonopoles. We note, however, that there are other possible decompositions not included in Table 2. For example, the M–A–M–A-chain could be decomposed into a pair of a charge 2 multimonopole and a charge -2 antimultimonopole.

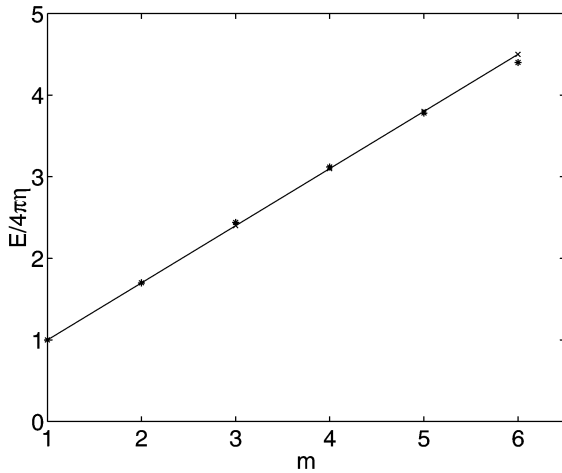


Fig. 1. The exact energy (asterisks) and the estimate for the energy (crosses) are shown for chains with $m = 1, \dots, 6$. The solid line demonstrates the linear dependence of $E_{\text{est}}^{(m)}$ on m .

To define the magnetic dipole moment for solutions with even m , we first transform to a gauge where the Higgs field is constant at infinity, $\Phi = \tau_z$. From the asymptotic expansion [16], we obtain $K_3 \rightarrow (1 - \cos(m\theta))/\sin(\theta) + C_3 \sin\theta/r$. Thus the gauge field assumes the asymptotic form

$$A_\mu dx^\mu = C_3 \frac{\sin^2 \theta}{2r} \tau_z d\varphi, \quad (10)$$

from which we read off the (dimensionless) magnetic dipole moment $\mu = C_3$. Solutions with odd m have vanishing magnetic dipole moment, since in this case the function K_3 is odd under the transformation $z \leftrightarrow -z$. Consequently, the asymptotic form of the gauge potential cannot contain terms like the r.h.s. of Eq. (10).

The magnetic dipole moment also increases with increasing m , as seen in Table 1. To obtain an estimate for the dipole moment we consider the magnetic charges as point charges located at the nodes of the

Table 2

The decompositions of the m -chains into subchains and their energies are given for $m = 1, \dots, 6$

m	Chain	$E[4\pi\eta]$	Decomposition	$E[4\pi\eta]$
2	M–A	$E^{(2)} = 1.70$	(M) + (A)	$2E^{(1)} = 2.00$
3	M–A–M	$E^{(3)} = 2.44$	(M–A) + (M) (M) + (A) + (M)	$E^{(2)} + E^{(1)} = 2.7$ $3E^{(1)} = 3.00$
4	M–A–M–A	$E^{(4)} = 3.12$	(M–A) + (M–A) (M–A–M) + (A) (M–A) + (M) + (A) (M) + (A) + (M) + (A)	$2E^{(2)} = 3.40$ $E^{(3)} + E^{(1)} = 3.44$ $E^{(2)} + 2E^{(1)} = 3.70$ $4E^{(1)} = 4.00$
5	M–A–M–A–M	$E^{(5)} = 3.78$	(M–A–M–A) + (M) (M–A–M) + (A–M) (M–A) + (M–A) + (M) (M–A–M) + (A) + (M) (M–A) + (M) + (A) + (M) (M) + (A) + (M) + (A) + (M)	$E^{(4)} + E^{(1)} = 4.12$ $E^{(3)} + E^{(2)} = 4.14$ $2E^{(2)} + E^{(1)} = 4.40$ $E^{(3)} + 2E^{(1)} = 4.44$ $E^{(2)} + 3E^{(1)} = 4.7$ $5E^{(1)} = 5.00$
6	M–A–M–A–M–A	$E^{(6)} = 4.40$	(M–A–M–A–M) + (A) (M–A–M–A) + (M–A) (M–A–M) + (A–M–A) (M–A) + (M–A) + (M–A) (M–A–M–A) + (M) + (A) (M–A–M) + (A–M) + (A) (M–A) + (M–A) + (M) + (A) (M–A–M) + (A) + (M) + (A) (M–A) + (M) + (A) + (M) + (A) (M) + (A) + (M) + (A) + (M) + (A)	$E^{(5)} + E^{(1)} = 4.78$ $E^{(4)} + E^{(2)} = 4.82$ $2E^{(3)} = 4.88$ $3E^{(2)} = 5.10$ $E^{(4)} + 2E^{(1)} = 5.12$ $E^{(3)} + E^{(2)} + E^{(1)} = 5.14$ $2E^{(2)} + 2E^{(1)} = 5.40$ $E^{(3)} + 3E^{(1)} = 5.44$ $E^{(2)} + 4E^{(1)} = 5.70$ $6E^{(1)} = 6.00$

Higgs field, yielding

$$\mu_{\text{est}}^{(m)} = \sum_{i=1,m} z_i P_i,$$

with charges $P_i = 1$ for monopoles and $P_i = -1$ for antimonopoles, respectively. For comparison the estimated magnetic dipole moments are also given in Table 1. The deviation from the exact values is on the order of 10%.

Concerning the nodes of the Higgs field we observe that the distances between the nodes increase with increasing m . Remarkably, the distances between the nodes do not vary much within a chain. For example, denoting the location of the nodes by z_i in decreasing order we find for the chain with θ winding number $m = 6$ from Table 1, $|z_1 - z_2| \approx 5.06$, $|z_2 - z_3| \approx 5.11$ and $|z_3 - z_4| \approx 4.92$.

In Fig. 2 we present the dimensionless energy density for the solutions with θ winding number $m = 1, \dots, 6$. The energy density of the m -chain possesses m maxima on the z -axis, and decreases with increasing ρ . The locations of the maxima are close to the nodes of the Higgs field, which are indicated by asterisks. For a given m the maxima are of similar magnitude, but their height decreases with increasing m . (Note that the scale for the $m = 1$ solution is different compared to the $m \geq 2$ solutions, and that the contour lines are distorted due to the different scaling of the ρ - and z -axis.)

5. Conclusion

We have obtained new static axially symmetric solutions of the SU(2) Yang–Mills–Higgs theory which represent monopole–antimonopole chains. They are characterized by the θ winding number m , which equals the number of nodes of the Higgs field, and the total number of monopoles and antimonopoles. Solutions with even m carry no magnetic charge but possess a non-vanishing magnetic dipole moment, whereas solutions with odd m carry unit magnetic charge but possess no magnetic dipole moment. The energy of these m -chains increases (almost) linearly with m .

We interpret the m -chains as equilibrium states of m monopoles and antimonopoles. As shown long ago [17], the force between monopoles is given

by twice the Coulomb force when the charges are unequal, and vanishes when the charges are equal, provided the monopoles are at large distances. Thus, monopoles and antimonopoles can only be in static equilibrium, if they are close enough to experience a repulsive force that counteracts the attractive Coulomb force. In other words, m -chains are essentially non-BPS solutions. To see this in another way let us rewrite the energy in the form

$$E = \int \left\{ \frac{1}{4} \text{Tr}((\varepsilon_{ijk} F_{ij} \pm D_k \Phi)^2) \mp \frac{1}{2} \varepsilon_{ijk} \text{Tr}(F_{ij} D_k \Phi) \right\} d^3r. \quad (11)$$

The second term is proportional to the topological charge and vanishes when m is even. The first term is just the integral of the square of the Bogomol'nyi equations. Thus, for even m the energy is a measure for the deviation of the solution from selfduality.

So far we have considered only solutions in the vacuum sector or the sector with unit topological charge. Generalizing the ansatz to φ winding number $n > 1$ leads to new solutions, which carry topological charge n if m is odd, and to solutions in the vacuum sector if m is even [18]. Recently, the $n = 2, m = 2$ solution has been constructed in an extended model [19], which includes the YMH theory as a special case. Chains of $n = 2$ multimonopoles and antimultimonopoles will be presented elsewhere [18]. For $n > 2$ a new phenomenon occurs. The zeros of the Higgs field no longer form a set of isolated points. Instead the Higgs field vanishes on rings around the z -axis [18]. (For odd m the node at the origin persists.)

Dyonic solutions can be readily obtained from the m -chains [16,20], as outlined in [21,22]. Interestingly, these solutions carry electric charge even for solutions in the vacuum sector.

When the gravitational interaction is included, we anticipate a different behaviour for m -chains with finite magnetic charge and those with vanishing magnetic charge. For magnetically charged solutions a degenerate horizon may form at a critical value of the gravitational parameter, as observed for monopoles [23] and multimonopoles [24]. On the other hand, no formation of a horizon was found for the gravitating monopole–antimonopole pair [25].

We expect that solutions analogous to the m -chains exist in Weinberg–Salam theory [26–28], generalizing

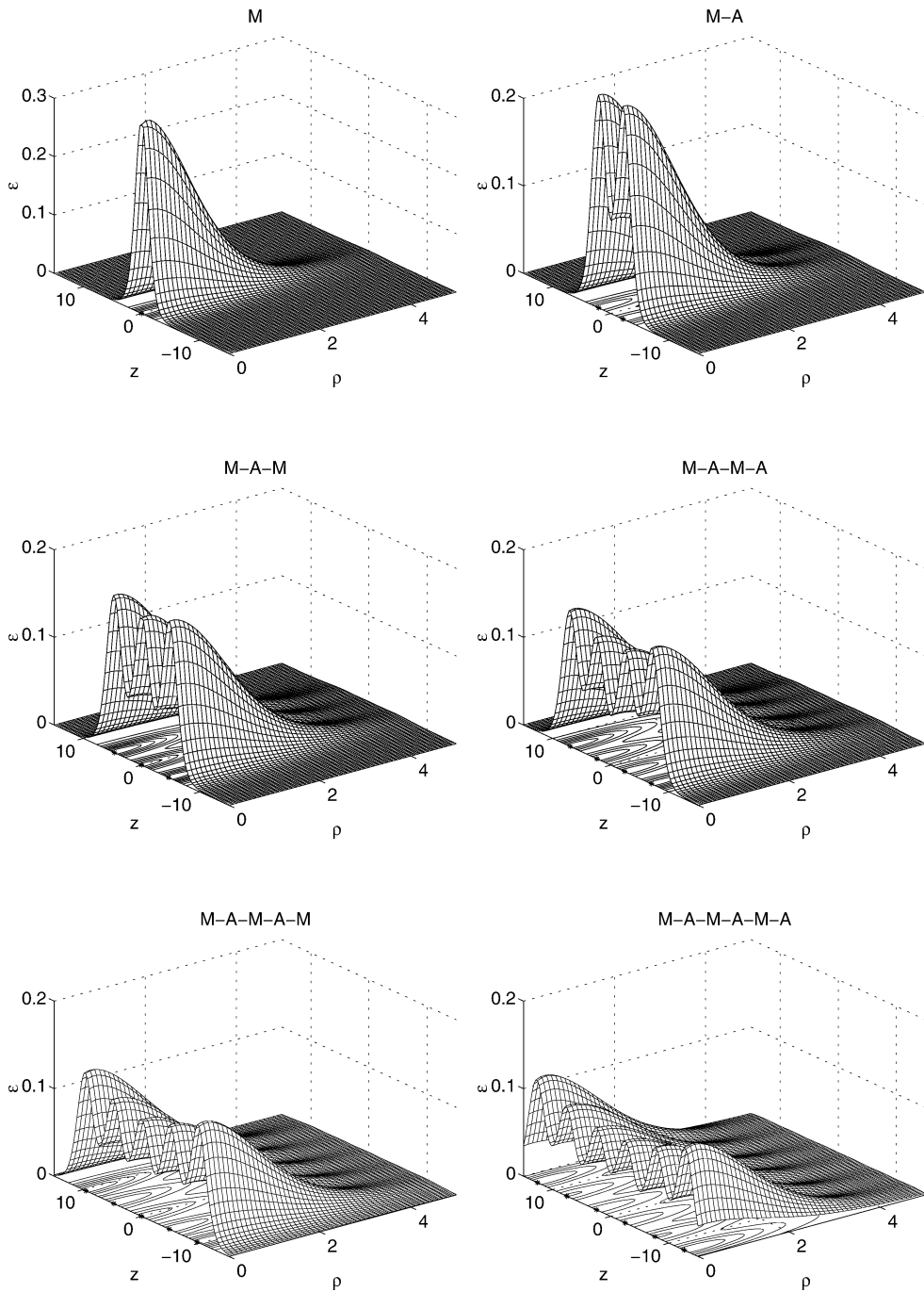


Fig. 2. The dimensionless energy density is shown as function of ρ and z for $m = 1, \dots, 6$. The asterisks indicate the nodes of the Higgs field.

the sphaleron–antisphaleron pair [27]. The axially symmetric ansatz with φ winding number n and θ winding number m [28] allows for (multi)sphaleron–anti(multi)sphaleron chains and solutions with rings of zeros.

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