Excess of a class of g-frames

Lili Zang, Wenchang Sun *, Dianfa Chen

Department of Mathematics and LPMC, Nankai University, Tianjin 300071, China

Abstract

In this paper, we study the relationship between frames for the super Hilbert space $\mathcal{H} \oplus \mathcal{H}$ and g-frames for $\mathcal{H}$ with respect to $\mathbb{C}^2$. We show that a g-frame associated with a frame for $\mathcal{H} \oplus \mathcal{H}$ remains a g-frame whenever any one of its elements is removed. Furthermore, we show that the excess of such a g-frame is at least $\dim \mathcal{H}$.

© 2008 Elsevier Inc. All rights reserved.

1. Introduction

The frame was first introduced by Duffin and Schaeffer [14] in the study of nonharmonic Fourier series in 1952. It is a generalization of the Riesz basis. Let $K$ be a subset of $\mathbb{Z}$. Recall that a sequence $\{\varphi_k : k \in K\}$ in a Hilbert space $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there are two positive constants $A$ and $B$ such that

$$A \|f\|^2 \leq \sum_{k \in K} |\langle f, \varphi_k \rangle|^2 \leq B \|f\|^2, \quad \forall f \in \mathcal{H}.$$ 

$A$ and $B$ are the lower and upper frame bounds, respectively. A frame that ceases to be a frame whenever any one of its elements is removed is said to be an exact frame. It is well known that exact frames and Riesz bases are identical (see [24]).

Let $\{\varphi_k : k \in K\}$ be a frame for $\mathcal{H}$. Then for every $f \in \mathcal{H}$, there is some $\{c_k : k \in K\} \in \ell^2$ such that

$$f = \sum_{k \in K} c_k \varphi_k.$$

(1.1)

Since a frame is usually not a basis, the coefficients in the above expansion are usually not unique. The excess of a sequence in a Hilbert space is the greatest number of elements that can be removed yet leave a set with the same closed span. For a frame set, the remaining set may not be a frame.

The frame possesses many nice properties which makes it very useful in wavelet analysis, irregular sampling theory, signal processing and many other fields. We refer to [3,4,9,11,12,17,24] for an introduction to the frame theory and its applications.

We observe that various generalizations of the frame have been proposed recently. For example, bounded quasi-projectors [15], frames of subspaces [2,7], fusion frames [8,16,20], pseudo-frames [21], oblique frames [10], outer frames [1]...
and a class of time–frequency localization operators [13]. All of these generalizations have been proved to be useful in many applications. One of the authors [22,23] introduced the concept of g-frames and gave some properties of g-frames.

Before introducing the concept of g-frames, we give some notations. In this paper, \( \mathcal{U} \) and \( \mathcal{V} \) are two Hilbert spaces and \( \{ \mathcal{V}_k: k \in \mathbb{K} \} \) is a sequence of closed subspaces of \( \mathcal{V} \). \( \mathcal{L}(\mathcal{U}, \mathcal{V}_k) \) is the collection of all bounded linear operators from \( \mathcal{U} \) to \( \mathcal{V}_k \).

**Definition 1.1.** We call a sequence of operators \( \{ A_k \in \mathcal{L}(\mathcal{U}, \mathcal{V}_k): k \in \mathbb{K} \} \) a g-frame for \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_k: k \in \mathbb{K} \} \), if there are two positive constants \( A \) and \( B \) such that

\[
A \| f \|^2 \leq \sum_{k \in \mathbb{K}} \| A_k f \|^2 \leq B \| f \|^2, \quad \forall f \in \mathcal{U}.
\]  

We call \( A \) and \( B \) the lower and upper g-frame bounds, respectively.

We call \( \{ A_k: k \in \mathbb{K} \} \) a g-frame for \( \mathcal{U} \) with respect to \( \mathcal{V} \) if \( \mathcal{V}_k = \mathcal{V} \), \( \forall k \in \mathbb{K} \).

By the Riesz representation theorem, for every \( \Lambda \in \mathcal{L}(\mathcal{H}, \mathbb{C}^2) \), there is some \( (\varphi, \psi) \in \mathcal{H} \oplus \mathcal{H} \) such that \( \Lambda f = (\langle f, \varphi \rangle, \langle f, \psi \rangle)^T \), \( \forall f \in \mathcal{H} \). Therefore, every g-frame \( \{ A_k: k \in \mathbb{K} \} \) for \( \mathcal{H} \) with respect to \( \mathbb{C}^2 \) is of the following form,

\[
A_k f = (\langle f, \varphi_k \rangle, \langle f, \psi_k \rangle)^T, \quad \forall f \in \mathcal{H}.
\]  

**Definition 1.2.** Let \( \{(\varphi_k, \psi_k): k \in \mathbb{K} \} \) be a sequence of elements of \( \mathcal{H} \oplus \mathcal{H} \) and \( A_k \) be defined by (1.3). If \( \{ A_k: k \in \mathbb{K} \} \) is a g-frame for \( \mathcal{H} \) with respect to \( \mathbb{C}^2 \), then we call it a g-frame associated with \( \{(\varphi_k, \psi_k): k \in \mathbb{K} \} \).

For any \((f_1, g_1), (f_2, g_2) \in \mathcal{H} \oplus \mathcal{H}\), we define their inner product by

\[
\langle (f_1, g_1), (f_2, g_2) \rangle = \langle f_1, g_1 \rangle + \langle f_2, g_2 \rangle.
\]

It can be verified easily that \( \mathcal{H} \oplus \mathcal{H} \) is a Hilbert space with respect to this inner product. Such Hilbert spaces are called as super Hilbert spaces in literatures [6,18,19] and have been widely studied recently. For example, Balan [6] introduced the concept of super frames and presented some density results for Weyl–Heisenberg super frames. In [19], Han and Larson derived necessary and sufficient conditions for the direct sum of two frames to be a super frame. And in [18], Gu and Han investigated the connection between decomposable Parseval wavelet frames and super wavelet frames, and gave some necessary and sufficient conditions for extendable Parseval wavelet frames.

Let \( \{(\varphi_k, \psi_k): k \in \mathbb{K} \} \) be a frame for \( \mathcal{H} \oplus \mathcal{H} \) and \( A_k \) be defined by (1.3). It is easy to see that \( \{ A_k: k \in \mathbb{K} \} \) is a g-frame for \( \mathcal{H} \) with respect to \( \mathbb{C}^2 \). In this paper, we show that \( \{ A_k: k \in \mathbb{K} \} \) remains a g-frame whenever any one of its elements is removed. Furthermore, we show that the excess of such a g-frame is at least \( \text{dim} \mathcal{H} \).

2. G-frames for Hilbert spaces

Let \( \mathbb{J} \) be a subset of \( \mathbb{Z} \), \( A_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j) \) and \( \{ e_{j,k}: k \in \mathbb{K}_j \} \) be an orthonormal basis for \( \mathcal{V}_j \), where \( \mathbb{K}_j \) is a subset of \( \mathbb{Z} \), \( j \in \mathbb{J} \). Let

\[
u_{j,k} = A_j^* e_{j,k}, \quad j \in \mathbb{J}, \ k \in \mathbb{K}_j.
\]  

It was proved in [22] that

\[
A_j f = \sum_{k \in \mathbb{K}_j} (f, \nu_{j,k}) e_{j,k}, \quad \forall f \in \mathcal{U}.
\]  

We call \( \{ \nu_{j,k}: j \in \mathbb{J}, \ k \in \mathbb{K}_j \} \) the sequence induced by \( \{ A_j: j \in \mathbb{J} \} \) with respect to \( \{ e_{j,k}: k \in \mathbb{K}_j \} \). The following proposition gives a characterization of g-frames.

**Proposition 2.1.** (See [22, Theorem 3.1]) Let \( A_j \in \mathcal{L}(\mathcal{U}, \mathcal{V}_j) \) and \( \{ \nu_{j,k}: j \in \mathbb{J}, \ k \in \mathbb{K}_j \} \) be defined as in (2.1). Then \( \{ A_j: j \in \mathbb{J} \} \) is a g-frame for \( \mathcal{U} \) with respect to \( \{ \mathcal{V}_j: j \in \mathbb{J} \} \) if and only if \( \{ \nu_{j,k}: j \in \mathbb{J}, \ k \in \mathbb{K}_j \} \) is a frame for \( \mathcal{U} \).

**Example 2.1.** Let \( \{(\varphi_k, \psi_k): k \in \mathbb{K} \} \in \mathcal{H} \oplus \mathcal{H} \) and \( A_k \) be defined as in (1.3), \( k \in \mathbb{K} \). By Proposition 2.1, \( \{ A_k: k \in \mathbb{K} \} \) is a g-frame for \( \mathcal{H} \) with respect to \( \mathbb{C}^2 \) if and only if \( \{ \varphi_k: k \in \mathbb{K} \} \cup \{ \psi_k: k \in \mathbb{K} \} \) is a frame for \( \mathcal{H} \).

G-frames have many similar properties with frames. For example, for a g-frame for \( \mathcal{U} \), we can define its frame operator \( S \) as follows:

\[
S f = \sum_{k \in \mathbb{K}} A_k^* A_k f.
\]
3. Excess of G-frames associated with frames for super Hilbert spaces

In this section, we study the excess of g-frames associated with frames for the super Hilbert space $H \oplus H$. Recall that the excess of a sequence in a Hilbert space is the greatest number of elements that can be removed yet leave a set with the same closed span.

First, we introduce a result on the excess of frames.

**Proposition 3.1.** (See [24, Lemma 4.6].) Let $\{f_k: k \in \mathbb{Z}\}$ be a frame for some Hilbert space $H$. Then for any $k_0 \in \mathbb{Z}$, $\{f_k: k \in \mathbb{Z}, k \neq k_0\}$ is either incomplete in $H$ or still a frame for $H$.

The following lemma gives a necessary condition for $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ to be a frame for $H \oplus H$, which is easy to check and we leave it to interested readers.

**Lemma 3.2.** Suppose that $H$ is a Hilbert space and that $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ is a frame for $H \oplus H$. Then for any $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 \neq 0$, $\{(\alpha \psi_k + \beta \psi_k): k \in \mathbb{K}\}$ is a frame for $H$. In particular, both $\{(\psi_k: k \in \mathbb{K}\}$ and $\{(\psi_k: k \in \mathbb{K}\}$ are frames for $H$.

By Lemma 3.2, we can easily find examples such that both $\{(\psi_k: k \in \mathbb{K}\}$ and $\{(\psi_k: k \in \mathbb{K}\}$ are frames for $H$ while $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ is not a frame for $H \oplus H$. In fact, if $\{(\psi_k: k \in \mathbb{K}\}$ is a frame for $H$, then $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ cannot be a frame for $H \oplus H$. Otherwise, by letting $(\alpha, \beta) = (1, -1)$ in the above lemma, we get that $\{0, 0, \ldots\}$ is a frame for $H$, which is impossible.

By Proposition 2.1 and Lemma 3.2, if $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ is a frame for $H \oplus H$, then $\{A_k: k \in \mathbb{K}\}$ defined by (1.3) is a g-frame for $H$ with respect to $C^2$. For this g-frame, we have the following.

**Theorem 3.3.** Suppose that $H$ is a Hilbert space and that $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ is a frame for $H \oplus H$. Then for any $k_0 \in \mathbb{K}$, $\{A_k: k \in \mathbb{K}, k \neq k_0\}$ is still a g-frame for $H$ with respect to $C^2$.

**Proof.** By Lemma 3.2, both $\{(\psi_k: k \in \mathbb{K}\}$ and $\{(\psi_k: k \in \mathbb{K}\}$ are frames for $H$.

Since $\{(\psi_k, \psi_k): k \in \mathbb{K}\}$ is a frame for $H \oplus H$, for any $k_0 \in \mathbb{K}$, there is some $\{c_k: k \in \mathbb{K}\} \in \ell^2$ such that

$$\sum_{k \in \mathbb{K}} c_k (\psi_k, \psi_k) = (\varphi_{k_0}, 0).$$

Hence

$$\sum_{k \in \mathbb{K}} c_k \psi_k = \varphi_{k_0} \quad (3.1)$$

and

$$\sum_{k \in \mathbb{K}} c_k \psi_k = 0. \quad (3.2)$$

There are two cases.

**Case 1.** $c_{k_0} = 0$.

In this case, we see from (3.1) that $\varphi_{k_0} = \sum_{k \neq k_0} c_k \psi_k$. Hence, $\overline{\text{span}}\{\varphi_k: k \in \mathbb{K}, k \neq k_0\} = \overline{\text{span}}\{\varphi_k: k \in \mathbb{K}\} = H$. By Proposition 3.1, $\{\varphi_k: k \in \mathbb{K}, k \neq k_0\}$ is a frame for $H$. On the other hand, it is easy to see that $\{\psi_k: k \in \mathbb{K}, k \neq k_0\}$ is a Bessel sequence. Hence $\{\varphi_k: k \in \mathbb{K}, k \neq k_0\} \cup \{\psi_k: k \in \mathbb{K}, k \neq k_0\}$ is a frame for $H$. By Proposition 2.1, $\{A_k: k \in \mathbb{K}, k \neq k_0\}$ is a g-frame for $H$ with respect to $C^2$. 

where $A_k^*$ is the adjoint operator of $A_k$. It can be proved that $S$ is well defined on $U$ and is a bounded, invertible, and linear operator.

Let

$$\tilde{A}_k = A_k S^{-1}.$$
Case 2. $c_{k_0} \neq 0$.

By (3.2), $\psi_{k_0} = -\sum_{k \in \mathbb{K}} k \neq k_0 (c_k / c_{k_0}) \psi_k$.

Similarly to Case 1 we can prove that $\{ \psi_k : k \in \mathbb{K}, k \neq k_0 \}$ is a frame for $\mathcal{H}$ and therefore, $\{ A_k : k \in \mathbb{K}, k \neq k_0 \}$ is a g-frame for $\mathcal{H}$ with respect to $C^2$. This completes the proof. \( \square \)

From the proof of Theorem 3.3 we know that if $\{ (\varphi_k, \psi_k) : k \in \mathbb{K} \}$ is a frame for $\mathcal{H} \oplus \mathcal{H}$, then for any $k_0$, at least one of $\{ \varphi_k : k \in \mathbb{K}, k \neq k_0 \}$ and $\{ \psi_k : k \in \mathbb{K}, k \neq k_0 \}$ is still a frame for $\mathcal{H}$. And therefore, $\{ A_k : k \in \mathbb{K}, k \neq k_0 \}$ is still a g-frame for $\mathcal{H}$ with respect to $C^2$.

In other words, Theorem 3.3 shows that the excess of a g-frame associated with a frame for a super Hilbert space is at least 1. Next we show that the excess of such a g-frame is at least $\dim H$. Specifically, we have the following.

**Theorem 3.4.** Let $\mathcal{H}$ be a Hilbert space, $\{ (\varphi_k, \psi_k) : k \in \mathbb{K} \}$ be a frame for $\mathcal{H} \oplus \mathcal{H}$ and $A_k$ be defined by (1.3). Then $\{ A_k : k \in \mathbb{K} \}$ has an excess of at least $\dim \mathcal{H}$.

Moreover, if $\dim \mathcal{H} < \infty$, then there is a partition $\{ \mathbb{K}_1, \mathbb{K}_2, \mathbb{K}_3 \}$ for $\mathbb{K}$, i.e., $\mathbb{K} = \mathbb{K}_1 \cup \mathbb{K}_2 \cup \mathbb{K}_3$ and $\mathbb{K}_i \cap \mathbb{K}_j = \emptyset$, $i \neq j$, such that $\# \mathbb{K}_1 = \# \mathbb{K}_2 = \dim \mathcal{H}$, and both $\{ \varphi_k : k \in \mathbb{K}_1 \}$ and $\{ \psi_k : k \in \mathbb{K}_2 \}$ are frames for $\mathcal{H}$. And therefore, both $\{ A_k : k \in \mathbb{K}_1 \}$ and $\{ A_k : k \in \mathbb{K}_2 \}$ are g-frames for $\mathcal{H}$.

Before giving the proof of Theorem 3.4, we introduce a lemma.

**Lemma 3.5.** Suppose that $\mathcal{H}$ is a Hilbert space and that $\{ \varphi_k : k \in \mathbb{K} \}$ is a Riesz basis for $\mathcal{H}$. Let $n$ be a non-negative integer, $\{ (f_i, g_i) : 1 \leq i \leq n \} \subset \mathcal{H} \oplus \mathcal{H}$, and $\{ \varphi_k : k \in \mathbb{K} \} \subset \mathcal{H}$ be a Bessel sequence. If $h_1, h_2, \ldots, h_{n+1} \in \mathcal{H}$ and $(0, h_1), (0, h_2), \ldots, (0, h_{n+1})$ can be linearly represented by $\{ (f_i, g_i) : 1 \leq i \leq n \}$, $\{ (\varphi_k, \psi_k) : k \in \mathbb{K} \}$, then $h_1, h_2, \ldots, h_{n+1}$ are linearly dependent in $\mathcal{H}$.

**Proof.** We prove the lemma by induction. First we assume that $n = 0$.

Assume that $h_1 \in \mathcal{H}$ and that $(0, h_1)$ can be linearly represented by $\{ (\varphi_k, \psi_k) : k \in \mathbb{K} \}$, i.e., there is some $c_k : k \in \mathbb{K} \in \ell^2$ such that

$$\sum_{k \in \mathbb{K}} c_k (\varphi_k, \psi_k) = (0, h_1). \tag{3.3}$$

That is,

$$\sum_{k \in \mathbb{K}} c_k \varphi_k = 0 \tag{3.4}$$

and

$$\sum_{k \in \mathbb{K}} c_k \psi_k = h_1. \tag{3.5}$$

Since $\{ \varphi_k : k \in \mathbb{K} \}$ is a Riesz basis for $\mathcal{H}$, we see from (3.4) that $c_k = 0, k \in \mathbb{Z}$. Consequently, $h_1 = 0$. Hence $h_1$ is linearly dependent.

Next, we assume that the conclusion is true for some $n - 1$ with $n \geq 1$ and that $(0, h_1), (0, h_2), \ldots, (0, h_{n+1})$ can be linearly represented by $\{ (f_i, g_i) : 1 \leq i \leq n \}$ and $\{ (\varphi_k, \psi_k) : k \in \mathbb{K} \}$. In this case, we can find some $\{ a_{i,j} : 1 \leq i \leq n + 1, 1 \leq j \leq n \}$ and $\{ c_{l,k} : 1 \leq l \leq n + 1, k \in \mathbb{K} \}$ such that

$$\begin{align*}
& a_{1,1}(f_1, g_1) + \cdots + a_{1,n}(f_n, g_n) + \sum_{k \in \mathbb{K}} c_{1,k} (\varphi_k, \psi_k) = (0, h_1), \\
& a_{2,1}(f_1, g_1) + \cdots + a_{2,n}(f_n, g_n) + \sum_{k \in \mathbb{K}} c_{2,k} (\varphi_k, \psi_k) = (0, h_2), \\
& \ldots \\
& a_{n+1,1}(f_1, g_1) + \cdots + a_{n+1,n}(f_n, g_n) + \sum_{k \in \mathbb{K}} c_{n+1,k} (\varphi_k, \psi_k) = (0, h_{n+1}).
\end{align*} \tag{3.6}$$

Denote $A = (a_{i,j})_{1 \leq i \leq n+1, 1 \leq j \leq n}$, an $(n + 1) \times n$ matrix.

There are two cases.

**Case 1.** $A$ is a zero matrix. In this case, every row of $A$ is zero. For any $i$ with $1 \leq i \leq n + 1$, we have

$$\sum_{k \in \mathbb{K}} c_{l,k} (\varphi_k, \psi_k) = (0, h_i).$$

Hence $c_{l,k} = 0, \forall k \in \mathbb{K}$. Therefore $h_i = 0$. Obviously, $h_1, h_2, \ldots, h_{n+1} \in \mathcal{H}$ are linearly dependent.
**Case 2.** $A$ is not a zero matrix. Without loss of generality, we assume that $a_{1,1} \neq 0$.

Multiplying both sides of the first equation in (3.6) by $-a_{1,1}/a_{1,1}$ and then adding them to both sides of the $i$th equation, respectively. We have that $(0, h_i - (a_{1,1}/a_{1,1})h_1), 2 \leq i \leq n + 1$ can be linearly represented by $(f_i, g_i): 2 \leq i \leq n) \cup \{(\varphi_k, \psi_k): k \in \mathbb{K}\}$.

By the induction, $(h_i - a_{1,1}/a_{1,1}h_1), 2 \leq i \leq n + 1$ are linearly dependent. Hence there are constants $c_i, 2 \leq i \leq n + 1$, not all of which are zeros, such that

$$\sum_{i=2}^{n+1} c_i \left( h_i - \frac{a_{1,1}}{a_{1,1}} h_1 \right) = 0.$$ 

Consequently,

$$-\sum_{i=2}^{n+1} c_i \frac{a_{1,1}}{a_{1,1}} h_1 + \sum_{i=2}^{n+1} c_i h_i = 0.$$ 

Therefore, $h_1, h_2, \ldots, h_{n+1}$ are also linearly dependent. Hence, there are arbitrary, we have $\dim H \leq n$. This completes the proof. $\square$

**The following is an immediate consequence.**

**Corollary 3.6.** Let $H$ be a Hilbert space and $\{(\varphi_k, \psi_k): k \in \mathbb{K}\}$ be a frame for $H \oplus H$. If $\{\varphi_k: k \in \mathbb{K}\}$ contains a Riesz basis, that is, there is some $K_1 \subset \mathbb{K}$ such that $\{\varphi_k: k \in K_1\}$ is a Riesz basis for $H$, then $\dim H \leq \#(\mathbb{K} \setminus K_1)$.

**Proof.** Denote $n = \#(\mathbb{K} \setminus K_1)$.

If $n = \infty$, the conclusion holds obviously.

Next we assume that $n < \infty$. Since $\{\varphi_k, \psi_k\}: k \in \mathbb{K}\}$ is a frame for $H \oplus H$, for any $h_1, h_2, \ldots, h_{n+1} \in H, (0, h_1), (0, h_2), \ldots, (0, h_{n+1})$ can be linearly represented by $\{(\varphi_k, \psi_k): k \in \mathbb{K}\} = \{(\varphi_k, \psi_k): k \in \mathbb{K} \setminus K_1\} \cup \{(\varphi_k, \psi_k): k \in K_1\}$. By Lemma 3.5, $h_1, h_2, \ldots, h_{n+1}$ are linearly dependent. Since $h_1, h_2, \ldots, h_{n+1}$ are arbitrary, we have $\dim H \leq n$. This completes the proof. $\square$

**Proof of Theorem 3.4.** Without loss of generality, we assume that $n := \dim H > 0$.

First, we prove that $\{A_k: k \in \mathbb{K}\}$ possesses an excess of at least $\dim H$. There are two cases.

**Case 1.** $\dim H < \infty$.

By Corollary 3.6, $\{\varphi_k, k \in \mathbb{K}\}$ is not a Riesz basis for $H$. Hence, there is some $K_1$ such that $\{\varphi_k, k \in \mathbb{K}, k \neq K_1\}$ is still a frame for $H$. If $n = 1$, the conclusion is proved.

If $n > 1$, using Corollary 3.6 again, we get that $\{\varphi_k, k \in \mathbb{K}, k \neq K_1\}$ is not a Riesz basis for $H$. Hence, there is some $K_2$ such that $\{\varphi_k, k \in \mathbb{K}, k \neq K_1, K_2\}$ is still a frame for $H$.

Repeating the above procedure, we can find some $K_1 \subset \mathbb{K}$ such that $\#(\mathbb{K} \setminus K_1) = \dim H = n$ and $\{\varphi_k, k \in \mathbb{K} \setminus K_1\}$ is a frame for $H$. Consequently $\{\varphi_k: k \in \mathbb{K} \setminus K_1\}$ is a g-frame for $H$ with respect to $\mathbb{C}^2$. Therefore, $\{A_k: k \in \mathbb{K}\}$ has an excess of at least $\dim H$.

**Case 2.** $\dim H = \infty$.

From the proof of Case 1, we know that we can remove arbitrary finitely many elements of $\{A_k: k \in \mathbb{K}\}$ such that the left is still a $g$-frame for $H$ with respect to $\mathbb{C}^2$. Hence, $\{A_k: k \in \mathbb{K}\}$ has an excess of infinity.

Next, we prove the “moreover” part.

Let $\{e_k: 1 \leq k \leq n\}$ be an orthonormal basis for $H$. Then $\dim H \oplus H = 2n$ and $\{e_k, 0: 1 \leq k \leq n\} \cup \{0, e_k: 1 \leq k \leq n\}$ is an orthonormal basis for $H \oplus H$. Since $\{(\varphi_k, \psi_k): k \in \mathbb{K}\}$ is a frame for $H \oplus H$ and $\dim H \oplus H = 2n < \infty$, we can choose a basis for $H \oplus H$ from $\{(\varphi_k, \psi_k): k \in \mathbb{K}\}$. Without loss of generality, we assume that $\mathbb{K} \supset \{1, 2, \ldots, 2n\}$ and $\{(\varphi_k, \psi_k): 1 \leq k \leq 2n\}$ is a basis for $H \oplus H$. Then there is an $2n \times 2n$ invertible matrix $A$ such that

$$\begin{bmatrix}
\varphi_1 & \psi_1 \\
\vdots & \vdots \\
\varphi_n & \psi_n \\
\varphi_{n+1} & \psi_{n+1} \\
\vdots & \vdots \\
\varphi_{2n} & \psi_{2n}
\end{bmatrix} = A \begin{bmatrix}
e_1 & 0 \\
\vdots & \vdots \\
\ne_n & 0 \\
0 & e_1 \\
\vdots & \vdots \\
0 & e_n
\end{bmatrix}.$$ 

(3.7)

Denote
where every $A_{ij}$ is a $1 \times n$ matrix, $1 \leq i \leq 2n$, $j = 1, 2$. Since $\det A \neq 0$, by Laplace Theorem, there is a partition of $\{1, 2, \ldots, 2n\}$, denoted by $\{i_1, \ldots, i_n\} \cup \{j_1, \ldots, j_n\}$, such that $(A_{i_1,1}, \ldots, A_{i_n,1})^T$ and $(A_{j_1,2}, \ldots, A_{j_n,2})^T$ are invertible.

Clearly, we have

\[
\begin{bmatrix}
\psi_{i_1} \\
\vdots \\
\psi_{i_n}
\end{bmatrix} = \begin{bmatrix}
A_{i_1,1} \\
\vdots \\
A_{i_n,1}
\end{bmatrix}
\begin{bmatrix}
e_1 \\
\vdots \\
e_n
\end{bmatrix}
\tag{3.9}
\]

and

\[
\begin{bmatrix}
\psi_{j_1} \\
\vdots \\
\psi_{j_n}
\end{bmatrix} = \begin{bmatrix}
A_{j_1,2} \\
\vdots \\
A_{j_n,2}
\end{bmatrix}
\begin{bmatrix}
e_1 \\
\vdots \\
e_n
\end{bmatrix}.
\tag{3.10}
\]

Hence both $(\psi_{i_1}, \ldots, \psi_{i_n})^T$ and $(\psi_{j_1}, \ldots, \psi_{j_n})^T$ are bases and therefore frames for $\mathcal{H}$. Let $\mathbb{K}_1 = \{i_1, \ldots, i_n\}$, $\mathbb{K}_2 = \{j_1, \ldots, j_n\}$ and $\mathbb{K}_3 = \mathbb{K} \setminus (\mathbb{K}_1 \cup \mathbb{K}_2)$. Then the conclusion follows. \(\square\)

**Remark 3.1.** By Lemma 3.2, if $\{(\psi_k, \phi_k) : k \in \mathbb{K}\}$ is a frame for $\mathcal{H} \oplus \mathcal{H}$, then both $\{(\phi_k, k \in \mathbb{K})\}$ and $(\psi_k : k \in \mathbb{K})$ are frames for $\mathcal{H}$. Now Theorem 3.4 shows that both $\{(\phi_k, k \in \mathbb{K})\}$ and $(\psi_k : k \in \mathbb{K})$ possess an excess no less than $\dim \mathcal{H}$.

We see from the proof of Theorem 3.4 that whenever $\dim \mathcal{H} = \infty$, we can find some $\mathbb{K}_1 \subset \mathbb{K}$ such that $\# \mathbb{K}_1 = \infty$ and for any $\mathbb{K}_1' \subset \mathbb{K}_1$ with finitely many elements, $\{(\phi_k, k \in \mathbb{K}_1 \setminus \mathbb{K}_1') : k \in \mathbb{K}_1 \}$ is a frame for $\mathcal{H}$. In particular, for any $k_1 \in \mathbb{K}_1$, $(\phi_k : k \in \mathbb{K}, k \neq k_1) \cap \mathcal{H}$ is a frame for $\mathcal{H}$. By [3, Theorem 5.4], we have the following consequence.

**Corollary 3.7.** Let $\mathcal{H}$ be a Hilbert space, $\{(\phi_k, \psi_k) : k \in \mathbb{K}\}$ be a frame for $\mathcal{H} \oplus \mathcal{H}$ and $\lambda_k$ be defined by (1.3). Suppose $\dim \mathcal{H} = \infty$ and that there exists some constant $L > 0$ and an infinite subset $\mathbb{K}_1$ of $\mathbb{K}$ such that for any $k_1 \in \mathbb{K}_1$, $L$ is a lower frame bound for $(\phi_k, k \in \mathbb{K}, k \neq k_1)$. Then there is an infinite subset $\mathbb{K}_1'$ of $\mathbb{K}_1$ such that $\{(\lambda_k, k \in \mathbb{K}_1 \setminus \mathbb{K}_1') : k \in \mathbb{K}_1 \}$ is a g-frame for $\mathcal{H}$.

**Proof.** By [3, Theorem 5.4], for $\varepsilon = L/2$, there is an infinite subset $\mathbb{K}_1' \subset \mathbb{K}_1$ such that $(\phi_k, k \in \mathbb{K} \setminus \mathbb{K}_1')$ is still a frame for $\mathcal{H}$ with lower frame bound $L - \varepsilon = L/2$. Consequently, $\{(\lambda_k, k \in \mathbb{K}_1 \setminus \mathbb{K}_1') : k \in \mathbb{K}_1 \}$ is a g-frame for $\mathcal{H}$. This completes the proof. \(\square\)

**Remark 3.2.** For the finite-dimensional case, the redundancy of a frame $\mathcal{F} = \{f_j\}_{j \in \mathbb{J}}$ is the ratio $r = M/N$, where $M = \# \mathbb{J}$ and $N$ is the dimension of the space spanned by the elements of $\mathcal{F}$ (see [5]). When $\dim \mathcal{H} < \infty$, we see from the “merder” part of Theorem 3.4 that the redundancy of $(\psi_k : k \in \mathbb{K} \setminus \mathbb{K}_1) \cup (\psi_k : k \in \mathbb{K})$ is at least $2$.

Next we study the linear dependence of $(\psi_k : k \in \mathbb{K} \setminus \mathbb{K}_1) \cup (\psi_k : k \in \mathbb{K})$ whenever $(\psi_k, \phi_k) : k \in \mathbb{K})$ is a frame for $\mathcal{H} \oplus \mathcal{H}$.

**Definition 3.1.** The rank of a set of vectors is defined as the dimension of its closed linear span.

**Theorem 3.8.** Let $\mathcal{H}$ be a Hilbert space. Suppose that $\dim \mathcal{H} = n$, $0 < n < \infty$, and that $\{(\psi_k, \phi_k) : k \in \mathbb{K}\}$ is a frame for $\mathcal{H} \oplus \mathcal{H}$. Then for any $I \subset \mathbb{K}$ with $\# I = m < \infty$, the rank of $(\psi_k : k \in \mathbb{K} \setminus I) \cup (\psi_k : k \in \mathbb{K} \setminus I)$ is no less than $n - \lfloor m/2 \rfloor$.

**Proof.** Denote $\Phi = \{(\psi_k, k \in \mathbb{K} \setminus I) \setminus (\psi_k : k \in \mathbb{K})$. First, we consider the case of $0 < n < \infty$.

Suppose that the rank of $\Phi$ is $l$ and that $\{\xi_i, \ldots, \xi_l\}$ is a basis for $\Phi$. Then $\{(\xi_i, 0) : 1 \leq i \leq l\} \cup \{(0, \xi_i) : 1 \leq i \leq l\}$ is linearly independent in $\mathcal{H} \oplus \mathcal{H}$ and its rank is $2l$. Obviously, $(\psi_k, \phi_k) : k \in \mathbb{K} \setminus I$ can be linearly represented by $((\xi_i, 0) : 1 \leq i \leq l) \cup \{(0, \xi_i) : 1 \leq i \leq l\}$. Hence it has a rank no greater than $2l$. On the other hand, since $\dim \mathcal{H} = n$ and $|(\phi_k, \psi_k) : k \in \mathbb{K} \setminus I$ is a frame for $\mathcal{H} \oplus \mathcal{H}$, the rank of $\{(\phi_k, \psi_k) : k \in \mathbb{K}\}$ is $2n$. Consequently, the rank of $(\psi_k, \phi_k) : k \in \mathbb{K} \setminus I)$ is no less than $2n - m$. It follows that $2n - m \leq 2l$. Hence $l \geq n - \lfloor m/2 \rfloor$.

Next we consider the case of $n = \infty$.

Let $I$ be a subset of $\mathbb{K}$ such that $\# I = m < \infty$. By Lemma 3.2, $\{(\psi_k, k \in \mathbb{K})$ is a frame for $\mathcal{H}$. Hence there are infinitely many linearly independent elements in $\{(\psi_k, k \in \mathbb{K})$. For arbitrary positive integer $N$, there are $N + m$ linearly independent elements in $\{(\psi_k, k \in \mathbb{K}), among which there are at least $N$ elements belonging to $\{(\psi_k, k \in \mathbb{K} \setminus I)$. Hence the rank of $(\psi_k, k \in \mathbb{K} \setminus I)$ is no less than $N$. Since $N$ is arbitrary, the rank of $(\psi_k, k \in \mathbb{K} \setminus I)$ is infinite. Consequently, $\Phi$ has a rank of infinity. \(\square\)

**Remark 3.3.** By Theorem 3.8, if $\{(\psi_k, \phi_k) : k \in \mathbb{K}\}$ is a frame for $\mathcal{H} \oplus \mathcal{H}$, then the rank of $(\psi_k, k \in \mathbb{K}) \cup (\psi_k : k \in \mathbb{K})$ keeps unchanged whenever any pair of $(\psi_k, \psi_k)$ is removed.
Acknowledgments

We thank the referees very much for valuable suggestions which helped to improve the paper.

References


