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Dowker spaces and paracompactness questions

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Abstract

We construct, in ZFC, a hereditarily collectionwise normal, hereditarily metaLindelöf, hereditarily realcompact Dowker space. This answers a question of R. Hodel (also asked by S. Watson and D. Burke) and another question of M.E. Rudin. © 2001 Elsevier Science B.V. All rights reserved.

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Introduction

This paper was motivated by two problems.

1. Back in 1972, R. Hodel [3] raised

Problem 1. Is every metaLindelöf, collectionwise normal space paracompact?

A space [2] X is called metaLindelöf, if every open cover of X has a point-countable open refinement. X is collectionwise normal, if every discrete collection $\langle F_i\rangle_{i\in I}$ of closed sets can be expanded to a pairwise disjoint open collection, i.e., if there is a pairwise disjoint collection $\langle U_i\rangle_{i\in I}$ of open sets such that $U_i \supset F_i$ for every $i \in I$.

The question was also asked by Watson [8] and Burke [2]. Watson [8] points out that the only known counterexample is a consistent example of a screenable Dowker space by Rudin [6] constructed in 1983.

2. In her 1971 paper constructing a Dowker space in ZFC, Rudin [5] asks

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[2] "Space" in this paper means regular T_1 topological space.

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Problem 2. Is there a realcompact Dowker space?

A space X is realcompact if every z-ultrafilter on X with the countable intersection property is fixed. X is a Dowker space, if X is a normal space which is not countably paracompact.

Again, consistent examples of realcompact Dowker spaces are known [4].

The aim of this paper is to show that both of these questions can be answered in ZFC ("no" to Problem 1 and "yes" to Problem 2). Moreover, the answers can be consolidated into a single example such that all subspaces of the construction have the desired properties (except the Dowker property).

Main Theorem. *There is a hereditarily collectionwise normal, hereditarily metaLindelöf, hereditarily realcompact Dowker space X.*

Sections 1–5 of this paper are devoted to the proof of the Main Theorem. The technique we use is in the same family as the technique of constructing a screenable Dowker space [2], i.e., we construct a natural default hereditarily collectionwise normal, metaLindelöf (and realcompact) space and build in enough diagonalization (through countable elementary submodels) to make the outcome space not countably paracompact. The reflection tricks, however, are different from those in [2].

Sections 1 and 2 deal with the construction and the basic properties of the space X. These sections are relatively easy to read, even without pencil and paper, and will give the reader an idea of what the space looks like. The hard part is to show that X is not countably paracompact. Readers who want to construct examples with a similar technique may want to work through Sections 3–5 containing the proof that X is not countably paracompact. The use of countable structures, the technique of complete neighborhoods and the way the reflection works for open separations of uncountable relatively closed discrete collections are the main ideas.

Our terminology and notation are the standard ones used in set theory and set-theoretic topology. In particular, $[Y]^{\leqslant\kappa}$ is the set of all subsets of Y of cardinality $\leqslant\kappa$. We are going to use the following characterization of a Dowker space.

Proposition. *A space X is a Dowker space if and only if X is normal and there is an increasing open cover $\langle W_n\rangle_{n\in\omega}$ of X with no countable closed refinement.*

1. The construction of X

The set of points of X is $\mathfrak{c}\times\omega$. Let us fix the notation $W_n=\mathfrak{c}\times n$ for the union of the first n rows and $C_\beta=\{\beta\}\times\omega$ for the βth column ($n\in\omega$, $\beta\in\mathfrak{c}$). Let $\pi:\mathfrak{c}\times\omega\to\mathfrak{c}$ denote the natural projection, let $\{q_n\colon n\in\omega\}$ be an open base for the Cantor space topology on $\mathfrak{c}=2^\omega$. We are going to start with the simple topology on $X=\mathfrak{c}\times\omega$ generated by

$$\mathcal{B}_0=\big\{W_n\colon n\in\omega\big\}\cup\big\{X\setminus\{x\}\colon x\in X\big\}\cup\big\{\pi^{\leftarrow}(q_n)\colon n\in\omega\big\}$$

as a subbase. This topology is T_1, and it would be easy to show that it is realcompact. Furthermore, $\{W_n\colon\ n \in \omega\}$ is an increasing open cover of X with no point-finite closed refinement in this topology. Unfortunately, this initial topology is not even Hausdorff, let alone (hereditarily collectionwise) normal. To achieve that we need $2^{\mathfrak{c}}$ more steps. In each step we consider either a potential relatively closed discrete collection to be separated by disjoint open sets, or an open collection to be given a point-countable open refinement. These collections will show up as Type I, respectively Type II sequences. Let us say that $S = \langle O, \langle F^\rho \rangle_{\rho<\mathfrak{c}} \rangle$ is a *Type I sequence* if $O \subset X$ and $\langle F^\rho \rangle_{\rho<\mathfrak{c}}$ is a sequence of pairwise disjoint subsets of O. $S = \langle U^\rho \rangle_{\rho<\mathfrak{c}}$ will be called a *Type II sequence* if $U^\rho \subset X$ for every $\rho < \mathfrak{c}$.

If $A \subset X$, then let

$$S\lceil A = \big\langle O \cap A, \langle F^\rho \cap A \rangle_{\rho \in \pi(A)} \big\rangle \quad \text{if } S \text{ is Type I,}$$

and let

$$S\lceil A = \langle U_\rho \cap A \rangle_{\rho \in \pi(A)} \quad \text{if } S \text{ is Type II.}$$

We are going to define our separations and refinements from countable chunks $S\lceil A$ of S via control triples defined below. Let $S(A) = \{S\lceil A\colon\ S \text{ is Type I}\}$.

Definition 1.1. $\langle A, D, u \rangle$ is a *control triple* if and only if the following conditions are satisfied:

(C_1) $A \in [X]^\omega$;
(C_2) $D \in [S(A)]^{\leqslant\omega}$;
(C_3) u is a function with $\mathrm{dom}(u) \in [A]^\omega$ such that $u(x) \in [S(A) \setminus D]^{\leqslant\omega}$ for every $x \in \mathrm{dom}(u)$;
(C_4) $x \neq x'$ in $\mathrm{dom}(u)$ implies $u(x) \cap u(x') = \emptyset$.

Let $\langle A_\beta,\ D_\beta,\ u_\beta \rangle_{\beta<\mathfrak{c}}$ list all control triples mentioning each $\mathfrak{c}$ times.

Now, let $\langle S_\xi \rangle_{\xi<2^{\mathfrak{c}}}$ list all Type I and Type II sequences mentioning each $2^{\mathfrak{c}}$ times.

We will construct an increasing sequence $\langle \mathcal{B}_\tau \rangle_{\tau<2^{\mathfrak{c}}}$ of subbases for topologies on X. Subsets of X which are open in the topology generated by $\mathcal{B}_\tau$ will be called *τ-open*.

$\mathcal{B}_0$ has already been constructed.

If $\tau < 2^{\mathfrak{c}}$ is a limit ordinal, then set $\mathcal{B}_\tau = \bigcup_{\xi<\tau} \mathcal{B}_\xi$.

If $\tau = \xi + 1 < 2^{\mathfrak{c}}$, then we consider several cases according to what S_ξ is.

Case 1. Suppose that $S_\xi = \langle O_\xi, \langle F_\xi^\rho \rangle_{\rho<\mathfrak{c}} \rangle$ is a Type I sequence, that O_ξ is ξ-open, that $\langle F_\xi^\rho \rangle_{\rho<\mathfrak{c}}$ is a relatively discrete sequence of relatively closed sets in the subspace O_ξ of the space X with the topology generated by $\mathcal{B}_\xi$, and that ξ is minimal in $2^{\mathfrak{c}}$ to satisfy all of the above conditions. Then we will define a pairwise disjoint expansion $\langle B_\xi^\rho \rangle_{\rho<\mathfrak{c}}$ of $\langle F_\xi^\rho \rangle_{\rho<\mathfrak{c}}$ by subsets of O_ξ and we will set

$$\mathcal{B}_\tau = \mathcal{B}_{\xi+1} = \mathcal{B}_\xi \cup \big\{B_\xi^\rho\colon\ \rho < \mathfrak{c}\big\}.$$

To do this, let us introduce the following notation:

$$F_\xi = \bigcup_{\rho<\mathfrak{c}} F_\xi^\rho; \qquad B_\xi = \bigcup_{\rho<\mathfrak{c}} B_\xi^\rho;$$

if $x \in F_\xi$ then $\bar{x}(\xi)$ is the unique $\rho < \mathfrak{c}$ with $x \in F_\xi^\rho$; if $x \in B_\xi$ then $x(\xi)$ is the unique $\rho < \mathfrak{c}$ with $x \in B_\xi^\rho$ for the sets B_ξ^ρ to be constructed at step ξ.

Which $\langle \beta, j\rangle \in O_\xi$ goes to which B_ξ^ρ (or to no B_ξ^ρ at all) will be decided by induction on β. So suppose that $\beta < \mathfrak{c}$, we have decided on points of $\bigcup_{\alpha<\beta} C_\alpha$, and consider the βth column $C_\beta = \{\beta\} \times \omega$.

Subcase 1.1. If $S_\xi \lceil A_\beta \in D_\beta$, then we decide on points of $O_\xi \cap C_\beta$ in the following way.

(a) If $F_\xi \cap C_\beta = \emptyset$, then $B_\xi \cap C_\beta = \emptyset$ (i.e., set $\langle \beta, j\rangle \notin B_\xi^\rho$ for each $\rho < \mathfrak{c}$ and $j \in \omega$).

(b) If $F_\xi \cap C_\beta \neq \emptyset$, then pick the smallest $j \in \omega$ such that $\langle \beta, j\rangle \in F_\xi \cap C_\beta$. Then for every $\langle \beta, i\rangle \in O_\xi \cap C_\beta$,

(i) if $\langle \beta, i\rangle \in F_\xi \setminus F_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$, then let $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, i\rangle}(\xi)}$;

(ii) if $\langle \beta, i\rangle \in (O_\xi \setminus F_\xi) \cup F_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$, then let $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$.

[In words, (i) and (ii) together say that if $\langle \beta, i\rangle \in F_\xi^\rho$ then we (have to) put $\langle \beta, i\rangle$ in B_ξ^ρ, but otherwise we put $\langle \beta, i\rangle$ in $B_\xi^{\overline{\langle \beta, j\rangle}(\xi)}$.]

Subcase 1.2. Suppose that there is an $x = \langle \alpha, n\rangle \in \operatorname{dom}(u_\beta)$ such that $\alpha < \beta$, $x \in B_\xi$ and $S_\xi \lceil A_\beta \in u_\beta(x)$. Note that $S_\xi \lceil A_\beta \notin D_\beta$ by (C_3) and there is only one such x by (C_4). Recall that $x(\xi)$ is the unique $\rho < \mathfrak{c}$ with $x \in B_\xi^\rho$.

Now, for each $\langle \beta, i\rangle \in O_\xi \cap C_\beta$,

(a) if $\langle \beta, i\rangle \in F_\xi \setminus F_\xi^{x(\xi)}$, then let $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, i\rangle}(\xi)}$;

(b) if $\langle \beta, i\rangle \in (O_\xi \setminus F_\xi) \cup F_\xi^{x(\xi)}$, then let $\langle \beta, i\rangle \in B_\xi^{x(\xi)}$.

[In words, (1.2a) and (1.2b) say that if $\langle \beta, i\rangle \in F_\xi^\rho$ for some $\rho < \mathfrak{c}$, then we (have to) set $\langle \beta, i\rangle \in B_\xi^\rho$, but otherwise we put $\langle \beta, i\rangle$ in $B_\xi^{x(\xi)}$.]

Subcase 1.3. Suppose neither Subcase 1.1 nor Subcase 1.2 holds. Then for every $\langle \beta, i\rangle \in O_\xi \cap C_\beta$, let $\langle \beta, i\rangle \in B_\xi^{\overline{\langle \beta, i\rangle}(\xi)}$ if $\langle \beta, i\rangle \in F_\xi$, and let $\langle \beta, i\rangle \in B_\xi^0$, if $\langle \beta, i\rangle \in O_\xi \setminus F_\xi$.

Case 2. Suppose that $S_\xi = \langle U_\xi^\rho\rangle_{\rho<\mathfrak{c}}$ is Type II *and* is a sequence of ξ-open sets. Then we are going to define $V_\xi^\rho \subset U_\xi^\rho$ in such a way that $\bigcup_{\rho<\mathfrak{c}} V_\xi^\rho = \bigcup_{\rho<\mathfrak{c}} U_\xi^\rho$ and $\langle V_\xi^\rho\rangle_{\rho<\mathfrak{c}}$ is point-countable. Then we set

$$\mathcal{B}_\tau = \mathcal{B}_{\xi+1} = \mathcal{B}_\xi \cup \{V_\xi^\rho\colon\ \rho < \mathfrak{c}\}.$$

For every $\langle \beta, i\rangle \in \bigcup_{\rho<\mathfrak{c}} U_\xi^\rho$ we must decide which sets V_ξ^ρ the point $\langle \beta, i\rangle$ will belong to. This is easily done in two subcases.

Subcase 2.1. Suppose there is a $\rho \in \pi(A_\beta)$ such that $\langle \beta, i\rangle \in U_\xi^\rho$. Then set

$$\langle \beta, i\rangle \in V_\xi^\rho \quad \text{if and only if} \quad \langle \beta, i\rangle \in U_\xi^\rho \text{ and } \rho \in \pi(A_\beta).$$

Subcase 2.2. Not Subcase 2.1. Then set $\langle \beta, i\rangle \in V_\xi^\rho$ if and only if $\rho < \mathfrak{c}$ is the smallest ordinal with $\langle \beta, i\rangle \in U_\xi^\rho$.

It is clear from the above definition that $\langle V_\xi^\rho\rangle_{\rho<\mathfrak{c}}$ is point-countable.

Case 3. Neither Case 1 nor Case 2 holds. Then let $\mathcal{B}_\tau = \mathcal{B}_{\xi+1} = \mathcal{B}_\xi$.

Finally, the topology of X is generated by $\mathcal{B} = \bigcup_{\xi<2^\mathfrak{c}} \mathcal{B}_\xi$ as a subbase. For ease of reference let $H_i = \{\xi < 2^\mathfrak{c}\colon\ \text{Case } i \text{ holds}\}$ $(i = 1, 2)$ and let $H = H_1 \cup H_2$.

2. X is hereditarily collectionwise normal, hereditarily metaLindelöf and hereditarily realcompact

Proposition 2.1. *X is hereditarily collectionwise normal and hereditarily metaLindelöf.*

Proof. To prove that X is hereditarily collectionwise normal, let O be an open subspace of X and let $\mathcal{F}$ be a relatively closed discrete collection in the subspace O. Let $\langle F^\rho\rangle_{\rho<\mathfrak{c}}$ list each nonempty member of $\mathcal{F}$ exactly once and possibly $\emptyset$ several times to make a sequence of length $\mathfrak{c}$. Note that "O is open and $\langle F^\rho\rangle_{\rho<\mathfrak{c}}$ is a relatively closed discrete collection in O" is witnessed by $\leqslant \mathfrak{c}$ subbasic open sets. Since each term of $\langle S_\xi\rangle_{\xi<2^\mathfrak{c}}$ is listed $2^\mathfrak{c}$ times and $\mathrm{cf}(2^\mathfrak{c}) > \mathfrak{c}$, it follows that there is a first $\xi < 2^\mathfrak{c}$ such that $S_\xi = \langle O, \langle F^\rho\rangle_{\rho<\mathfrak{c}}\rangle$, O is ξ-open, and $\langle F^\rho\rangle_{\rho<\mathfrak{c}}$ is a relatively discrete sequence of relatively closed sets in the subspace O of the space X with the topology generated by $\mathcal{B}_\xi$. Then $\langle B_\xi^\rho\rangle_{\rho<\mathfrak{c}}$ is an open expansion of $\langle F_\xi^\rho\rangle_{\rho<\mathfrak{c}} = \langle F^\rho\rangle_{\rho<\mathfrak{c}}$ is O_ξ.

The proof that X is hereditarily metaLindelöf follows similarly, making use of Type II sequences.

Proposition 2.2. *X hereditarily realcompact.*

Proof. Let Y be a subspace of X and let $\mathcal{Z}$ be a z-ultrafilter on Y with $\bigcap \mathcal{Z} = \emptyset$. We need to show that there is a countable $\mathcal{Z}' \subset \mathcal{Z}$ such that $\bigcap \mathcal{Z}' = \emptyset$. Let

$$\mathcal{E} = \{\pi^{\leftarrow}(q_n) \cap Y\colon n \in \omega\}.$$

To prove the existence of $\mathcal{Z}'$, we will consider two cases.

Case 1. Suppose that for every $y \in Y$ there is an E_y such that $y \in E_y \in \mathcal{E}$ and $Y \setminus E_y \in \mathcal{Z}$. Then $\mathcal{Z}' = \{Y \setminus E_y\colon y \in Y\}$ is as required.

Case 2. Suppose Case 1 does not hold, i.e., there is a $y \in Y$ such that whenever $y \in E \in \mathcal{E}$, then $Y \setminus E \notin \mathcal{Z}$. Since $\mathcal{Z}$ is a z-ultrafilter and each E is clopen, this implies that $\mathcal{E}_y = \{E \in \mathcal{E}\colon y \in E\} \subset \mathcal{Z}$. Note that if $y = \langle\beta, m\rangle$, then $\bigcap \mathcal{E}_y = (\{\beta\} \times \omega) \cap Y$ is countable. Since $\bigcap \mathcal{Z} = \emptyset$ we can add countably many more members of $\mathcal{Z}$ to $\mathcal{E}_y$ to obtain a countable $\mathcal{Z}' \subset \mathcal{Z}$ with $\bigcap \mathcal{Z}' = \emptyset$. □

3. Complete neighborhoods

Let $x \in X$. A finite intersection of subbasic sets from $\bigcup_{1\leqslant\xi<2^\mathfrak{c}} B_\xi$ is described by a finite function t such that $\mathrm{dom}(t) \in [H]^{<\omega}$ and

(a) $\xi \in \mathrm{dom}(t) \cap H_1$ implies $t(\xi) \in \mathfrak{c}$ and $x \in B_\xi^{t(\xi)}$ (i.e., $t(\xi) = x(\xi)$);

(b) $\xi \in \mathrm{dom}(t) \cap H_2$ implies $\emptyset \neq t(\xi) \in [\mathfrak{c}]^{<\omega}$ and $x \in \bigcap_{\rho\in t(\xi)} V_\xi^\rho$.

Call a finite function t as above *compatible with* x. For t compatible with x, let us set

$$B_\xi^t = \begin{cases} B_\xi^{x(\xi)}, & \text{if } \xi \in \mathrm{dom}(t) \cap H_1, \\ \bigcap_{\rho\in t(\xi)} V_\xi^\rho, & \text{if } \xi \in \mathrm{dom}(t) \cap H_2. \end{cases}$$

Note that the sets

$$V_{t,K,n}(x)=\bigcap_{\xi\in\mathrm{dom}(t)} B_\xi^t\cap\pi^{\leftarrow}(q_n)\setminus K,$$

where t is compatible with x, $\pi(x)\in q_n$ and $K\in[X\setminus\{x\}]^{<\omega}$ form an open neighborhood base for x.

For every $\xi\in H_1$ with $x\in O_\xi$, let

$$O_\xi(x)=\begin{cases} O_\xi\setminus F_\xi, & \text{if } x\in O_\xi\setminus F_\xi,\\ (O_\xi\setminus F_\xi)\cup F_\xi^{\bar{x}(\xi)}, & \text{if } x\in F_\xi.\end{cases}$$

Given a basic open neighborhood $V_{t,K,n}(x)$ of x, let $U_\xi^t(x)=O_\xi(x)$ if $\xi\in\mathrm{dom}(t)\cap H_1$, and $U_\xi^t(x)=\bigcap_{\rho\in t(\xi)}U_\xi^\rho$ if $\xi\in\mathrm{dom}(t)\cap H_2$.

Definition 3.1. We will say that a basic open neighborhood $V_{t,k,n}(x)$ of x is *complete* if for every $\xi\in\mathrm{dom}(t)$, $V_{t\lceil\xi,K,n}(x)\subset U_\xi^t(x)$.

Completeness Lemma 3.2. *Every point $x\in X$ has a neighborhood basis consisting of complete neighborhoods.*

Proof. For an incomplete neighborhood $V_{t,K,n}(x)$, let $\xi_{t,K,n}$ denote the biggest $\xi\in\mathrm{dom}(t)$ such that $V_{t\lceil\xi,K,n}(x)\not\subset U_\xi^t(x)$. Our lemma follows from the following

Claim. *For every incomplete neighborhood $V_{t,K,n}(x)$ there is a neighborhood $V_{t',K',n'}(x)\subset V_{t,K,n}(x)$ such that either $V_{t',K',n'}(x)$ is complete or $\xi_{t',K',n'}<\xi_{t,K,n}$.*

To prove the claim, let $\eta=\xi_{t,K,n}$. Since $x\in U_\eta^t(x)$ and $U_\eta^t(x)$ is η-open, there are t'', K'', n'' such that $\mathrm{dom}(t'')\subset H\cap\eta$ and $x\in V_{t'',K'',n''}(x)\subset U_\eta^t(x)$.

Now let $\mathrm{dom}(t')=\mathrm{dom}(t)\cup\mathrm{dom}(t'')$ and set

$$t'(\xi)=\begin{cases} t(\xi), & \text{if } \xi\in\mathrm{dom}(t)\setminus\mathrm{dom}(t''),\\ t''(\xi), & \text{if } \xi\in\mathrm{dom}(t'')\setminus\mathrm{dom}(t),\\ x(\xi)(=t(\xi)=t''(\xi)), & \text{if } \xi\in\mathrm{dom}(t)\cap\mathrm{dom}(t'')\cap H_1,\\ t(\xi)\cup t''(\xi), & \text{if } \xi\in\mathrm{dom}(t)\cap\mathrm{dom}(t'')\cap H_2.\end{cases}$$

Let $K'=K\cup K''$ and $n'\in\omega$ be such that $x\in\pi^{\leftarrow}(q_{n'})\subset\pi^{\leftarrow}(q_n)\cap\pi^{\leftarrow}(q_n'')$. Then $x\in V_{t',K',n'}(x)\subset V_{t,K,n}(x)$, and for $\xi\in\mathrm{dom}(t')\setminus\eta=\mathrm{dom}(t)\setminus\eta$,

(a) $\xi=\eta$ implies $V_{t'\restriction\xi,K',n'}(x)\subset V_{t'',k''n''}(x)\subset U_\eta^t(x)=U_n^{t'}(x)$;

(b) $\xi>\eta$ implies $V_{t'\restriction\xi,K',n'}(x)\subset V_{t\restriction\xi,K,n}(x)\subset U_\xi^t(x)=U_\xi^{t'}(x)$. □

4. X is not countably paracompact: finding and reflecting β

Our goal is to show that $\langle W_m\rangle_{m\in\omega}$ is an open cover without a point-finite closed refinement. The proof will take up both this section and Section 5.

So suppose for contradiction that there is a sequence $\langle Z_m\rangle_{m\in\omega}$ of closed subsets of X such that $X=\bigcup_{m\in\omega}Z_m$ and $Z_m\subset W_m$ for every $m\in\omega$. For each $m\in\omega$ consider the unique $\xi_m \in H_1$ with $\langle O_{\xi_m}, \langle F^\rho_{\xi_m}\rangle_{\rho<\mathfrak{c}}\rangle$ satisfying $O_{\xi_m} = X$, $F^0_{\xi_m} = Z_m$, $F^1_{\xi_m} = X \setminus W_m$ and $F^\rho_{\xi_m} = \emptyset$ for $\rho \in \mathfrak{c} \setminus 2$. Recall that $\langle B^\rho_{\xi_m}\rangle_{\rho<\mathfrak{c}}$ is a pairwise disjoint open expansion of $\langle F^\rho_{\xi_m}\rangle_{\rho<\mathfrak{c}}$ and that $B_{\xi_m} = \bigcup_{\rho<\mathfrak{c}} B^\rho_{\xi_m}$ is an open subset of $O_{\xi_m} = X$ containing $F_{\xi_m} = \bigcup_{\rho<\mathfrak{c}} F^\rho_{\xi_m} = Z_m \cup (X \setminus W_m)$. For every $x = \langle\beta, m\rangle \in X$, let $V(x) = V_{t(x),K(x),n(x)}(x)$ be a basic neighborhood of x such that

(T_0) $\{\xi_j\colon j \leqslant m\} \subset \mathrm{dom}(t(x))$

and thus, by $x \in F^1_{\xi_m}$, $V(x) \subset B^1_{\xi_m} \subset X \setminus Z_m$. Let $t_1(x) = \mathrm{dom}(t(x)) \cap H_1$ (note that, unlike $t(x)$, $t_1(x)$ is just a finite subset of H_1, not a function), and for every subset $A \subset X$, introduce the notation $t_1(A) = \bigcup_{x\in A} t_1(x)$. By passing to a smaller neighborhood, if necessary we can (and will) assume that the neighborhoods $V(x) = V_{t(x),K(x),n(x)}(x)$ also satisfy the following properties:

(T_1) if $j < m < \omega$, $\xi \in t_1(\beta, j)$ and $\langle\beta, m\rangle \in B_\xi$, then $\xi \in t_1(\beta, m)$;

(T_2) each $V_{t(x),K(x),n(x)}$ is a complete basic open neighborhood.

Now let M, N be countable elementary submodels of $H((2^{2^\mathfrak{c}})^+) = \{$all sets whose transitive closure has cardinality $\leqslant 2^{2^\mathfrak{c}}\}$ in such a way that

$$\mathfrak{c}, \langle S_\xi\rangle_{\xi<2^\mathfrak{c}}, \ \langle B_\xi\rangle_{\xi<\mathfrak{c}}, \ H, H_1, t : X \to Fn(H,\mathfrak{c}) \cup Fn\big(H, [\mathfrak{c}]^{<\omega}\big),$$

$$\langle\xi_m\rangle_{m\in\omega}, t_1 : X \to [H]^{<\omega}, \quad K : X \to [X]^{<\omega}, \quad \langle x(\xi)\rangle_{\langle\xi,x\rangle\in H_1\times X} \in M \in N.$$

Let $A = N \cap X (= (N \cap \mathfrak{c}) \times \omega)$, $D = \{S_\xi\lceil A : \xi \in M \cap H_1\}$.

Proposition 4.1. *There is a function u satisfying (C_3) and (C_4) in the definition of a control triple and such that whenever $v : X \to [H_1 \setminus M]^{<\omega}$ is an infinite partial function, $v \in N$ and $x \neq x'$ in A implies $v(x) \cap v(x') = \emptyset$, then there are infinitely many $x \in \mathrm{dom}(v) \cap \mathrm{dom}(u)$ such that*

$$u(x) = \big\{S_\xi\lceil A\colon \xi \in v(x)\big\}.$$

Proof. Let $\langle v_j\rangle_{j\in\omega}$ enumerate all functions $v \in N$ as in Proposition 4.1 mentioning each infinitely many times. By induction on j pick distinct $\{x_j\colon j \in \omega\} \subset N \cap X$ in such a way that $j < m < \omega$ implies $v_j(x_j) \cap v_m(x_m) = \emptyset$. Set dom $(u) = \{x_j\colon j \in \omega\}$ and $u(x_j) = \{S_\xi\lceil A\colon \xi \in v_j(x_j)\}$. To show that ($C_3$) and ($C_4$) hold we only need to show that

(a) $u(x_j) \cap D = \emptyset$ for every $j \in \omega$;

(b) $j < m < \omega$ implies $u(x_j) \cap u(x_m) = \emptyset$.

Suppose indirectly that $u(x_j) \cap D \neq \emptyset$, i.e., there are $\xi \in v_j(x_j)$ and $\eta \in M \cap H_1$ such that $S_\xi\lceil A = S_\eta\lceil A$. We are going to show first that $\xi, \eta \in N$. Indeed, $\eta \in N$ follows from $\eta \in M$. To see $\xi \in N$, note that by $v_j, x_j \in N$, if follows that $v_j(x_j) \in N$. Since $v_j(x_j)$ is a finite set, $\xi \in v_j(x_j) \subset N$. Now since $\xi, \eta \in N$ and $S_\xi\lceil A = S_\eta\lceil A$, it follows that $S_\xi = S_\eta$. Since $\xi, \eta \in H_1$, this by the minimality condition in Case 1 implies $\xi = \eta$. Then $\xi \in v_j(x_j) \cap (M \cap H_1) = \emptyset$, contradiction.

The proof of (b) is similar. □

Pick and fix a u as in Proposition 4.1. For the rest of this section and Section 5, fix a $\beta\in\mathfrak{c}$ such that $\beta > \pi(A)$ and $\langle A, D, u\rangle = \langle A_\beta, D_\beta, u_\beta\rangle$.

Reflection Lemma 4.2. *Let $\theta \in M \cap H_1$, $k \in \omega$. Then there is an $x = \langle\alpha, k\rangle \in \mathrm{dom}(u)$ with the following properties*:

(R_0) $n(x) = n(\beta, k)$;

(R_1) $t_1(x) \cap M = t_1(\beta, k) \cap M$;

(R_2) $\forall\xi \in t_1(\beta, k) \cap M(\langle\beta, k\rangle(\xi) \in M$ *implies* $\langle\beta, k\rangle(\xi) = x(\xi))$;

(R_3) $\langle\beta, k\rangle \in B_\theta$ *if and only if* $x \in B_\theta$;

(R_4) *if* $\langle\beta, k\rangle \in B_\theta$, *then either* $\langle\beta, k\rangle(\theta) \in M$ *or* $\langle\beta, k\rangle(\theta) \neq x(\theta)$;

(R_5) $x \in \mathrm{dom}(u)$ *and* $u(x) = \{S_\xi\lceil A\colon\ \xi \in t_1(x) \setminus M\}$.

Proof. Let us introduce the notation $n = n(\beta, k)$, $r = t_1(\beta, k) \cap M$, $r^1 = \{\xi \in r : \langle\beta, k\rangle(\xi) \in M\}$ and $f(\xi) = \langle\beta, k\rangle(\xi)$ for every $\xi \in r^1$. Let $i = 1$ if $\langle\beta, k\rangle \in B_\theta$ and $i = 0$ if $\langle\beta, k\rangle \notin B_\theta$. Note $n, r, r^1, f, i \in M$. Let $\varphi(\alpha)$ denote the statement "$n(\alpha, k) = n$ and $t_1(\alpha, k) \supset r$, for every $\xi \in r^1$, $\langle\alpha, k\rangle(\xi) = f(\xi)$ and $\langle\alpha, k\rangle \in B_\theta$ if and only if $i = 1$". Note that all the parameters of $\varphi(\alpha)$ are from M, and that $\varphi(\beta)$ is true.

Let $\psi(E)$ denote the statement "$E \subset \mathfrak{c}$ and $\forall\alpha \in E$ $\varphi(\alpha)$, and $\alpha \neq \gamma$ in E implies $(t_1(\alpha, k)\backslash r) \cap (t_1(\alpha, k)\backslash r) = \emptyset$". Let $\chi(E)$ denote "$E \subset \mathfrak{c}$ and $\alpha \neq \gamma$ in E implies $\langle\alpha, k\rangle, \langle\beta, k\rangle \in B_\theta$ and $\langle\alpha, k\rangle(\theta) \neq \langle\gamma, k\rangle(\theta)$".

Claim.

(a) *There is an uncountable $E \in M$ such that $\psi(E)$ holds.*

(b) *Moreover, if $\langle\beta, k\rangle \in B_\theta$ and $\langle\beta, k\rangle(\theta) \notin M$, then there is an uncountable $E \in M$ such that $\psi(E)$ and $\chi(E)$ both hold.*

Proof. We will prove (b) only. The proof of (a) is similar (and simpler). To prove (b), note first that by Zorn's Lemma there is a maximal E such that $\psi(E)$ and $\chi(E)$ both hold. Since all the parameters in $\psi(E)$ and $\chi(E)$ are from M, we can (and will) assume that $E \in M$. Suppose indirectly that E is countable. Then $E \subset M$. Consider $E' = D \cup \{\beta\} \supsetneq E$. $\psi(E')$ holds, because $\forall\alpha \in E$,

$$\big(t_1(\alpha, k) \setminus r\big) \cap \big(t_1(\beta, k) \setminus r\big) \subset t_1(\alpha, k) \cap \big(t_1(\beta, k) \setminus M\big) \subset M \cap \big(t_1(\beta, k) \setminus M\big) = \emptyset.$$

$\chi(E')$ holds, because by the assumption in (b), $\langle\beta, k\rangle \in B_\theta$ and $\langle\beta, k\rangle(\theta) \notin M$, whereas $\langle\alpha, k\rangle(\theta) \notin M$ for every $\alpha \in E$. But then E' contradicts the maximality of E, finishing the proof of the claim. □

Now, if $\langle\beta, k\rangle \in B_\theta$ and $\langle\beta, k\rangle(\theta) \notin M$, then fix an E satisfying (b); otherwise fix an E satisfying (a). Let

$$E_1 = \big\{\alpha \in E\colon\ (t_1(\alpha, k) \setminus r) \cap M = \emptyset\big\}.$$

Note that $E_1 \in N$ and E_1 is infinite (even uncountable). Let us define a function v by setting $\mathrm{dom}(v) = E_1 \times \{k\}$ and $v(\alpha, k) = t_1(\alpha, k) \setminus r$ for every $\langle\alpha, k\rangle \in \mathrm{dom}(v)$. Note that $v \in N$ is as required in the statement of Proposition 4.1, so there are infinitely many $x = \langle\alpha, k\rangle \in \mathrm{dom}(u) \cap \mathrm{dom}(v)$ such that $u(x) = \{S_\xi \restriction A\colon\ \xi \in v(x)\}$. If $\langle\beta, k\rangle \in B_\theta$ and $\langle\beta, k\rangle(\theta) \notin M$, then the $x(\theta)$'s for these x are all distinct, so we can pick x so that $\langle\beta, k\rangle(\theta) \neq x(\theta)$ is also satisfied. This $x = \langle\alpha, k\rangle$ then satisfies $(R_0), (R_2), (R_3)$ by $\varphi(\alpha)$, (R_1) by $\varphi(\alpha)$ and $(t_1(\alpha, k) \setminus r) \cap M = \emptyset$ and $(R_4), (R_5)$ by its definition.

5. X is not countably paracompact: homogeneity of β

Let us say that β (as defined in the previous section) is *ξ-homogeneous* iff $\xi \in H_1$ and

(H) $\quad xi \in t_1(C_\beta)$ implies that there is a $\gamma \in M \cap \mathfrak{c}$ such that $O_\xi \cap C_\beta \subset B_\xi^\gamma$.

Proposition 5.1. *There is an $m \in \omega$ such that β is not ξ_m-homogeneous.*

Proof. Recall the definition of Z_m and ξ_m from the beginning of the previous section. Since $X = \bigcup_{m\in\omega} Z_m$, we can pick and fix an $m \in \omega$ such that $Z_m \cap C_\beta \neq \emptyset$. Note that $\xi_m \in t_1(\beta, m) \subset t_1(C_\beta)$. On the other hand, $O_{\xi_m} \cap C_\beta = X \cap C_\beta = C_\beta$ is not contained in any $B_{\xi_m}^\gamma$, because both $B_{\xi_m}^0 \supset Z_m$ and $B_{\xi_m}^1 \supset X \setminus W_m$ intersect C_β. $\square$

By $\xi_m \in M \cap H_1$, a contradiction (to "X is countably paracompact") will follow once we prove the following result.

Main Lemma 5.2. *β is ξ-homogeneous for every $\xi \in M \cap H_1$.*

The rest of this section will be devoted to the proof of the Main Lemma. Suppose indirectly that there is a minimal $\theta \in M \cap H_1$ such that β is not θ-homogeneous, i.e., $\theta \in t_1(C_\beta)$ yet there is no $\gamma \in M \cap \mathfrak{c}$ such that $O_\theta \cap C_\beta \subset B_\theta^\gamma$.

Then let us pick k so big that $\langle\beta, k\rangle \in B_\theta \cap C_\beta$ and with the notation $y[k] = \{\langle\beta, j\rangle\colon\ j \leqslant k\}$, the following conditions are satisfied:

(1_k) if $F_\theta \cap C_\beta \neq \emptyset$, then $F_\theta \cap y[k] \neq \emptyset$;

(2_k) if there are at least two $\rho \in \mathfrak{c}$ such that $F_\theta^\rho \cap C_\beta \neq \emptyset$, then there are at least two $\rho \in \mathfrak{c}$ such that $F_\theta^\rho \cap y[k] \neq \emptyset$;

(3_k) if there are at least two $\rho \in \mathfrak{c}$ such that $B_\theta^\rho \cap C_\beta \neq \emptyset$, then there are at least two $\rho \in \mathfrak{c}$ such that $B_\theta^\rho \cap y[k] \neq \emptyset$;

(4_k) $\theta \in t_1(\beta, k)$.

By (T_1) in Section 4, there is a $k \in \omega$ satisfying (1_k)–(4_k).

Now, let us fix an $x = \langle\alpha, k\rangle$ as in the Reflection Lemma 4.2 (for our θ above).

Lemma 5.3. $y[k] \subset V_{t(x)\restriction\theta, K(x), n(x)}(x)$.

Proof. Since $x \in N$, it follows that $K(x) \in N$ and by the finiteness of $K(x)$, $K(x) \subset N$. Since $\beta \notin N$, it follows that $C_\beta \cap K(x) \subset C_\beta \cap N = \emptyset$. Since by (R_0), $y[k] \subset C_\beta \subset \pi^{\leftarrow}(q_{n(x)})$, we conclude that $y[k] \subset V_{\emptyset, K(x), n(x)}(x)$. $\square$

By induction on ξ we are going to show that for every $\xi \in \mathrm{dom}(t(x)) \cap \theta$,

$$(I_\xi) \qquad y[k] \subset B_\xi^{t(x)}(x)$$

holds.

Suppose $\xi \in \mathrm{dom}(t(x)) \cap \theta$ and we have proved (I_η) for $\eta \in \mathrm{dom}(t(x)) \cap \xi$.

Then by completeness of $V_{t(x),K(x),n(x)}(x)$,

$$y[k] \subset V_{t(x)\lceil\xi,K(x),n(x)}(x) \subset U_\xi^t(x). \tag{$*$}$$

The proof of (I_ξ) from $(*)$ will be split into several cases depending on $\xi \in \mathrm{dom}(t(x)) \cap\ \theta$.

Case 1(a). Suppose that $\xi \in H_1 \cap M$. Then by $\xi < \theta$ and the minimality of θ it follows that β is ξ-homogeneous. Now $\xi \in \mathrm{dom}(t(x)) \cap H_1 \cap M = t_1(x) \cap M$. By (R_1) from the Reflection Lemma, $\xi \in t_1(\beta,k) \cap M \subset t_1(C_\beta)$. Hence by the definition of ξ-homogeneity there is a $\gamma \in M \cap \mathfrak{c}$ with $O_\xi \cap C_\beta \subset B_\xi^\gamma$. Making use of $(*)$, it follows that

$$y[k] \subset U_\xi^t(x) \cap C_\beta = O_\xi(x) \cap C_\beta \subset O_\xi \cap C_\beta \subset B_\xi^\gamma.$$

In particular, $\gamma = \langle\beta,k\rangle(\xi)$. By (R_2), $\gamma = x(\xi)$. Hence $y[k] \subset B_\xi^\gamma = B_\xi^{x(\xi)} = B_\xi^t(x)$.

Case 1(b). Suppose $\xi \in H_1 \setminus M$. (Recall also that $\xi \in \mathrm{dom}(t(x)) \cap \theta$.) Then $\xi \in t_1(x) \setminus M$, so by (R_5), $S_\xi \lceil A_\beta \in u_\beta(x)$. By 1.2(b) in the definition of $\langle B_\xi^\rho\rangle_{\rho<\mathfrak{c}}$, to prove that $y[k] \subset B_\xi^{x(\xi)} = B_\xi^t(x)$, we only need to show that $y[k] \subset (O_\xi \setminus F_\xi) \cup F_\xi^{x(\xi)}$. This follows from the fact that by $(*)$, $y[k] \subset U_\xi^t(x) = O_\xi(x)$. (Recall that $O_\xi(x) = O_\xi \setminus F_\xi$, if $x \in O_\xi \setminus F_\xi$, and $O_\xi(x) = (O_\xi \setminus F_\xi) \cup F_\xi^{\bar{x}(\xi)}$ if $x \in F_\xi$, and that in the latter case, $\bar{x}(\xi) = x(\xi)$.)

Case 2. Suppose $\xi \in H_2$. Then by Subcase 2.1 in the definition of $\langle V_\xi^\rho\rangle_{\rho<\mathfrak{c}}$, to show

$$(I_\xi) \qquad y[k] \subset B_\xi^t(x) = \bigcap_{\rho\in t(\xi)} V_\xi^\rho,$$

it is enough to show that

(a) $y[k] \subset U_\xi^t(x) = \bigcap_{\rho\in t(\xi)} U_\xi^\rho$ and

(b) $t(\xi) \subset N$.

(a) follows by $(*)$, and (b) follows because $\xi \in \mathrm{dom}(t(x))$ and by $x \in N$, $\mathrm{dom}(t(x)) \subset N$. $\square$

6. The end of the proof of Main Lemma 5.2

Recall that at the beginning of the proof of the Main Lemma, we assumed indirectly that there was a minimal $\theta \in M \cap H_1$ such that β was not θ-homogeneous and then we chose $\langle\beta,k\rangle \in B_\theta \cap C_\beta$ with k big enough to satisfy (1_k)–(4_h). To arrive at a contradiction we will show that β is θ-homogeneous.

To do this, note first that by (R_1) and (4_k), $\theta \in t_1(\beta,k) \cap M = t_1(x) \cap M$. Thus by Lemma 5.3 and the completeness of $V_{t(x),K(x),n(x)}(x)$, we conclude that

$$y[k] \subset V_{t(x)\lceil\theta,K(x),n(x)}(x) \subset O_\theta(x).$$

Claim 1. $F_\theta \cap C_\beta \neq \emptyset$.

To see that Claim 1 is true, suppose indirectly that $F_\theta \cap C_\beta = \emptyset$. Note that by $\theta \in M \cap H_1$, $S_\theta \restriction A_\beta \in D_\beta$. Hence by (1.1a) of the definition of $\langle B_\theta^\rho \rangle_{\rho<\mathfrak{c}}$ it follows that $B_\theta \cap C_\beta = \emptyset$. On the other hand, by $\theta \in t_1(\beta, k)$ we have $\langle \beta, k\rangle \in B_\theta \cap C_\beta$, contradiction.

Claim 2. *There is precisely one $\gamma < \mathfrak{c}$ such that $F_\theta^\gamma \cap C_\beta \neq \emptyset$.*

By Claim 1, there is at least one such γ. Suppose there are at least two. Then by (2_k), there are at least two $\gamma < \mathfrak{c}$ such that $F_\theta^\gamma \cap y[k] \neq \emptyset$. On the other hand, $y[k] \subset O_\theta(x) =$ either $O_\theta \setminus F_\theta$ or $(O_\theta \setminus F_\theta) \cup F_\theta^{\bar{x}(\theta)}$, contradiction.

Next, let $j \in \omega$ be minimal with $\langle \beta, j\rangle \in F_\theta \cap C_\beta = F_\theta^\gamma \cap C_\beta$. Note that by (1_k), $\langle \beta, j\rangle \in y[k]$.

Claim 3. $x \in F_\theta^\gamma$.

To prove Claim 3, note that by $\langle \beta, j\rangle \in F_\theta \cap y[k]$ we can't have $y[k] \subset O_\theta \setminus F_\theta$. Hence $x \in F_\theta$ and $O_\theta(x) = (O_\theta \setminus F_\theta) \cup F_\theta^{\bar{x}(\theta)}$. Since $\langle \beta, j\rangle \in F_\theta \cap y[k] \subset F_\theta \cap O_\theta(x)$, it follows that $\langle \beta, j\rangle \in F_\theta^{\bar{x}(\theta)}$. Since $\langle \beta, j\rangle \in F_\theta^\gamma$ by the definition of $\langle \beta, j\rangle$, it follows that $\gamma = \bar{x}(\theta)$, so $x \in F_\theta^\gamma$.

Claim 4. $y[k] \subset B_\theta^\gamma$.

To prove Claim 4, recall from the proof of Claim 3, that by $\langle \beta, j\rangle \in F_\theta^\gamma$, $\overline{\langle \beta, j\rangle}(\theta) = \gamma = \bar{x}(\theta)$ and thus $y[k] \subset O_\theta(x) = (O_\theta \setminus F_\theta) \cup F_\theta^{\overline{\langle \beta, j\rangle}(\theta)}$. Thus $y[k] \subset B_\theta^\gamma$ follows from (1.1b(ii)) of the definition of $\langle B_\theta^\rho \rangle_{\rho<\mathfrak{c}}$.

To finish the proof that β is θ-homogeneous, note first that by Claim 4, $\langle \beta, k\rangle(\theta) = \gamma = \bar{x}(\theta) = x(\theta)$. Thus by (R_4), $\gamma \in M$. Further, by (3_k), B_θ^γ is the only member of $\langle B_\theta^\rho \rangle_{\rho<\mathfrak{c}}$ which intersects C_β, i.e., $B_\theta \cap C_\beta \subset B_\theta^\gamma$. Finally note that by (1.1b) in the definition of $\langle B_\theta^\rho \rangle_{\rho<\mathfrak{c}}$ we have $B_\theta \cap C_\beta = O_\theta \cap C_\beta$. Thus β is θ-homogeneous contrary to our assumption that θ is a minimal counterexample.

7. Final remarks

(1) By adding the sets $(\mathfrak{c} \setminus \alpha) \times \omega$, $\alpha < \mathfrak{c}$, to $\mathcal{B}_0$ at the beginning of the construction (and leaving the rest of the proof unchanged) we can make X left-separated.

(2) Several questions remain open.

Question 1. Is there a paraLindelöf, collectionwise normal Dowker space?

Even just the following well-known problem [8] is hard.

Question 2. Is there a paraLindelöf Dowker space?

Question 3. Is there a metaLindelöf, collectionwise normal and first countable Dowker space?

Question 4 (D. Burke). Is there a metaLindelöf, collectionwise normal and countably paracompact space which is *not* paracompact?

Not even consistency answers are known to Question 1–4.

Question 5. Is there a first countable Dowker space in ZFC?

There are, of course, many consistent examples of first countable Dowker spaces.

References

[1] Z. Balogh, A normal, screenable nonparacompact space in ZFC, Proc. Amer. Math. Soc., 126 pp. 1835–1844.
[2] D.K. Burke, Invited Lecture given at the 1996 Spring Topology Conference.
[3] R. Hodel, TOPO72, Lecture Notes in Math., Vol. 378, Springer, Berlin, 1972, p. 649, Problem 5.
[4] I. Juhasz, K. Kunen, M.E. Rudin, Two more hereditarily separable, non-Lindelof spaces, Canad. J. Math. 28 (1976) 998–1005.
[5] M.E. Rudin, A normal space X for which $X \times I$ is not normal, Fund. Math. 73 (1971) 179–186.
[6] M.E. Rudin, A normal, screenable, nonparacompact space, Gen. Topology Appl. 15 (1983) 313–322.
[7] M.E. Rudin, Dowker spaces, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set-Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 761–769.
[8] S. Watson, Problems I wish I could solve, in: J. van Mill, G.M. Reed (Eds.), Open Problems in Topology, North-Holland, Amsterdam, 1990, pp. 39–70.