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Quasiconjugates of Functions, Duality Relationship between Quasiconvex Minimization under a Reverse Convex Constraint and Quasiconvex Maximization under a Convex Constraint, and Applications*

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In this paper we introduce a concept of quasiconjugate for functions defined on R^n whose values are in $\bar{R}$. The conjugacy correspondence between functions and their quasiconjugates is one-to-one and symmetric in a class of quasiconvex functions whose minimizer on R^n is located at the origin. By using the concept of quasiconjugate we obtain a duality relationship between Quasiconvex Minimization under a Reverse Convex Constraint and Quasiconvex Maximization under a Convex Constraint. This duality relationship allows us to establish a primal–dual pair in a class of nonconvex optimization problems without the duality gap. Several applications are given.

1. INTRODUCTION

In Global Optimization theory there are two typical problems that are convex (or more generally, quasiconvex) maximization over a convex set and convex (or more generally, quasiconvex) minimization over the complementary of a convex set. These two problems are often called Concave Program and Reverse Convex Program, respectively. In Concave Program due to the objective function a local optimum may not be a global one whereas in Reverse Convex Constraint due to the constraint a local optimum may not be a global one. By an additional variable a concave program can be converted into a reverse convex program. Therefore, Reverse Convex Program is seemingly more complicated than Concave

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Program. But in this paper we shall see that Concave Program and Reverse Convex Program actually have the same level of difficulty.

Concave Program was studied first by H. Tuy in 1964 (see [34]). Up to now Concave Program has attracted numerous algorithmic studies (see, e.g., Zwart [42, 43], Taha [26], Tuy [35], Thoai and Tuy [31], Hoffman [12], Falk and Hoffman [5], Mukhamediev [17], Horst [14, 15], Tuy, Thieu, and Thai [36], Rosen [20], Rosen and Pardalos [21], and their references). Reverse Convex Program was studied later (see, e.g., Hillestad and Jacobsen [9, 10], Singer [22], Tuy [37], Tuy and Thuong [38, 39], Muu [18], Thach [27], Thoai [32], Fulop [7]). In [37] Tuy show that under the stability condition a reverse convex program can be systematically reduced to a sequence of linearly constrained convex maximization problems.

The purpose of this paper is to present a duality relationship between Concave Program and Reverse Convex Program. A concave program corresponds to the dual problem (in the dual space) which is a reverse convex program and a reverse convex program corresponds to the dual problem (in the dual space) which is a concave program. The correspondence is symmetric. If an optimal solution of the dual problem has been known then by solving an ordinary convex program we can obtain an optimal solution of the primal problem. In some cases, by the existing methods, the dual problem is much easier than the primal one and hence instead of solving the primal we can solve the dual. By this way we obtain a new approach for algorithmic studies for Concave Program and Reverse Convex Program. The duality relationship is based on a concept of quasiconjugate of functions.

The paper is organized as follows. In Section 2 we introduce a concept of quasiconjugate for functions defined on R^n whose values are in $\bar{R}$ ($\bar{R} = R \cup \{\pm\infty\}$) and give several illustrative examples. In Section 3 we give some basic properties of quasiconjugates and conjugacy correspondences between a function and its quasiconjugate. In Section 4 we introduce a relation between quasiconjugates and quasiconvex hulls of functions. In Section 5 we establish a duality relationship between Concave Program and Reverse Convex Program. In Section 6 we give some applications. Finally, we devote Section 7 to discussions.

2. Quasiconjugates of Functions

DEFINITION 2.1. Let $f: R^n \to \bar{R}$ be an arbitrary function. We call the quasiconjugate of f, denoted by f^H, a function defined as

$$f^H(v) = \begin{cases} -\inf\{f(x): \langle x, v\rangle \geqslant 1\} & \text{if } v \in R^n \setminus \{0\} \quad (1) \\ -\sup\{f(x): x \in R^n\} & \text{if } v = 0. \quad (2) \end{cases}$$

By Definition 2.1, if $v \neq 0$ then

$$f^H(v) = -\inf\{f(x): \langle x, v\rangle \geqslant 1\} \geqslant -\sup\{f(x): x \in R^n\} = f^H(0).$$

Therefore, the quasiconjugate function f^H has always a minimizer at 0, i.e.,

$$f^H(0) = \min\{f^H(v): v \in R^n\}. \tag{3}$$

Let us consider several examples.

EXAMPLE 2.1. $f(x) = c$. $f^H(v) = -c$ (c is a constant).

EXAMPLE 2.2. $f(x) = \|x\|^2$.

$$f^H(v) = \begin{cases} -1/\|v\|^2 & \text{if } v \neq 0 \\ -\infty & \text{if } v = 0 \end{cases}$$

($\|\cdot\|$ denotes the euclidean norm).

EXAMPLE 2.3. $f(x) = x^T A x$, where A is a positive definite $n \times n$-matrix and T is the transpose.

$$f^H(v) = \begin{cases} -u(v)^T A u(v) & \text{if } v \neq 0 \\ -\infty & \text{if } v = 0, \end{cases}$$

where $u(v) = (A + A^T)^{-1} v / v^T (A + A^T)^{-1} v$.

Examples 2.1–2.3 can easily be checked.

EXAMPLE 2.4. Let Y be a compact convex set in R^n containing 0 (assuming $Y \neq \{0\}$) and

$$f(x) = \max\{\langle y, x\rangle, y \in Y\}.$$

Since Y is a compact convex set and $0 \in Y$, one has $Y^{00} = Y$ (where Y^0 denotes the polar of Y: $Y^0 = \{x \in R^n: \langle x, y\rangle \leqslant 1 \ \forall y \in Y\}$, and Y^{00} the bipolar of Y). Therefore, $f(\cdot)$ is the minkowski functional of Y^0. Since Y is compact, Y^0 contains 0 in its interior. One has

$$f^H(0) = -\sup\{f(x): x \in R^n\} = -\infty.$$

Suppose that $v \neq 0$ and

$$-\alpha = f^H(v) = -\inf\{f(x): \langle x, v\rangle \geqslant 1\}.$$

Since $f(\cdot)$ is the minkowski functional of the convex set containing 0 in its interior, one has $\alpha > 0$. By the duality principle in Tuy [37] one has

$$\alpha = \inf\{f(x): \langle x, v\rangle \geqslant 1\} \Leftrightarrow 1 = \max\{\langle x, v\rangle: f(x) \leqslant \alpha\}. \tag{4}$$

Since $f(\cdot)$ is the minkowski functional of Y^0, one has

$$\{x: f(x) \leqslant \alpha\} = \alpha Y^0.$$

From (4) this implies

$$1 = \max\{\langle \alpha x, v\rangle: x \in Y^0\} = \alpha \max\{\langle x, v\rangle: x \in Y^0\}.$$

So,

$$\alpha = 1/\max\{\langle x, v\rangle: x \in Y^0\}$$

or

$$f^H(v) = -1/\max\{\langle x, v\rangle: x \in Y^0\}.$$

From the above examples we can obtain many others by noting that

$$(\lambda f)^H = \lambda \cdot f \qquad \forall \lambda \geqslant 0 \tag{5}$$

$$(f + \alpha)^H = f^H - \alpha \qquad \forall \alpha. \tag{6}$$

3. Basic Properties of Quasiconjugate Functions and Correspondence Between Functions and Their Quasiconjugates

Theorem 3.1. *Let $f: R^n \to \bar{R}$ be an arbitrary function. The quasiconjugate function f^H is quasiconvex on R^n and satisfies*

$$f^H(v) \geqslant f^H(\lambda v) \qquad \forall v \in R^n, \forall \lambda \in [0, 1]. \tag{7}$$

Proof. It is obvious that (7) is true if $v = 0$. In view of (3) we also see that (7) is true for $\lambda = 0$. Now let $v \neq 0$ and $\lambda \in (0, 1]$. Then, one has

$$\begin{aligned}\{x: \langle v, x\rangle \geqslant 1\} &\supseteq \{x: \lambda\langle v, x\rangle \geqslant 1\}\\ &\Rightarrow \inf\{f(x): \langle v, x\rangle \geqslant 1\} \leqslant \inf\{f(x): \langle \lambda v, x\rangle \geqslant 1\}\\ &\Rightarrow f^H(v) \geqslant f^H(\lambda v).\end{aligned}$$

Thus, (7) has been proved. We are going to prove that f^H is quasiconvex, i.e.,

$$f^H(\lambda v_1 + (1 - \lambda) v_2) \leqslant \max\{f^H(v_1), f^H(v_2)\} \tag{8}$$

for all v_1, v_2 and all $\lambda \in [0, 1]$. If either $v_1 = 0$ or $v_2 = 0$ then from (3) and (7) it follows (8) for all $\lambda \in [0, 1]$. If both v_1 and v_2 are nonzero vectors then one has

$$\begin{aligned}
&\{x\colon \langle \lambda v_1 + (1-\lambda) v_2, x\rangle < 1\} \\
&\qquad \supseteq \{x\colon \langle v_1, x\rangle < 1\} \cap \{x\colon \langle v_2, x\rangle < 1\} \\
&\quad \Rightarrow \{x\colon \langle \lambda v_1 + (1-\lambda) v_2, x\rangle \geqslant 1\} \\
&\qquad \subseteq \{x\colon \langle v_1, x\rangle \geqslant 1\} \cup \{x\colon \langle v_2, x\rangle \geqslant 1\} \\
&\quad \Rightarrow \inf\{f(x)\colon \langle \lambda v_1 + (1-\lambda) v_2, x\rangle \geqslant 1\} \\
&\qquad \geqslant \min\{\inf\{f(x)\colon \langle v_1, x\rangle \geqslant 1\}, \inf\{f(x)\colon \langle v_2, x\rangle \geqslant 1\} \\
&\quad \Rightarrow -\inf\{f(x)\colon \langle \lambda v_1 + (1-\lambda) v_2, x\rangle \geqslant 1\} \\
&\qquad \leqslant -\min\{\inf\{f(x)\colon \langle v_1, x\rangle \geqslant 1\}, \inf\{f(x)\colon \langle v_2, x\rangle \geqslant 1\}\} \\
&\qquad = \max\{-\inf\{f(x)\colon \langle v_1, x\rangle \geqslant 1\}, -\inf\{f(x)\colon \langle v_2, x\rangle \geqslant 1\}\} \\
&\quad \Rightarrow f^H(\lambda v_1 + (1-\lambda) v_2) \leqslant \max\{f^H(v_1), f^H(v_2)\}. \quad \blacksquare
\end{aligned}$$

DEFINITION 3.1. We say that a function $f\colon R^n \to \bar{R}$ achieves the maximum value at the infinite if $f(x_n) \to \sup\{f(x)\colon x \in R^n\}$ for any sequence $\{x_n\}$ such that $\|x_n\| \to +\infty$.

LEMMA 3.1. *Assume that $f\colon R^n \to \bar{R}$ achieves the maximum value at the infinite. If f is lower semi-continuous (lsc) then it has a minimizer on every nonempty closed subset of R^n.*

Proof. Let M be a nonempty closed set in R^n. Since f is lsc, it has a minimizer on any compact set. Therefore, if f has no minimizer on M then there exists a sequence $\{x_n\} \subseteq M$ such that $\|x_n\| \to +\infty$ and $f(x_n) \to \inf\{f(x)\colon x \in M\}$. Thus, by Definition 3.1 one has

$$\sup\{f(x)\colon x \in R^n\} = \lim_{n\to\infty} f(x_n) = \inf\{f(x)\colon x \in M\}.$$

Therefore, $f(x) = \text{const}$ for every $x \in M$. This conflicts with the fact that $f(\cdot)$ has no minimizer on M. $\blacksquare$

THEOREM 3.2. *If f is continuous at 0 and*

$$f(0) = \inf\{f(x)\colon x \in R^n\} \tag{9}$$

then f^H achieves the maximum value at the infinite. And if f achieves the maximum value at the infinite then f^H is continuous at 0 and

$$f^H(0) = \inf\{f^H(v)\colon v \in R^n\}.$$

Proof. Assume that f is continuous at 0 and (9) occurs. Then,

$$\begin{aligned}\sup_{v \in R^n} f^H(v) &= \sup_{v \in R^n} -\inf_x\{f(x): \langle v, x\rangle \geqslant 1\} \\ &= \sup_{v \in R^n} \sup_x \{-f(x): \langle v, x\rangle \geqslant 1\} \\ &= \sup_{x \in R^n\setminus\{0\}} -f(x) = -\inf_{x \in R^n\setminus\{0\}} f(x) \\ &= -\inf_{x \in R^n} f(x) = -f(0).\end{aligned}$$

Let $\{v_n\}$ be a sequence of vectors in R^n such that $\|v_n\| \to \infty$ $(n \to \infty)$. Then,

$$f^H(v_n) = -\inf_x\{f(x): \langle x, v_n\rangle \geqslant 1\} \geqslant -f(v_n/\|v_n\|^2).$$

Since $\|v_n/\|v_n\|^2\| = 1/\|v_n\| \to 0$ $(n \to \infty)$, this implies that

$$\varliminf_{n \to \infty} f^H(v_n) \geqslant -f(0) = \sup_{v \in R^n} f^H(v).$$

So, f^H achieves the maximum value at the infinite. In order to prove the second assertion it remains to prove that f^H is continuous at 0 when f achieves the maximum value at the infinite. Suppose that $\{v_n\}$ is a sequence of vectors in R^n such that $v_n \to 0$ $(n \to \infty)$. For each n there must exist a point x_n such that

$$\langle x_n, v_n\rangle \geqslant 1 \tag{10}$$

$$f(x_n) \leqslant \inf\{f(x): \langle x, v_n\rangle \geqslant 1\} + 1/n. \tag{11}$$

Since $v_n \to 0$ $(n \to \infty)$, from (10) it follows that $\|x_n\| \to \infty$ $(n \to \infty)$. Therefore, $f(x_n)$ tends to $\sup\{f(x): x \in R^n\}$. From (11) it follows that

$$\lim_{n \to \infty} |\inf\{f(x): \langle x, v_n\rangle \geqslant 1\} - f(x_n)| = 0.$$

So, one has

$$\begin{aligned}\lim_{n \to \infty} f^H(v_n) &= \lim_{n \to \infty} \{-\inf\{f(x): \langle x, v_n\rangle \geqslant 1\}\} \\ &= \lim_{n \to \infty} -f(x_n) = -\lim_{n \to \infty} f(x) \\ &= -\sup\{f(x): x \in R^n\} = f^H(0).\end{aligned}$$

Thus, f^H is continuous at 0. ∎

THEOREM 3.3. *If f is upper semi-continuous (usc) then f^H is lsc. If f is lsc and achieves the maximum value at the infinite then f^H is usc.*

Proof. Since the point-to-set map $v \mapsto \{x: \langle v, x\rangle \geqslant 1\}$ is lsc at any $v \neq 0$, from the upper semicontinuity of f it follows that the function

$$v \mapsto \inf\{f(x): \langle v, x\rangle \geqslant 1\}$$

is usc at any $v \neq 0$ (see, e.g., Fiacco [6, Theorem 2.2.1]), and hence the function f^H is lsc at any $v \neq 0$. Further, from (3) it follows that f^H is always lsc at 0. Thus, f^H is lsc on R^n when f is usc. Now, assume that f is lsc and achieves the maximum value at the infinite. Suppose that $\{v_n\} \to \bar{v}$. We need prove that

$$\overline{\lim}\, f^H(v_n) \leqslant f^H(\bar{v}). \tag{12}$$

By Lemma 3.1, for each n there exists x_n such that

$$\begin{aligned}&\langle v_n, x_n\rangle \geqslant 1\\ &f^H(v_n) = -\inf\{f(x): \langle v_n, x\rangle \geqslant 1\} = -f(x_n).\end{aligned}$$

For any subsequence $\{x_{n_s}\}$ such that $\|x_{n_s}\| \to \infty$ one has

$$\begin{aligned}\lim f^H(v_{n_s}) &= \lim\, -f(x_{n_s}) = -\lim f(x_{n_s})\\ &= -\sup\{f(x): x \in R^n\} = f^H(0) \leqslant f^H(\bar{v}).\end{aligned} \tag{13}$$

On the other hand, for any subsequence $\{x_{n_s}\} \to \bar{x}$ we have $\langle \bar{x}, \bar{v}\rangle \geqslant 1$. Therefore, by virtue of the lower semicontinuity of f one has

$$\begin{aligned}&\underline{\lim} f(x_{n_s}) \geqslant f(\bar{x})\\ &\quad\Rightarrow -\underline{\lim} f(x_{n_s}) \leqslant -f(\bar{x})\\ &\quad\Rightarrow \overline{\lim}\, -f(x_{n_s}) \leqslant -f(\bar{x})\\ &\quad\Rightarrow f^H(\bar{v}) \geqslant -f(\bar{x}) \geqslant \overline{\lim}\, -f(x_{n_s}) = \overline{\lim}\, f^H(v_{n_s}).\end{aligned} \tag{14}$$

From (13) and (14) it follows (12).

DEFINITION 3.2. A function $f: R^n \to \bar{R}$ is said to be strictly quasiconvex in the weak sense at $a \in R^n$ if for all $x \in R^n$ satisfying

$$f(a) < f(x) < \sup\{f(z): z \in R^n\}$$

one has

$$f(\lambda x + (1-\lambda)\, a) < f(x) \qquad \forall \lambda \in (0, 1).$$

DEFINITION 3.3. A function $f: R^n \to \bar{R}$ is said to be strictly quasiconvex in the weak sense on R^n if it is strictly quasiconvex in the weak sense at each point in R^n.

It can be easily seen that, in the general case, the strict quasiconvexity (see, e.g., Mangasarian [16]) implies the strict quasiconvexity in the weak sense. But, if either $\sup\{f(x): x \in R^n\} = \infty$ or $f(\cdot)$ has no maximizer on R^n then the strict quasiconvexity in the weak sense is equivalent to the strict quasiconvexity.

THEOREM 3.4. *Assume that a lsc function f achieves the maximum value at the infinite and $f(0) = \inf\{f(x): x \in R^n\}$. If f is strictly quasiconvex in the weak sense at 0 then f^H is strictly quasiconvex in the weak sense on R^n.*

Proof. First, we prove that f^H is strict quasiconvex in the weak sense at 0. Let $\bar{v}$ be a vector in R^n such that

$$f^H(0) < f^H(v) < \sup\{f^H(v): v \in R^n\}$$

and let $\lambda \in (0, 1)$. If $f^H(\lambda\bar{v}) \leqslant f^H(0)$ then $f^H(\bar{v}) > f^H(\lambda\bar{v})$. Now, suppose that $f^H(\lambda\bar{v}) > f^H(0)$. By Lemma 3.1 there is $\bar{x}$ such that

$$\begin{aligned} &\langle \bar{x}, \lambda\bar{v}\rangle \geqslant 1 \\ &f^H(\lambda\bar{v}) = -\inf\{f(x): \langle \lambda\bar{v}, x\rangle \geqslant 1\} = -f(\bar{x}). \end{aligned} \tag{15}$$

In view of (7) we see that

$$\begin{aligned} -f(\bar{x}) = f^H(\lambda\bar{v}) \leqslant f^H(\bar{v}) &< \sup\{f^H(v): v \in R^n\} \\ &= \sup_v\{-\inf_x\{f(x): \langle x, v\rangle \geqslant 1\}\} \\ &= -\inf_v \inf_v\{f(x): \langle x, v\rangle \geqslant 1\} \\ &\leqslant -\inf_x\{f(x): x \in R^n\} = -f(0). \end{aligned}$$

So, $f(\bar{x}) > f(0)$. Since $f^H(\lambda\bar{v}) > f^H(0)$, one has

$$\sup\{f(x): x \in R^n\} = -f^H(0) > -f^H(\lambda\bar{v}) = f(\bar{x}).$$

Therefore,

$$f(0) < f(\bar{x}) < \sup\{f(x): x \in R^n\}.$$

Since f is strictly quasiconvex in the weak sense at 0, this implies that

$$-f(\bar{x}) < -f(\lambda\bar{x}) \leqslant -\inf\{f(x): \langle x, \bar{v}\rangle \geqslant 1\} = f^H(\bar{v}).$$

Combining this and (15) yields $f^H(\bar{v}) > f^H(\lambda\bar{v})$. So, f^H is strictly quasiconvex in the weak sense at 0.

Now, let v_1 and v_2 be two vectors in R^n such that

$$f^H(v_1) < f^H(v_2) < \sup\{f^H(v): v \in R^n\}.$$

Assume that there is $\lambda \in (0, 1)$ satisfying

$$f^H(v_1 + \lambda(v_2 - v_1)) = f^H(v_2).$$

Then, from the quasiconvexity of f^H it follows that

$$f^H(v_1 + \theta(v_2 - v_1)) = f^H(v_2) \qquad \forall \theta \in [\lambda, 1].$$

Denote by S the set $\{v \in R^n: f^H(v) < f^H(v_2)\}$ and by M the line segment $[v_1 + \lambda(v_2 - v_1), v_2]$. Since f is lsc and achieves the maximum value at the infinite, by Theorem 3.3 S is an open set. It is clear that $S \cap M = \varnothing$. Therefore, S and M can be separated by hyperplane $\{v: l(v) = 0\}$, i.e.,

$$l(v) < 0 \qquad \forall v \in S \tag{16}$$

$$l(v) \geqslant 0 \qquad \forall v \in M, \tag{17}$$

where $l(\cdot)$ is an affine function on R^n (see, e.g., Tuy [40], Holmes [13]). On the other hand, since

$$\sup_{v \in R^n} f^H(v) > f^H(v_2) = f^H(v_1 + \lambda(v_2 - v_1)) > f^H(v_1) \geqslant f^H(0),$$

from the strict quasiconvexity in the weak sense of f^H at 0 it follows that

$$f^H(\theta v_2) < f^H(v_2) \qquad \forall \theta \in (0, 1)$$

$$f^H(\theta(v_1 + \lambda(v_2 - v_1))) < f^H(v_2) \qquad \forall \theta \in (0, 1).$$

This implies that

$$\theta v_2 \in S \qquad \forall \theta \in (0, 1)$$

$$\theta(v_1 + \lambda(v_2 - v_1)) \in S \qquad \forall \theta \in (0, 1).$$

Therefore,

$$l(\theta v_2) < 0 \qquad \forall \theta \in (0, 1)$$

$$l(\theta(v_1 + \lambda(v_2 - v_1))) < 0 \qquad \forall \theta \in (0, 1).$$

Letting $\theta \to 1$ we obtain $l(v_2) \leqslant 0$ and $l(v_1 + \lambda(v_2 - v_1)) \leqslant 0$. So, from (17) one has $l(v_2) = 0$ and $l(v_1 + \lambda(v_2 - v_1)) = 0$. This means that the hyperplane $\{v: l(v) = 0\}$ contains the line passing through v_1 and v_2. Therefore,

$l(v_1)=0$. Since $v_1 \in S$, we arrive at a contradiction with (16). So, we must have $f^H(v_1+\lambda(v_2-v_1))<f^H(v_2)$ for all $\lambda \in (0,1)$. Thus, f^H is strictly quasiconvex in the weak sense on R^n. ▮

4. Quasiconvex Hulls and Biquasiconjugates of Functions

First, we introduce a concept of quasiconvex hull of functions.

Definition 4.1. Let $f: R^n \to \bar{R}$ be an arbitrary function. A function h is called a quasiconvex hull of f if

$$\{x: h(x)<\alpha\}=\operatorname{conv}\{x: f(x)<\alpha\} \qquad \forall \alpha \in \bar{R}. \tag{18}$$

Proposition 4.1. *For any function $f: R^n \to \bar{R}$, a quasiconvex hull of f always exists and it is unique.*

Proof. For any $\alpha \in \bar{R}$, one has

$$\{x: f(x)<\alpha\} = \bigcup_{\alpha>\beta} \{x: f(x)<\beta\}.$$

So,

$$\operatorname{conv}\{x: f(x)<\alpha\} = \bigcup_{\alpha>\beta} \operatorname{conv}\{x: f(x)<\beta\}. \tag{19}$$

Define

$$h(x)=\inf\left\{\gamma: x \in \bigcup_{\gamma>\beta} \operatorname{conv}\{x: f(x)<\beta\}\right\}. \tag{20}$$

Then, we obtain the function $h: R^n \to \bar{R}$. We are going to prove (18). Suppose that $h(\bar{x})<\alpha$. From (20) this implies that there exists γ smaller than α such that $\bar{x} \in \bigcup_{\alpha>\beta} \operatorname{conv}\{x: f(x)<\beta\}$. Since $\gamma<\alpha$, from (19) it follows that $\bar{x} \in \operatorname{conv}\{x: f(x)<\alpha\}$. Conversely, suppose that $\bar{x} \in \operatorname{conv}\{x: f(x)<\alpha\}$. From (19) it follows that there is γ smaller than α such that $\bar{x} \in \operatorname{conv}\{x: f(x)<\gamma\}$. Then, (20) implies that $h(\bar{x}) \leqslant \gamma$. So, $h(\bar{x})<\alpha$.

To complete the proof it remains to prove that a quasiconvex hull of f is unique. Suppose that h and g are quasiconvex hulls of f. By the definition of quasiconvex hull one has

$$\{x: h(x)<\alpha\}=\{x: g(x)<\alpha\} \qquad \forall \alpha.$$

This implies that $h(x)=g(x)$ for all $x \in R^n$, i.e., $h=g$. ▮

Proposition 4.2. *The quasiconvex hull of a function f is the greatest quasiconvex function majorized by f.*

Proof. This proposition can readily be deduced from the definition. ∎

We go on with the definition of biquasiconjugate of functions.

DEFINITION 4.2. Let $f: R^n \to \bar{R}$ be an arbitrary function. The quasiconjugate of the function f^H is called the biquasiconjugate of f and denoted by f^{HH}.

The following theorem will give a relation between the quasiconvex hull and the biquasiconjugate of a function.

THEOREM 4.1. *Let $f: R^n \to \bar{R}$ be an usc function satisfying*

$$f(0) = \inf\{f(x): x \in R^n \setminus \{0\}\}. \tag{21}$$

Then, the biquasiconjugate of f coincides with its quasiconvex hull.

Proof. We need first the following lemma.

LEMMA 4.1. *Let $f: R^n \to \bar{R}$ be an arbitrary function. Then, one has*

$$\begin{aligned} f^{HH}(0) &= \inf\{f^{HH}(x): x \in R^n \setminus \{0\}\} \\ &= \inf\{f(x): x \in R^n \setminus \{0\}\} \end{aligned} \tag{22}$$

$$f(x) \geqslant f^{HH}(x) \qquad \forall x \in R^n \setminus \{0\}. \tag{23}$$

Proof. Let $x \in R^n \setminus \{0\}$. One has

$$\begin{aligned} f^{HH}(x) &= -\inf_v \{f^H(v): \langle v, x\rangle \geqslant 1\} \\ &= -\inf_v \{-\inf_z\{f(z): \langle v, z\rangle \geqslant 1\}: \langle v, x\rangle \geqslant 1\} \\ &= \sup_v \{\inf_z\{f(z): \langle v, z\rangle \geqslant 1\}: \langle v, x\rangle \geqslant 1\} \\ &\leqslant \sup_v \{f(x): \langle v, x\rangle \geqslant 1\} = f(x). \end{aligned}$$

Thus, (23) has been proved. By definition one has further

$$\begin{aligned} f^{HH}(0) &= -\sup_v \{f^H(v): v \in R^n\} \\ &= -\sup_v \{-\inf_x\{f(x): \langle v, x\rangle \geqslant 1\} \\ &= \inf_v\{\inf_x\{f(x): \langle x, v\rangle \geqslant 1\}\} \\ &= \inf_x\{f(x): x \in R^n \setminus \{0\}\}. \end{aligned} \tag{24}$$

Combining this and (23) yields

$$f^{HH}(0) \geqslant \inf_x\{f^{HH}(x): x \in R^n \setminus \{0\}\}. \tag{25}$$

But by virtue of (3) one has

$$f^{HH}(0) = \inf\{f^{HH}(x): x \in R^n\}. \tag{26}$$

From (24), (25), and (26) it follows (22). ▌

Now, we turn to prove Theorem 4.1. From Lemma 4.1 and (21) it follows that

$$\{x: f(x) < \alpha\} \subseteq \{x: f^{HH}(x) < \alpha\} \qquad \forall \alpha \in \bar{R}.$$

Moreover, since f^{HH} is quasiconvex (Theorem 3.1), the set $\{x: f^{HH}(x) < \alpha\}$ is convex and hence it must contain $\operatorname{conv}\{x: f(x) < \alpha\}$. We are going to prove the inverse inclusion, i.e., $\{x: f^{HH}(x) < \alpha\} \subseteq \operatorname{conv}\{x: f(x) < \alpha\}$. Indeed, let $\bar{x} \notin \operatorname{conv}\{x: f(x) < \alpha\}$. Since one has

$$f^{HH}(0) = \inf\{f^{HH}(x): x \in R^n\} = \inf\{f(x): x \in R^n\} = f(0),$$

if $\alpha \leqslant f(0)$ then $\{x: f^{HH}(x) < \alpha\} = \varnothing$ and hence $x \notin \{x: f^{HH}(x) < \alpha\}$. Now, suppose that $\alpha > f(0)$, i.e.,

$$0 \in \{x: f(x) < \alpha\} \subseteq \operatorname{conv}\{x: f(x) < \alpha\}. \tag{27}$$

Since $\bar{x}$ does not belong to the open convex set $\operatorname{conv}\{x: f(x) < \alpha\}$, there is a hyperplane separating $\bar{x}$ from $\operatorname{conv}\{x: f(x) < \alpha\}$ (see, e.g., Tuy [40] or Holmes [13]). Furthermore, from (27) it follows that the separting hyperplane can be taken as a form $\{x: \langle \bar{v}, x\rangle = 1\}$ where $\bar{v}$ satisfies

$$\langle \bar{v}, \bar{x}\rangle \geqslant 1 \tag{28}$$

$$\langle \bar{v}, x\rangle < 1 \qquad \forall x \in \operatorname{conv}\{x: f(x) < \alpha\}. \tag{29}$$

From (29) it follows that $\inf\{f(x): \langle x, \bar{v}\rangle \geqslant 1\} \geqslant \alpha$. So,

$$f^{HH}(x) = -\inf\{f^H(v): \langle v, \bar{x}\rangle \geqslant 1\} \geqslant -f^H(\bar{v})$$
$$= \inf\{f(x): \langle x, \bar{v}\rangle \geqslant 1\} \geqslant \alpha.$$

Therefore, $\bar{x} \notin \{x: f^{HH}(x) < \alpha\}$. Thus,

$$\{x: f^{HH}(x) < \alpha\} \subseteq \operatorname{conv}\{x: f(x) < \alpha\}$$

and hence

$$\{x: f^{HH}(x) < \alpha\} = \operatorname{conv}\{x: f(x) < \alpha\}$$

(for all $\alpha \in \bar{R}$). By the definition, f^{HH} is the quasiconvex hull of f.

COROLLARY 4.1. *Let $f: R^n \to \bar{R}$ be an usc function satisfying* (21). *If f is quasiconvex then $f^{HH} = f$.*

Proof. Since the quasiconvex hull of f is the greatest quasiconvex function majorized by f, the corollary is immediately deduced from Theorem 4.1. ∎

COROLLARY 4.2. *If f is an usc quasiconvex function satisfying* (21) *then f^H is a quasiconvex function satisfying*

$$f^H(0) = \inf\{f^H(v): v \in R^n \setminus \{0\}\}. \tag{30}$$

Proof. Since $f^{HH} = f$, it follows that $f^H = (f^{HH})^H = (f^H)^{HH}$. Then, by virtue of Lemma 4.1 one has (30). ∎

THEOREM 4.2. *Let $f: R^n \to \bar{R}$ be a function satisfying* (21). *If f is lsc and achieves the maximum value at the infinite then f^{HH} is the quasiconvex hull of f and*

$$\{x: f^{HH}(x) \leqslant \alpha\} = \operatorname{conv}\{x: f(x) \leqslant \alpha\}. \tag{31}$$

Proof. By (21) and Lemma 4.1, one has

$$f^{HH}(0) = \inf\{f^{HH}(x): x \in R^n \setminus \{0\}\} = \inf\{f(x): x \in R^n \setminus \{0\}\} = f(0)$$

and

$$\{x: f^{HH}(x) \leqslant \alpha\} \supset \{x: f(x) \leqslant \alpha\} \qquad \forall \alpha. \tag{32}$$

Since $\{x: f^{HH}(x) \leqslant \alpha\}$ is convex, this implies

$$\{x: f^{HH}(x) \leqslant \alpha\} \supset \operatorname{conv}\{x: f(x) \leqslant \alpha\} \qquad \forall \alpha. \tag{33}$$

Now we prove the inverse inclusion. Let $x \notin \operatorname{conv}\{x: f(x) \leqslant \alpha\}$. Since $\{x: f(x) \leqslant \alpha\}$ is a compact set, $\operatorname{conv}\{x: f(x) \leqslant \alpha\}$ is closed. Therefore, there is vector x^* such that

$$\langle x^*, x\rangle \geqslant 1 \tag{34}$$

$$\langle x^*, y\rangle < 1 \qquad \forall y: f(y) \leqslant \alpha. \tag{35}$$

By Lemma 3.1 this implies

$$\inf\{f(y): \langle x^*, y\rangle \geqslant 1\} > \alpha. \tag{36}$$

So,

$$\begin{aligned} f^{HH}(x) &= -\inf\{f^H(v): \langle v, x\rangle \geqslant 1\} \geqslant -f^H(y) \\ &= \inf\{f(y): \langle x^*, y\rangle \geqslant 1\} > \alpha. \end{aligned} \tag{37}$$

Therefore, one obtains (31). Let h be a quasiconvex function majorized by f. For any α, one has

$$\begin{aligned} &\{x: h(x) \leqslant \alpha\} \supset \{x: f(x) \leqslant \alpha\} \\ &\quad \Rightarrow \{x: h(x) \leqslant \alpha\} \supset \operatorname{conv}\{x: f(x) \leqslant \alpha\} = \{x: f^{HH}(x) \leqslant \alpha\}. \end{aligned}$$

Thus $h(x) \leqslant f^{HH}(x)$. So, f^{HH} is the quasiconvex hull of f. ∎

COROLLARY 4.3. *Let f be a lsc, quasiconvex function satisfying* (21). *If f achieves the maximum value at the infinite then $f^{HH} = f$.*

5. DUALITY RELATIONSHIP BETWEEN QUASICONVEX MAXIMIZATION UNDER A CONVEX CONSTRAINT AND QUASICONVEX MINIMIZATION UNDER A REVERSE CONVEX CONSTRAINT

We consider a quasiconvex maximization over a convex set

$$\max\{f(x): x \in D\}, \tag{P}$$

where $f(\cdot)$ is an usc quasiconvex function and D a compact convex set. Even in a special case where f is a convex quadratic function and D is defined by a finite number of linear inequations, this problem is NP-hard.

DEFINITION 5.1. A quasiconvex maximization over a convex set, (P), is said to be in the standard form if D contains 0 and

$$f(0) = \inf\{f(x): x \in R^n\}.$$

Note that any quasiconvex maximization problem (P) can be easily transformed into the standard form. Indeed, let $z \in D$. Set $\bar{D} = D - z$, $\bar{f}(x) = \max\{f(z), f(x+z)\}$. Then $0 \in \bar{D}$, $\bar{f}$ is usc, and $\bar{f}(0) = \inf\{\bar{f}(x): x \in R^n\}$. Problem (P) is equivalent to $\max\{\bar{f}(x): x \in \bar{D}\}$ which is in the standard form.

Now we consider a quasiconvex minimization over the complementary of a convex set

$$\min\{g(x): x \in R^n \backslash \operatorname{int} G\}, \tag{Q}$$

where $g(\cdot)$ is a lsc quasiconvex function achieving the maximum value at the infinite and G is a closed convex set with the nonempty interior.

Definition 5.2. A quasiconvex minimization over the complementary of a convex set (Q) is said to be in the standard form if $0 \in \operatorname{int} G$ and

$$g(0) = \inf\{g(x): x \in R^n \backslash \{0\}\}.$$

We see that problem (P) can be regarded as a special case of a more general one

$$\min\{g(x): x \in M \backslash \operatorname{int} G\}, \tag{38}$$

where $g(\cdot)$ is a lsc quasiconvex function achieving the maximum value at the infinite and M, G are closed convex sets (M is not singleton). Problem (38) is often called a d.c. programming (see Tuy [41]). Even in a special case where $g(\cdot)$ is a constant function, M is defined by a finite number of linear equations and G is a sphere, problem (38) is NP-Complete.

Suppose that by minimizing function $g(\cdot)$ on M we obtain a solution z. If $z \notin \operatorname{int} G$ then we are done: z is also an optimal solution to (38) (the reverse convex constraint is not essential). Otherwise we can transform (38) into a quasiconvex minimization over the complementary of a convex set in the standard form. Indeed, by setting

$$\bar{G} = G - z, \qquad \bar{g}(x) = \min\{g(z), g(x+z) + \delta(x+z \mid M)\}$$

problem (38) can be transformed into

$$\min\{\bar{g}(x): x \notin \operatorname{int} \bar{G}\},$$

where $0 \in \operatorname{int} \bar{G}$, $\bar{g}$ is lsc, quasiconvex, achieves the maximum value at the infinite and satisfies $\bar{g}(0) = \inf\{\bar{g}(x): x \in R^n \backslash \{0\}\}$.

In the sequel we introduce definitions of dual problems of (P) and (Q), respectively.

Definition 5.3. (i) Suppose that (P) is in the standard form. The problem

$$\min\{f^H(v): v \in R^n \backslash \operatorname{int} D^0\} \tag{P*}$$

is called the dual problem of (P).

(ii) Suppose that (Q) is in the standard form. The problem

$$\max\{g^H(v): v \in G^0\} \tag{Q*}$$

is called the dual of (Q).

We see that if a quasiconvex maximization over a convex set (P) is in the standard form then its dual is a quasiconvex minimization over the complement of a convex set and the dual is also in the standard form. Indeed, since D is a compact set, D^0 contains 0 in its interior and since f is usc, quasiconvex and satisfies

$$f(0) = \inf\{f(x): x \in R^n \setminus \{0\}\},$$

it is continuous at 0. By Theorems 3.2, 3.3, and Corollary 4.2, f^H is a lsc quasiconvex function achieving the maximum value at the infinite and satisfies

$$f^H(0) = \inf_{x R^n} f_0(x).$$

Analogously, the dual of (Q) is a quasiconvex maximization over a convex set in the standard form. Furthermore, since $D^{00} = D$ and $f^{HH} = f$ (see Corollaries 4.1 and 4.3), we see that the dual problem of (P*), denoted by (P**) is nothing but (P). Analogously, one has (Q**) is the same as (Q). So, we obtain an one-to-one correspondence between a class of quasiconvex maximization problems with a convex constraint in the standard form and a class of quasiconvex minimization problems with a reverse convex constraint in the standard form. The correspondence is symmetric. Before giving the duality relationship between (P) and (P*), let us recall the definition of a normal cone.

DEFINITION 5.4. Let C be a closed convex set in R^n, x a point in R^n (x does not necessarily belong to C). The cone

$$\{v \in R^n: \langle v, z - x\rangle \leqslant 0 \ \forall z \in C\}$$

is called the normal cone to C at x and is denoted by $N(C, x)$.

THEOREM 5.1. *Let* (P) *be a quasiconvex maximization over a closed convex set in the standard form and* (P*) *the dual of* (P). *One has the following duality relationship.*

(i) $-\sup(\text{P}) = \inf(\text{P*})$.

(ii) *If $\bar{x}$ is an optimal solution to* (P) *then every minimizer of f^H on the halfspace $\{v \in R^n: \langle \bar{x}, v\rangle \geqslant 1\}$ is an optimal solution to* (P*).

(iii) *If $\bar{v}$ is an optimal solution to* (P*) *then for any $\bar{x} \in N(D^0, \bar{v}) \setminus \{0\}$ the vector $\bar{x}/\langle \bar{v}, \bar{x}\rangle$ is an optimal solution to* (P).

Proof. (i) One has

$$\begin{aligned}-\sup(\mathrm{P}) &= -\sup\{f(x): x\in D\}\\ &= -\sup\{f^{HH}(x): x\in D\}\\ &= -\sup_x\{-\inf_v\{f^H(v): \langle v, x\rangle \geqslant 1\}: x\in D\}\\ &= \inf_{x\in D}\inf_v\{f^H(v): \langle v, x\rangle \geqslant 1\}\\ &= \inf_v\inf_{x\in D}\{f^H(v): \langle v, x\rangle \geqslant 1\}.\end{aligned}$$

For every $v\in \operatorname{int} D^0$ we have $\langle v, x\rangle < 1\ \forall x\in D$. Therefore, if $v\in \operatorname{int} D^0$ then $\inf_{x\in D}\{f^H(v): \langle v, x\rangle \geqslant 1\} = \inf\varnothing = +\infty$. So,

$$\begin{aligned}-\sup(\mathrm{P}) &= \inf_v\inf_{x\in D}\{f^H(v): \langle v, x\rangle \geqslant 1\}\\ &= \inf_{v\notin \operatorname{int} D^0} f^H(v) = \inf(\mathrm{P}^*).\end{aligned}$$

(ii) Suppose that $\bar{x}$ is an optimal solution to (P), i.e.,

$$\bar{x}\in D; \qquad f(\bar{x}) = \sup(\mathrm{P}).$$

Let $\bar{v}$ be a minimizer of f^H on the halfspace $\{v\in R^n: \langle \bar{x}, v\rangle \geqslant 1\}$. Since $\bar{x}\in D$, the set $\{v\in R^n: \langle \bar{x}, v\rangle \geqslant 1\}$ is contained in the feasible set $\{v: v\notin \operatorname{int} D^0\}$ of (P*). Furthermore,

$$\begin{aligned}\inf(\mathrm{P}^*) &= -\sup(\mathrm{P}) = -f(\bar{x}) = -f^{HH}(\bar{x})\\ &= \inf\{f^H(v): \langle \bar{x}, v\rangle \geqslant 1\} = f^H(\bar{v}).\end{aligned}$$

Therefore, $\bar{v}$ is an optimal solution to (P*).

(iii) Suppose that $\bar{v}$ is an optimal solution to (P*). Let $\bar{x}\in N(D^0, \bar{v})\backslash\{0\}$. Then by the definition of a normal cone one has

$$\langle \bar{x}, v-\bar{v}\rangle \leqslant 0 \qquad \forall v\in D^0.$$

Since $\bar{x}\neq 0$ and $0\in \operatorname{int} D^0$, this implies that $\langle \bar{x}, \bar{v}\rangle > 0$. So,

$$\begin{aligned}&\langle \bar{x}/\langle \bar{x}, \bar{v}\rangle, \bar{v}\rangle = 1\\ &0\geqslant \langle \bar{x}/\langle \bar{x}, \bar{v}\rangle, v-\bar{v}\rangle = \langle \bar{x}/\langle \bar{x}, \bar{v}\rangle, v\rangle - 1 \qquad \forall v\in D^0.\end{aligned}$$

Thus, $1 \geqslant \langle \bar{x}/\langle \bar{x}, \bar{v}\rangle, v\rangle$ $\forall v \in D^0$. Therefore, $\bar{x}/\langle \bar{x}, \bar{v}\rangle$ belongs to D. On the other hand one has

$$-f(\bar{x}/\langle \bar{x}, \bar{v}\rangle) = -f^{HH}(\bar{x}/\langle \bar{x}, \bar{v}\rangle) = \inf_v\{f^H(v): \langle \bar{x}/\langle \bar{x}, \bar{v}\rangle, v\rangle \geqslant 1\}$$

$$\leqslant f^H(\bar{v}) = \inf(\mathrm{P}^*) = -\sup(\mathrm{P}).$$

So, $\bar{x}/\langle \bar{x}, \bar{v}\rangle$ is an optimal solution of (P). ▮

Remark 5.1. If we have obtained a solution of the primal problem (P), by Theorem 5.1(ii), we can obtain a solution of the dual (P*) by minimizing a quasiconvex function over a halfspace (i.e., by solving a convex program). Conversely, if we have obtained a solution of the dual problem, by Theorem 5.1(iii), we can obtain a solution of the primal by finding a vector in a convex set. This requires us to solve a convex program.

6. Applications

Application 6.1. Let A be a symmetric positive definite $n \times n$-matrix. Since $\{x: \|x\| \leqslant 1\}^0 = \{v: \|v\| \leqslant 1\}$, by virtue of Example 2.3 we have the following primal–dual pair

Primal	Dual
$\max\{x^TAx: \|x\| \leqslant 1\}$	$\min\{-1/v^TA^{-1}v: \|v\| \geqslant 1\}$.

Since $-1/v^TA^{-1}v = \alpha \Leftrightarrow v^TA^{-1}v = -1/\alpha$, the dual problem can be rewritten as

$$\min\{v^TA^{-1}v: \|v\| \geqslant 1\}. \tag{40}$$

If we denote by $-\alpha$ the optimal value in the dual problem then $1/\alpha$ will be the optimal value in (40). By Theorem 5.1(i) one has

$$\max\{x^Tax: \|x\| \leqslant 1\} = \alpha = 1/\min\{v^TA^{-1}v: \|v\| \geqslant 1\}.$$

This equality says that if α is the greatest eigenvalue of A then $1/\alpha$ is the smallest eigenvalue of A^{-1}.

Application 6.2. Let $a_1, a_2, ..., a_k$ be k vectors in R^n such that $0 \in \mathrm{int}(\mathrm{conv}\{a_1, a_2, ..., a_k\})$. Since

$$(\mathrm{conv}\{a_1, a_2, ..., a_k\})^0 = \{v: \langle a_i, v\rangle \leqslant 1\ \forall i = 1, ..., k\},$$

by virtue of Example 2.2 we have the following primal–dual pair

Primal	Dual
minimize $\|x\|^2$	maximize $-1/\|v\|^2$
s.t. $x \in R^n \backslash \mathrm{int}(\mathrm{conv}\{a_1, ..., a_k\})$	s.t. $\langle a_i, v\rangle \leqslant 1\ \forall i = 1, ..., k.$

The dual can be rewritten as

$$\max\{\|v\|^2 \colon \langle a_i, v\rangle \leqslant 1\ \forall i = 1, ..., k\}. \tag{41}$$

This is a linearly constrained quadratic concave minimization problem and it has attracted a lot of algorithmic studies (see Tuy *et al.* [34–36], Rosen *et al.* [20, 21], ...). By Theorem 5.1, if $\bar{v}$ is a solution of the dual then a minimizer of $\|x\|^2$ on the halfspace $\{x \colon \langle \bar{v}, \bar{x}\rangle \geqslant 1\}$ is a solution of the primal. This minimizer is $\bar{v}/\|\bar{v}\|^2$.

Application 6.3. Let Y be a compact convex set containing 0 in its interior and X a closed convex set containing 0 in its interior. By virtue of Example 2.4, one has the following primal–dual pair

Primal	Dual
$\min\limits_{x \notin \mathrm{int}\, X} \max\limits_{y \in Y} \langle y, x\rangle$	$\max\limits_{v \in X^0} \{-1/\max\limits_{z \in Y^0} \{\langle z, v\rangle\}.$

The maximization of the function $v \mapsto -1/\max\{\langle v, x\rangle \colon x \in V\}$ is equivalent to the maximization of the function $v \mapsto \max\{\langle v, x\rangle \colon x \in V\}$. So, the dual can be rewritten as

$$\max_{v \in X^0} \max_{z \in Y^0} \langle z, v\rangle. \tag{42}$$

This problem is a bilinear programming. Thus, we have the duality relationship between a minimax problem with a reverse convex constraint and a bilinear programming.

Since in finite dimension cases the dimension of the initial space is equal to the dimension of the dual space, the number of variables in the primal problem is equal to the number of variables in the dual problem. But in the following application we shall show that for some time the dimension of the dual problem can be strongly reduced and much smaller than the dimension of the primal.

Application 6.4. We consider the primal problem

$$\text{minimize } \|x\|^2, \qquad \text{s.t. } g(x) \geqslant 0, \tag{43}$$

where

$$g(x) = \sup\{y^T Bx - 1 \colon h(y) \leqslant 0,\ y \in R^m\} \tag{44}$$

with B being a $m \times n$-matrix and $h(\cdot)$ a convex function defined on R^m such that $\{y: h(y) \leqslant 0\}$ is bounded. It is obvious that $g(\cdot)$ is a finite convex function and $g(0) = -1 < 0$. If denote

$$G = \{x: g(x) \leqslant 0\} \tag{45}$$

then problem (43) can be rewritten as $\min\{\|x\|^2: x \in R^n \setminus \operatorname{int} G\}$. By virtue of (44) and (45) we can see that $G^0 = \{v \in R^n: v = B^T y$ for some y satisfying $h(y) \leqslant 0\}$. Now the dual of (43) can be stated as

$$\text{maximize } -1/\|v\|^2, \qquad \text{s.t. } v = B^T y \text{ for some } y: h(y) \leqslant 0,$$

or equivalently,

$$\text{maximize } \|v\|^2, \qquad \text{s.t. } v = B^T y \text{ for some } y: h(y) \leqslant 0. \tag{46}$$

If we replace variable v by variable y then (46) becomes

$$\text{maximize } \|B^T y\|^2, \qquad \text{s.t. } h(y) \leqslant 0,\ y \in R^m. \tag{47}$$

This is a convex maximization on a convex set in R^m. Thus, the dimension of the primal problem is n whereas the dimension of the dual (47) is m. If $m \ll n$ then it will be much more appropriate to solve (47) (in R^m) than to solve directly (43) (in R^n).

7. Discussion

In this section we discuss the relations between our results with the previous one.

In Singer [23] a concept of general conjugation was introduced. An operator c which associates each function $f: R^n \to \bar{R}$ with a function $f^c: R^n \to \bar{R}$ is called a conjugation if

$$(\inf_{i \in I} f_i)^c = \sup_{i \in I} f_i^c \tag{48}$$

$$(f + \alpha)^c = f^c - \alpha, \tag{49}$$

where $\alpha \in R$, $f_i: R^n \to \bar{R}$ $\forall i \in I$. The conjugate operator H (Definition 2.1) always satisfies (49) (see (6)), whereas it does not satisfy (48). Indeed, it is easy to check that $(\inf_{i \in I} f_i)^H$ coincides with $\sup_{i \in I} f_i^H$ at every point except the origin. Thus, the quasiconjugate operator H is not a conjugation in the sense given in [23].

In Greenberg *et al.* [8], Crouzeix [3], Atteia *et al.* [1], and Singer [24] several attempts have been made to represent the lower semi-continuous quasiconvex hull of f (i.e., the greatest lower semi-continuous quasiconvex fuction majorized by f) as a second conjugate of f, in some sense. For instance, Singer [24] has introduced, for any $f: R^n \to \bar{R}$ and $\lambda \in R$, the "λ-semi-conjugate" f^θ_λ of f, as the function defined by

$$f^\theta_\lambda(v) = \lambda - 1 - \inf\{f(x): \langle v, x \rangle \geqslant \lambda - 1\} \tag{50}$$

and he has proved that the function

$$f^{\theta\theta}(x) := \sup_{\lambda \in R} (f^\theta_\lambda)^\theta_\lambda(x) \tag{51}$$

coincides with the lower semi-continuous quasiconvex hull of f. In those papers a class of quite general functions has been considered. But in order to obtain the semi-continuous quasiconvex hull the previous works use more than one operator. For example, in Singer [24] we have to use the "λ-semi-conjugate" operator (see (50)) and the "normalized second semi-conjugate" (see (51)). In this paper a new definition of quasiconjugate and a definition of quasiconvex hull are introduced. Let f be a function satisfying

$$f(0) = \inf\{f(x): x \in R^n \setminus \{0\}\}.$$

By Theorems 4.1 and 4.2 if either f is usc or f is lsc and achieves the maximum value at the infinite, then the quasiconvex hull of f can be obtained by using only the quasiconjugate and it is exactly the biquasiconjugate of f. Of course, if f is lsc and achieves the maximum value at the infinite then the quasiconvex hull of f is lsc (Theorem 4.2) and hence it is the semi-continuous quasiconvex hull of f as well.

In recent years, Duality Theory in Nonconvex Optimization, especially, in D.C. Minimization has attracted attention from several researchers (see, e.g., Pshenichnyyi [19], Toland [33], Hiriart-Urruty *et al.* [4, 11]). The approaches in the papers mentioned above are based on the formula

$$(g-h)^*(v) = \sup_{u \in \operatorname{dom} h^*} \{g^*(v+u) - h^*(u)\} \qquad \forall v \in R^n,$$

where h is a convex function and h^*, g^* denote the conjugates of h, g, respectively. In Hiriart-Urruty [11], the dual of a convex maximization problem with a convex constraint also is obtained that is a d.c. minimization problem (a problem of minimizing a d.c. function). For example, if we consider the primal problem given in Application 5.1 then the dual can be determined as

$$\min\{(1/4)\, v^T a^{-1} v - \|v\|: v \in R^n\}$$

(see [11]). But, by the approaches in those papers we could not obtain a dual problem for a general reverse convex program, especially, we could not obtain a duality relationship between Concave Program and Reverse Convex Program.

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