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Control Theory of Hyperbolic Equations Related to Certain Questions in Harmonic Analysis and Spectral Theory*

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1. INTRODUCTION

In a 1967 paper "Nonharmonic Fourier Series in the Control Theory of Distributed Parameter Systems" [14] we have shown that the classical results of Paley and Wiener [12], Levinson [10], Schwartz [17], and others can be used to advantage in studying the controllability of the wave equation in a single space dimension. The purpose of the present article goes beyond such a simple application of existing results in harmonic analysis to control problems. We wish to show in addition that the study of control problems for certain hyperbolic partial differential equations leads to some interesting, and perhaps unexpected, consequences in harmonic analysis. Thus there is a two-way interplay between these two subjects, only recently becoming apparent, and we may hope for deeper studies of this relationship in the future.

Because our purpose is to uncover this relationship, we will not attempt great generality in our presentations. Many of the results which we will obtain are valid for any second order linear hyperbolic partial differential equation in two independent variables x and t whose coefficients depend only upon x. However, such a complete treatment would introduce complications which would obscure our main points. Hence we shall focus our attention in this paper on systems related to partial differential equations of the form

$$\rho(x)\frac{\partial^2 w}{\partial t^2} - p(x)\frac{\partial^2 w}{\partial x^2} + q(x)\frac{\partial w}{\partial t} + r(x)\frac{\partial w}{\partial x} = 0, \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t < \infty, \tag{1.1}$$

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where the coefficient functions ρ, p, q and r are twice continuously differentiable for $0 \leqslant x \leqslant 1$ and

$$\rho(x) \geqslant \rho_0 > 0, \qquad p(x) > p_0 > 0, \qquad 0 \leqslant x \leqslant 1.$$

If (1.1) is thought of as a model for small vibrations of a flexible string, ρ is the linear mass density and p is the modulus of elasticity.

We shall impose boundary conditions of the form

$$A_0 \frac{\partial w}{\partial t}(0, t) + B_0 \frac{\partial w}{\partial x}(0, t) = 0, \qquad 0 \leqslant t < \infty, \tag{1.2}$$

$$A_1 \frac{\partial w}{\partial t}(1, t) + B_1 \frac{\partial w}{\partial x}(1, t) = f(t), \qquad 0 \leqslant t < \infty, \tag{1.3}$$

with the proviso that

$$\frac{A_0}{B_0} \neq \pm \left(\frac{\rho(0)}{p(0)}\right)^{1/2}, \qquad \frac{A_1}{B_1} = \pm \left(\frac{\rho(1)}{p(1)}\right)^{1/2}. \tag{1.4}$$

If we again use the physical analogy of the flexible string, the boundary condition (1.2) corresponds to a fixed end ($B_0 = 0$), an end free to move in the direction of the w axis ($A_0 = 0$), or an end free to move but with positive or negative friction ($A_0 \neq 0$, $B_0 \neq 0$). The reason for the restrictions (1.4) will become clear later. The boundary condition (1.3) at $x = 1$ can be interpreted similarly with $f(t)$ a "control" force at our disposal with which we attempt to influence the evolution of solutions of (1.1).

We will find it convenient to put our problem in a certain standard form. The change of independent variable

$$\xi = \int_0^x \left(\frac{\rho(s)}{p(s)}\right)^{1/2} ds$$

carries (1.1) into an equation of the form

$$\begin{gathered} \frac{\partial^2 w}{\partial t^2} - \frac{\partial^2 w}{\partial \xi^2} + a(\xi) \frac{\partial w}{\partial t} + b(\xi) \frac{\partial w}{\partial \xi} = 0, \\ 0 \leqslant \xi \leqslant \ell \equiv \xi(1), \qquad 0 \leqslant t < \infty. \end{gathered} \tag{1.5}$$

The coefficients $a(\xi)$, $b(\xi)$ are now continuously differentiable functions of ξ. This second-order scalar equation can be replaced by the first-order two-dimensional system

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a(\xi) & b(\xi) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{1.6}$$

where $u = \partial w/\partial t$, $v = \partial w/\partial \xi$. Every solution of (1.5) in class C^m, $m \geqslant 2$, corresponds to a solution of (1.6) of class C^{m-1}. It should be noted, however, that two solutions of (1.5) differing by a nonzero constant are carried into the same solution of (1.6). Otherwise the correspondence is complete in both directions. The appropriate boundary conditions are now

$$\alpha_0 u(0, t) + \beta_0 v(0, t) \equiv 0, \tag{1.7}$$

$$\alpha_1 u(\ell, t) + \beta_1 v(\ell, t) \equiv f(t), \tag{1.8}$$

with the condition

$$\frac{\alpha_0}{\beta_0} \neq \pm 1, \qquad \frac{\alpha_1}{\beta_1} \neq \pm 1. \tag{1.9}$$

We have arrived at the system (1.6) because we wished to introduce our topic by means of the familiar scalar equation (1.1). But all of the work which we do is done just as easily if we generalize (1.6) slightly to

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{1.10}$$

where the real coefficients $a_{ij}(\xi)$ are continuously differentiable for $0 \leqslant \xi \leqslant \ell$. We retain the boundary conditions (1.7), (1.8).

By studying the controllability of the system (1.6)–(1.8) we will be able to prove certain theorems about the operator

$$L\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{d\xi}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} \tag{1.11}$$

with boundary conditions of the type (1.7), (1.8). In particular, if the (in general complex) eigenvalues of L are $\{\lambda_k\}$, we will be able to establish that $\{e^{\lambda_k t}\}$ form a Riesz basis for $L^2[0, 2\ell]$ in a way very different from that pursued by Paley and Wiener, Levinson, Schwartz, and others. Moreover, by showing that the controls of which bring solutions of (1.10), (1.7), (1.8) to zero at time $t = 2\ell$ can be synthesized by means of a linear feedback control law, we prove a rather unusual characterization of the dual basis of $L^2[0, 2\ell]$ relative to $\{e^{\lambda_k t}\}$ which has possible application to numerical computation of the functions $\{q_k(t)\}$ which are biorthogonal to $\{e^{\lambda_k t}\}$.

2. Principal Results

In this section we state our theorems for the system (1.10), (1.7), (1.8) and supply proofs where they are reasonably short. The proofs of Theorems 1 and 3 are long and are given in Sections 3 and 4.

The basis of our work is the question of finite time controllability. This topic has been studied earlier by the author [14, 15] and in a thesis by J. Grainger [7]. The present work begins with a statement of these results in terms of "finite energy" solutions, i.e., generalized solutions of (1.10), (1.7), (1.8) for which

$$\int_0^{\ell} [| u(\xi, t)|^2 + | v(\xi, t)|^2] \, d\xi < \infty, \qquad t \geqslant 0.$$

Appropriate existence, uniqueness and regularity theorems for such solutions may be found in [9] and [11]. Although we have taken all of the coefficients in our partial differential equation and boundary conditions to be real, we will find it convenient to consider complex solutions.

THEOREM 1. *Let initial and terminal states* (u_0, v_0) *and* (u_1, v_1) *be given at the times* $t = 0$ *and* $t = 2\ell$, *respectively, with* u_0, v_0, u_1, v_1 *all in* $L^2[0, \ell]$. *Then there is exactly one function* $f \in L^2[0, 2\ell]$ *such that the solution* (u, v) *of* (1.10), (1.7), (1.8) *which satisfies*

$$u(\xi, 0) = u_0(\xi), \qquad v(\xi, 0) = v_0(\xi) \qquad \text{a.e. } in\ [0, \ell] \tag{2.1}$$

also satisfies

$$u(\xi, 2\ell) = u_1(\xi), \qquad v(\xi, 2\ell) = v_1(\xi) \qquad \text{a.e. } in\ [0, \ell] \tag{2.2}$$

and there is a positive constant P, *independent of* u_0, v_0, u_1, v_1 *such that*

$$\int_0^{2\ell} | f(t)|^2 \, dt \leqslant P \int_0^{\ell} (| u_0(\xi)|^2 + | v_0(\xi)|^2 + | u_1(\xi)|^2 + | v_1(\xi)|^2) \, d\xi. \tag{2.3}$$

Moreover, there is a second positive constant $\hat{P}$ *such that when* $u_1 = v_1 = 0$

$$\int_0^{\ell} (| u_0(\xi)|^2 + | v_0(\xi)|^2) \, d\xi \leqslant \hat{P} \int_0^{2\ell} | f(t)|^2 \, dt. \tag{2.4}$$

Also, when $u_1 = v_1 = 0$ *the condition* (1.9) *can be replaced by the weaker restriction*

$$\frac{\alpha_0}{\beta_0} \neq 1, \qquad \frac{\alpha_1}{\beta_1} \neq -1$$

and the existence of f *satisfying* (2.3) *is still assured. However,* (2.4) *cannot be proved in this case.*

The proof will be given in Section 3. The time period 2ℓ is "critical." We have shown in [14] and [15] that it is in general impossible to satisfy the given initial and terminal conditions if less time is allowed, while the control f is not unique if more time is allowed.

The system (1.10), (1.7), (1.8) with $f \equiv 0$ has the form

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = L\begin{pmatrix} u \\ v \end{pmatrix},$$

where L is the differential operator defined by (1.11) with domain Δ in $L^2[0, \ell] \oplus L^2[0, 2\ell]$ consisting of pairs of functions (u, v) whose first derivatives, taken in the sense of the theory of distributions, lie in $L^2[0, \ell]$ and which satisfy

$$\alpha_0 u(0) + \beta_0 v(0) = \alpha_1 u(\ell) + \beta_1 v(\ell) = 0. \tag{2.5}$$

The adjoint of L is the operator

$$L^*\begin{pmatrix} w \\ z \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{d\xi}\begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{21}(\xi) \\ a_{12}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}$$

defined on the domain Δ^* which differs from Δ in that (w, z) belonging to it satisfy

$$\alpha_0 w(0) - \beta_0 z(0) = \alpha_1 w(\ell) - \beta_1 z(\ell) = 0. \tag{2.6}$$

Very general results due to Birkhoff [1], Schwartz [16], Kramer [8], and others show that L is a *spectral operator*; in particular it has a sequence of complex simple eigenvalues $\{\lambda_k\}$ such that the associated normalized eigenvectors (φ_k, ψ_k) form a Riesz basis in $L^2[0, \ell] \otimes L^2[0, \ell]$, i.e., each (u, v) in that space has a unique development

$$\begin{pmatrix} u \\ v \end{pmatrix} = \sum c_k \begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix} \tag{2.7}$$

with

$$m_1 \sum |c_k|^2 \leqslant \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| \leqslant m_2 \sum |c_k|^2 \tag{2.8}$$

for fixed positive constants m_1, m_2. The adjoint operator L^* has eigenvalues $\{\bar{\lambda}_k\}$ which are the complex conjugates of the $\{\lambda_k\}$ and eigenvectors (φ_k^*, ψ_k^*) such that

$$\left(\begin{pmatrix} \varphi_k \\ \psi_k \end{pmatrix}, \begin{pmatrix} \varphi_\ell^* \\ \psi_\ell^* \end{pmatrix}\right)_{L^2[0,\ell]\otimes L^2[0,\ell]} = \delta_{k\ell} = \begin{cases} 1, & k = \ell \\ 0, & k \neq \ell. \end{cases} \tag{2.9}$$

Now let (u, v) be a (possibly complex) solution of (1.10), (1.7), (1.8) and (w, z) a (possibly complex) solution of

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} a_{11}(\xi) & a_{21}(\xi) \\ a_{12}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix}, \tag{2.10}$$

satisfying boundary conditions of the form (2.6). If u_0, v_0, u_1, v_1 all belong to $C^1[0, \ell]$ one easily justifies the following computation in the rectangle $D = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, 0 \leqslant t \leqslant 2\ell\}$:

$$\begin{aligned}
0 &= \iint_D \left\{ \left(\begin{pmatrix} u \\ v \end{pmatrix}, \left[\frac{\partial}{\partial t} \begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} \right] \right) \right. \\
&\quad \left. + \left(\left[\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right], \begin{pmatrix} w \\ z \end{pmatrix} \right) \right\} d\xi\, dt \\
&= \iint_D \left\{ \frac{\partial}{\partial t} (u\bar{w} + v\bar{z}) - \frac{\partial}{\partial \xi} (u\bar{z} + v\bar{w}) \right\} d\xi\, dt \qquad (2.11) \\
&= \int_0^\ell [(u(\xi, 2\ell)\, \bar{w}(\xi, 2\ell) + v(\xi, 2\ell)\, \bar{z}(\xi, 2\ell)) \\
&\quad - (u(\xi, 0)\, \bar{w}(\xi, 0) + v(\xi, 0)\, \bar{z}(\xi, 0))]\, d\xi \\
&\quad + \int_0^{2\ell} [(u(0, t)\, \bar{z}(0, t) + v(0, t)\, \bar{w}(0, t)) \\
&\quad - (u(\ell, t)\, \bar{z}(\ell, t) + v(\ell, t)\, \bar{w}(\ell, t))]\, dt.
\end{aligned}$$

If we expand the solution $(u(\xi, t), v(\xi, t))$ as in (2.7),

$$\begin{pmatrix} u(\xi, t) \\ v(\xi, t) \end{pmatrix} = \sum c_k(t) \begin{pmatrix} \varphi_k(\xi) \\ \psi_k(\xi) \end{pmatrix},$$

and note that

$$e^{-\bar{\lambda}_k(T-t)} \begin{pmatrix} \varphi_k{}^*(\xi) \\ \psi_k{}^*(\xi) \end{pmatrix} = \begin{pmatrix} w(\xi, t) \\ z(\xi, t) \end{pmatrix}$$

is a solution of (2.10), we may substitute in (2.11) and use (2.9) and the boundary conditions satisfied by (u, v) and (w, z) at 0 and ℓ to see that

$$c_k(T) - c_k(0)\, e^{-\lambda_k T} = \begin{cases} \displaystyle\int_0^{2\ell} \frac{\overline{\varphi_k{}^*(\ell)}}{\beta_1}\, e^{-\lambda_k(T-t)} f(t)\, dt, & \beta_1 \neq 0 \\ \displaystyle\int_0^{2\ell} \frac{\overline{\psi_k{}^*(\ell)}}{\alpha_1}\, e^{-\lambda_k(T-t)} f(t)\, dt, & \beta_1 = 0. \end{cases} \qquad (2.12)$$

Now let

$$\begin{pmatrix} u(\xi, 0) \\ v(\xi, 0) \end{pmatrix} = \begin{pmatrix} u_0(\xi) \\ v_0(\xi) \end{pmatrix} = \sum c_k(0) \begin{pmatrix} \varphi_k(\xi) \\ \psi_k(\xi) \end{pmatrix} \qquad (2.13)$$

be steered by means of the control f to the zero terminal state

$$u_1(\xi) \equiv v_1(\xi) \equiv 0.$$

Then in (2.12) $c_k(T) = 0$ and thus

$$-c_k(0) = \begin{cases} \displaystyle\int_0^{2\ell} \frac{\overline{\varphi_k^*(\ell)}}{\beta_1} e^{\lambda_k t} f(t)\, dt, & \beta_1 \neq 0, \\ \displaystyle\int_0^{2\ell} \frac{\overline{\psi_k^*(\ell)}}{\beta_1} e^{\lambda_k t} f(t)\, dt, & \beta_1 = 0. \end{cases} \tag{2.14}$$

for all k.

In order to perform the calculations (2.11) we assumed u_0, v_0 belong to $C^1[0, \ell]$. One may verify easily, however, that (2.14) will also hold for u_0, v_0 in $L^2[0, \ell]$ through approximation of (2.13) by finite partial sums and use of the inequalities of Theorem 1. We leave this to the reader. Then it follows from (2.8) that if $\{c_k(0)\}$ is any sequence of complex numbers with

$$\sum | c_k(0)|^2 < \infty$$

the *moment problem* (2.14) has a solution $f \in L^2[0, 2\ell]$. Moreover, using (2.3), (2.4) and (2.8) we see that there are positive numbers K_1 and K_2, independent of $\{c_k(0)\}$, such that

$$K_1 \sum | c_k(0)|^2 \leqslant \int_0^{2\ell} | f(t)|^2\, dt \leqslant K_2 \sum | c_k(0)|^2. \tag{2.15}$$

It is an easy consequence of Lemma 2.1 in the proof (Section 3) of Theorem 1 that when $\beta_1 \neq 0$, $| \varphi_k^*(\ell)|$ is bounded away from 0 and ∞ and when $\beta_1 = 0$, $| \psi_k^*(\ell)|$ is bounded away from 0 and ∞. Then (2.14) and (2.15) together prove

THEOREM 2. *Let the eigenvalues of L be $\{\lambda_k\}$ and let $\{c_k\}$ be any sequence of complex numbers with $\sum | c_k |^2 < \infty$. Then the moment problem*

$$c_k = \int_0^{2\ell} e^{\lambda_k t} f(t)\, dt \qquad \textit{for all } k, \tag{2.16}$$

has a unique solution of $f \in L^2[0, 2\ell]$ such that

$$K_3 \sum | c_k |^2 \leqslant \int_0^{2\ell} | f(t)|^2\, dt \leqslant K_4 \sum | c_k |^2$$

for certain positive constants K_3 and K_4 independent of $\{c_k\}$.

This theorem implies that the functions $\{e^{\lambda_k t}\}$ form a Riesz basis for the space $L^2[0, 2\ell]$, i.e., every function $g \in L^2[0, 2\ell]$ has an expansion

$$g(t) = \sum \gamma_k e^{\lambda_k t},$$

convergent in $L^2[0, 2\ell]$, with the property

$$K_5 \sum |\gamma_k|^2 \leqslant \|g\|_{L^2[0,2\ell]} \leqslant K_6 \sum |\gamma_k|^2$$

for positive constants K_5 and K_6. The coefficients γ_k are given by

$$\gamma_k = \int_0^{2\ell} g(t)\, \bar{q}_k(t)\, dt,$$

where $q_k(t)$ is the solution of (2.16) with $c_k = 1$, $c_\ell = 0$, $\ell \neq k$. The sequence $\{q_k\}$ is the *biorthogonal sequence* for $\{e^{\lambda_k t}\}$, or the dual basis for $L^2[0, 2\ell]$ relative to the basis $\{e^{\lambda_k t}\}$. Another interpretation is that q_k is the unique control function steering the initial state

$$\begin{pmatrix} u_0(\xi) \\ v_0(\xi) \end{pmatrix} = -r_k \begin{pmatrix} \varphi_k(\xi) \\ \psi_k(\xi) \end{pmatrix}, \qquad r_k = \begin{cases} \dfrac{\beta_1}{\bar{\varphi}_k{}^*(\ell)}, & \beta_1 \neq 0 \\[2ex] \dfrac{\alpha_1}{\bar{\psi}_k{}^*(\ell)}, & \beta_1 = 0, \end{cases} \tag{2.17}$$

to zero in time $t = 2\ell$.

Now one could also prove all of these results by the Fourier-transform methods of Paley and Wiener [12], Levinson [10] and Schwartz [17], provided one had sufficiently good asymptotic estimates of the location of the eigenvalues $\{\lambda_k\}$. In this respect the interesting thing about Theorem 2 is that it has been proved without detailed reference to the location of these eigenvalues. Even the necessary information that L is a spectral operator can be proved rather easily with the partial differential equations methods we employ together with a general theorem in [16]. Of course our work is quite special since it applies only to sequences $\{\lambda_k\}$ consisting of eigenvalues of operators L defined above, whereas the work of the authors cited applies to much more general sequences.

The familiar results to the effect that the functions $\{e^{\lambda_k t}\}$ are excessive in $L^2[0, T]$ if $T < 2\ell$ and deficient but linearly independent in $L^2[0, T]$ if $T > 2\ell$ can also be proved using methods like these. How this would be done should be clear from the work in [14] and [15] together with what we have already written here so we will not go into details.

While the proof of Theorem 1 in Section 3 is constructive, the method used is not particularly well adapted to computation. Thus it is significant that this control f can be synthesized by means of a linear-feedback control law, provided a linear relationship holds between the initial and terminal states. This, and other consequences, follow from

THEOREM 3. *Let u_0 and v_0 lie in $L^2[0, \ell]$ and let γ be any real number. Let*

$$\sigma^+ = \exp\left(-\tfrac{1}{2}\int_0^\ell [a_{11}(\xi) + a_{12}(\xi) + a_{21}(\xi) + a_{22}(\xi)]\, d\xi\right), \tag{2.18}$$

$$\sigma^- = \exp\left(-\tfrac{1}{2}\int_0^\ell [-a_{11}(\xi) + a_{12}(\xi) + a_{21}(\xi) - a_{22}(\xi)]\, d\xi\right), \tag{2.19}$$

where the $a_{ij}(\xi)$ are the coefficients appearing in (1.10). *Let (u, v) be a solution of* (1.10), (1.7) *and* (1.8). *Then*

$$u(\xi, 2\ell) \equiv \gamma u_0(\xi), \qquad v(\xi, 2\ell) \equiv \gamma v_0(\xi) \tag{2.20}$$

if and only if the solution (u, v) satisfies the boundary condition

$$\begin{aligned}\left(\frac{\sigma^+}{\beta_0 - \alpha_0} - \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}\right) u(\ell, t) + \left(\frac{\sigma^+}{\beta_0 - \alpha_0} + \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}\right) v(\ell, t) \\ = \int_0^\ell [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, d\xi,\end{aligned} \tag{2.21}$$

where h_1 and h_2 are certain continuous functions depending only upon the a_{ij}, α_0, β_0 and γ. When the a_{ij} are all zero, h_1 and h_2 vanish identically. When $\gamma = 0$ it is sufficient to assume $\alpha_0/\beta_0 \neq 1$ and the term $\gamma\sigma^-/(\beta_0 + \alpha_0)$ disappears.

An immediate consequence of Theorem 3 is the feedback law for the control f. If we put

$$\alpha_2 = \frac{\sigma^+}{\beta_0 - \alpha_0} - \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}, \qquad \beta_2 = \frac{\sigma^+}{\beta_0 - \alpha_0} + \frac{\gamma\sigma^-}{\beta_0 + \alpha_0} \tag{2.22}$$

we verify readily that

$$(\alpha_2)^2 + (\beta_2)^2 = 2\left[\frac{(\sigma^+)^2}{(\beta_0 - \alpha_0)^2} + \frac{(\gamma\sigma^-)^2}{(\beta_0 + \alpha_0)^2}\right] > 0.$$

If the vector (α_2, β_2) is a multiple of (α_1, β_1), say $(\alpha_2, \beta_2) = c(\alpha_1, \beta_1)$, $c \neq 0$, then (1.8) and (2.21) together yield

$$f(t) = \int_0^\ell \left[\frac{h_1(\xi)}{c} u(\xi, t) + \frac{h_2(\xi)}{c} v(\xi, t)\right] d\xi.$$

If (α_2, β_2) and (α_1, β_1) are linearly independent, then one can find a third vector (α_3, β_3) in R^2 such that (α_2, β_2) and (α_3, β_3) are linearly independent and

$$(\alpha_2, \beta_2) = c_1(\alpha_1, \beta_1) - c_2(\alpha_3, \beta_3)$$

with $c_1 \neq 0$. Then

$$f(t) = \frac{c_2\alpha_3}{c_1} u(\ell, t) + \frac{c_2\beta_3}{c_1} v(\ell, t) + \int_0^\ell \left[\frac{h_1(\xi)}{c_1} u(\xi, t) + \frac{h_2(\xi)}{c_1} v(\xi, t)\right] d\xi.$$

Thus we have proved

THEOREM 4. *Let initial and terminal conditions* (u_0, v_0) *and* (u_1, v_1) *be given satisfying* (2.20). *Then the control* f *steering the solution* (u, v) *of* (1.10), (1.7), (1.8) *from* (u_0, v_0) *to* $(u_1, v_1) = \gamma(u_0, v_0)$ *satisfies a feedback law*

$$f(t) = \mu u(\ell, t) + \nu v(\ell, t) + \int_0^\ell [k_1(\xi) u(\xi, t) + k_2(\xi) v(\xi, t)] d\xi,$$

where k_1 *and* k_2 *lie in* $C^1[0, \ell]$ *and* (μ, ν) *is either the zero vector or else* (α_1, β_1) *and* (μ, ν) *are linearly independent.*

If we take $\gamma = 0$ in Theorem 3 we see that a solution of (1.10), (1.7) and

$$u(\ell, t) + v(\ell, t) = \frac{\beta_0 - \alpha_0}{\sigma^+} \int_0^\ell [h_1(\xi) u(\xi, t) + h_2(\xi) v(\xi, t)] d\xi \qquad (2.23)$$

always satisfies $u(\xi, 2\ell) \equiv v(\xi, 2\ell) \equiv 0$. Then from (2.17) and the remarks accompanying it we see that the functions $\{q_k(t)\}$ biorthogonal to $\{e^{\lambda_k t}\}$ can be computed by solving (1.10), (1.7), (2.23) with the initial state (2.17) and then using (1.8). Since the computation of h_1 and h_2 can be carried out once and for all (see Section 4) by solving a relatively simple partial differential equation, we have here a possible method for the numerical calculation of the functions $\{q_k(t)\}$. We remark that (1.10), (1.7) and (2.23) is a system whose solutions can be approximated rather easily using the method of characteristics [3].

Now we will make some comments about the implications of Theorem 3 in a general mathematical sense, not particularly related to control problems. Fixing γ as in Theorem 3, we consider the unbounded operator L_1 defined in the Hilbert space $L^2[0, \ell] \oplus L^2[0, \ell]$ by (1.11) but with domain Δ_1 consisting of pairs of functions (u, v) satisfying

$$\alpha_0 u(0) + \beta_0 v(0) = 0 \qquad (2.24)$$

and [cf. (2.21), (2.22)]

$$\alpha_2 u(\ell) + \beta_2 v(\ell) = \int_0^\ell [h_1(\xi) u(\xi) + h_2(\xi) v(\xi)] d\xi \qquad (2.25)$$

and having first derivatives in $L^2[0, \ell]$. It is not difficult to verify that Δ_1 is dense in $L^2[0, \ell] \oplus L^2[0, \ell]$.

Solutions (u, v) of (1.10), (2.24), (2.25) have the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = e^{L_1 t} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix},$$

where $e^{L_1 t}$ is the strongly continuous semigroup (group if $\gamma \neq 0$) generated by the operator L_1. Theorem 3 gives us certain information about this semigroup (group) which in turn indicates some interesting properties of the operator L_1.

If $\gamma = 0$ the semigroup $e^{L_1 t}$ has the property

$$e^{L_1(2\ell)} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = 0$$

for all (u_0, v_0) in $L^2[0, \ell] \oplus L^2[0, \ell]$. Thus

$$e^{L_1 t} = 0$$

for $t \geqslant 2\ell$. Thus we have a somewhat unusual example of a strongly continuous semigroup which vanishes identically after a certain time, in this case 2ℓ. From results in [5] we see that the spectrum of L_1 must be empty in this case.

This result can be proved more or less directly when the a_{ij} are all zero (so that h_1 and h_2 are also zero). In this case the boundary condition (2.25) becomes

$$u(\ell) + v(\ell) = 0. \tag{2.26}$$

When the a_{ij} are not all identically zero the properties of the operator L with a right-hand boundary condition of the form (2.26) are rather elusive. This is one of the singular cases encountered by Birkhoff [1] and others in their pioneering work on the spectral properties of such operators. The significance of our work lies in the fact that we have shown that if in this singular case we replace the boundary condition (2.26) by

$$u(\ell) + v(\ell) = \frac{\beta_0 - \alpha_0}{\sigma^+} \int_0^\ell [h_1(\xi)\, u(\xi) + h_2(\xi)\, v(\xi)]\, d\xi$$

$$\left[\alpha_2 = \beta_2 = \frac{\sigma^+}{\beta_0 - \alpha_0} \text{ if } \gamma = 0\right],$$

then once again we have an operator whose spectrum is empty. We remark that one can give examples to show that this is not generally true for the boundary condition (2.26) when the a_{ij} are nonzero.

Now we take up the case $\gamma \neq 0$. Theorem 3 then shows that the group $e^{L_1 t}$ has the property

$$e^{L_1(2\ell)} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \gamma \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}$$

so that $e^{L_1(2\ell)} = \gamma I$. Letting

$$\rho = \frac{\log |\gamma|}{2\ell}$$

it is clear that the group $e^{(L_1 - \rho I)t}$ is *periodic* with period 2ℓ when $\gamma > 0$:

$$e^{(L_1 - \rho I)2\ell} = I, \qquad \gamma > 0, \tag{2.27}$$

and *antiperiodic* when $\gamma < 0$, i.e.,

$$e^{(L_1 - \rho I)2\ell} = -I, \qquad \gamma < 0.$$

Consider the case $\gamma > 0$. We define a new inner product $\langle\, , \rangle$ in $L^2[0, \ell] \oplus L^2[0, \ell]$ by

$$\left\langle \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \right\rangle = \int_0^{2\ell} \left(e^{(L_1 - \rho I)t} \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}, e^{(L_1 - \rho I)t} \begin{pmatrix} \hat{u}_0 \\ \hat{v}_0 \end{pmatrix} \right) dt,$$

where $(\, ,)$ is the usual inner product in that space. Because the operators $e^{(L_1 - \rho I)t}$ are uniformly bounded and have uniformly bounded inverses [the latter a consequence of (2.27)] we see that the norm $\langle\!\langle\ \rangle\!\rangle$ associated with the inner product $\langle\, , \rangle$ is equivalent to the usual norm $\|\ \|$ associated with $(\, ,)$ in the sense that

$$r_1 \left\langle\!\!\left\langle \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle\!\!\right\rangle \leqslant \left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| \leqslant r_2 \left\langle\!\!\left\langle \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle\!\!\right\rangle$$

for certain fixed positive constants r_1, r_2. The periodicity of the group $e^{(L_1 - \rho I)t}$ when $\gamma > 0$ shows that the inner product $\langle\, , \rangle$ is invariant under the action of the group. Thus $e^{(L_1 - \rho I)t}$ is a *unitary* group with respect to this inner product in $L^2[0, \ell] \oplus L^2[0, \ell]$. Stone's theorem [13] then shows that $L_1 - \rho I$ is anti-Hermitian with respect to this inner product with a representation

$$L_1 - \rho I = \int_{-\infty}^{\infty} i\mu \, dE(\mu),$$

where $E(\mu)$ is the spectral measure associated with $L_1 - \rho I$. Since $e^{(L_1-\rho I)t}$ is periodic, however, we can show easily that the support of $E(\mu)$ must be a subset of the points

$$0, \pm\frac{k\pi}{\ell}, \qquad k = 1, 2, 3, \ldots .$$

Thus, with respect to the usual inner product $(\,,\,)$, L_1 is a spectral operator with spectrum a subset of the points

$$\rho, \rho \pm i\frac{k\pi}{\ell}, \qquad k = 1, 2, 3, \ldots . \tag{2.28}$$

When $\gamma < 0$ we can argue in much the same way to show that L_1 is a spectral operator whose spectrum is a subset of the points

$$\rho \pm i\frac{(k-\frac{1}{2})\pi}{\ell}, \qquad k = 1, 2, 3, \ldots . \tag{2.29}$$

When the a_{ij} are all zero, which implies $\sigma^+ = \sigma^- = 1$ and h_1 and h_2 are zero, one can verify directly that the spectrum of the operator L with boundary conditions

$$\begin{gathered} \alpha_0 u(0) + \beta_0 v(0) = 0, \\ \left(\frac{1}{\beta_0 - \alpha_0} - \frac{\gamma}{\beta_0 - \alpha_0}\right) u(\ell) + \left(\frac{1}{\beta_0 - \alpha_0} + \frac{\gamma}{\beta_0 + \alpha_0}\right) v(\ell) = 0 \end{gathered} \tag{2.30}$$

consists of precisely the points (2.28) or (2.29), depending upon whether $\gamma > 0$ or $\gamma < 0$, respectively, and that each such point is an eigenvalue of single multiplicity. If the a_{ij} are not zero and we consider the operator L with boundary conditions (2.30), the eigenvalues are again simple and approach the values (2.28) or (2.29) asymptotically. The perturbation in L brought about by introducing the nonzero a_{ij} gives rise to a perturbation in the eigenvalues. Thus it is of some interest to be able to prove that this perturbation of the eigenvalues can be "undone," not by removing the a_{ij}, but by changing the right-hand boundary condition. Specifically, our result is

THEOREM 5. *There exist continuous functions h_1 and h_2 such that the operator*

$$L_1\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{d}{d\xi}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

with boundary conditions

$$\alpha_0 u(0) + \beta_0 v(0) = 0, \tag{2.31}$$

$$\begin{aligned}\left(\frac{\sigma^+}{\beta_0 - \alpha_0} - \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}\right) u(\ell) + \left(\frac{\sigma^+}{\beta_0 - \alpha_0} + \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}\right) v(\ell) \\ = \int_0^\ell [h_1(\xi)\, u(\xi) + h_2(\xi)\, v(\xi)]\, d\xi, \qquad \gamma \text{ real,}\end{aligned} \tag{2.32}$$

is a spectral operator whose spectrum coincides (multiplicity included) with that of the operator

$$L_0 \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{d}{d\xi} \begin{pmatrix} u \\ v \end{pmatrix}$$

with boundary conditions (2.31) *and*

$$\left(\frac{1}{\beta_0 - \alpha_0} - \frac{\gamma}{\beta_0 + \alpha_0}\right) u(\ell) + \left(\frac{1}{\beta_0 - \alpha_0} + \frac{\gamma}{\beta_0 + \alpha_0}\right) v(\ell) = 0. \tag{2.33}$$

Remarks. When $a_{11}(\xi) + a_{22}(\xi) \equiv 0$, $\sigma^+ = \sigma^-$ and the boundary conditions (2.32) and (2.33) differ only by an integral term.

If we want the boundary condition (2.33) to have a given form

$$\alpha u(\ell) + \beta v(\ell) = 0 \tag{2.34}$$

we can do so by setting

$$\gamma = \frac{\beta - \alpha}{\beta + \alpha} \left(\frac{\beta_0 + \alpha_0}{\beta_0 - \alpha_0}\right).$$

The only boundary condition (2.33) which cannot be realized in this way is

$$u(\ell) - v(\ell) = 0.$$

Proof of Theorem 5. We have already established that the spectrum of L_1 is a subset of the spectrum of L_0. When $\gamma = 0$ the spectra of L_1 and L_0 have been shown to be empty in both cases so there is nothing to prove. Hence we need only show that when $\gamma \neq 0$ each point in (2.28) or (2.29) belongs to the spectrum of L_1 and that each of these points is a simple eigenvalue.

Let us consider a boundary-value control system

$$\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = 0, \tag{2.35}$$

$$\alpha_0 u(0, t) + \beta_0 v(0, t) = 0, \tag{2.36}$$

$$\alpha_2 u(\ell, t) + \beta_2 v(\ell, t) - \int_0^\ell [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, d\xi = g(t), \tag{2.37}$$

where α_2 and β_2 are given by (2.22) and $g \in L^2[0, 2\ell]$.

Now consider the following adjoint system:

$$\frac{\partial}{\partial t}\begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{21}(\xi) \\ a_{12}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} w \\ z \end{pmatrix} + \frac{w(\ell, t)}{\beta_2}\begin{pmatrix} h_1(\xi) \\ h_2(\xi) \end{pmatrix} = 0, \tag{2.38}$$

$$\alpha_0 w(0) - \beta_0 z(0) = 0, \tag{2.39}$$

$$\alpha_2 w(\ell) - \beta_2 z(\ell) = 0. \tag{2.40}$$

[If $\beta_2 = 0$ we replace $w(\ell, t)/\beta_2$ in (2.38) by $z(\ell, t)/\alpha_2$.] Then we compute, using (2.36) and (2.39),

$$\frac{d}{dt}\left(\begin{pmatrix} u(\cdot, t) \\ v(\cdot, t) \end{pmatrix}, \begin{pmatrix} w(\cdot, t) \\ z(\cdot, t) \end{pmatrix}\right)$$

$$= \int_0^\ell \left[\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{12}(\xi) \\ a_{21}(\xi) & a_{22}(\xi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix}\right)\right.$$

$$\left. + \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} w \\ z \end{pmatrix} + \begin{pmatrix} a_{11}(\xi) & a_{21}(\xi) \\ a_{12}(\xi) & a_{22}(\xi) \end{pmatrix} - \frac{w(\ell, t)}{\beta_2}\begin{pmatrix} h_1(\xi) \\ h_2(\xi) \end{pmatrix}\right)\right] d\xi$$

$$= \int_0^\ell \left[\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix}\right) + \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} w \\ z \end{pmatrix}\right)\right] d\xi$$

$$- \frac{\overline{w(\ell, t)}}{\beta_2}\int_0^\ell [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, d\xi = \frac{\overline{w(\ell, t)}}{\beta_2}\, g(t),$$

the last equality following immediately when we integrate the term

$$\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} w \\ z \end{pmatrix}\right)$$

by parts and then use (2.37) and (2.40). [Again, if $\beta_2 = 0$ we replace $\overline{w(\ell, t)}/\beta_2$ by $\overline{z(\ell, t)}/\alpha_2$.] Thus, if $\binom{u}{v}$ and $\binom{w}{z}$ satisfy (2.35) and (2.38) and the given boundary conditions, we have

$$\left(\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix}, \begin{pmatrix} w(\cdot, 2\ell) \\ z(\cdot, 2\ell) \end{pmatrix}\right) - \left(\begin{pmatrix} u(\cdot, 0) \\ v(\cdot, 0) \end{pmatrix}, \begin{pmatrix} w(\cdot, 0) \\ v(\cdot, 0) \end{pmatrix}\right) = \int_0^{2\ell} \frac{\overline{w(\ell, t)}}{\beta_2}\, g(t)\, dt. \tag{2.41}$$

Suppose now we set $u(\xi, 0) \equiv v(\xi, 0) \equiv 0$ and consider the following problems:

(a) Letting $\binom{u}{v}$ solve (2.35)–(2.37) for these zero initial data and for arbitrary $g \in L^2[0, 2\ell]$, are the terminal states $(u(\cdot, 2\ell), v(\cdot, 2\ell))$ dense in $L^2[0, T]$?

(b) Can the zero state $(u(\cdot, 2\ell), v(\cdot, 2\ell)) = (0, 0)$ be reached using some $g \neq 0$ in $L^2[0, 2\ell]$?

We will show that the answer to (a) is "yes" and the answer to (b) is "no." Assuming this for the moment, we can complete the proof of Theorem 2.

Suppose λ_j were an eigenvalue of L_1 with multiplicity greater than 1. Since L_1 has been shown to be similar to an anti-Hermitian operator, there must then exist two independent eigenvectors (w_j, z_j) and $(\hat{w}_j, \hat{z}_j)$ of L_1^*—corresponding to the eigenvalue $\bar{\lambda}_j$ of L_1^*. Then both

$$\begin{pmatrix} w(\xi, t) \\ z(\xi, t) \end{pmatrix} = e^{\bar{\lambda}_j(2\ell - t)} \begin{pmatrix} w_j(\xi) \\ z_j(\xi) \end{pmatrix} \tag{2.42}$$

and

$$\begin{pmatrix} \hat{w}(\xi, t) \\ \hat{z}(\xi, t) \end{pmatrix} = e^{\bar{\lambda}_j(2\ell - t)} \begin{pmatrix} \hat{w}_j(\xi) \\ \hat{z}_j(\xi) \end{pmatrix} \tag{2.43}$$

solve

$$\frac{d}{dt} \begin{pmatrix} w \\ z \end{pmatrix} + L_1^* \begin{pmatrix} w \\ z \end{pmatrix} = 0,$$

which is the abstract form of (2.38)–(2.40). Indeed, (see [2] for related material) L_1^* is the operator

$$L_1^* \begin{pmatrix} w \\ z \end{pmatrix} = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} w \\ z \end{pmatrix} - \begin{pmatrix} a_{11}(\xi) & a_{21}(\xi) \\ a_{12}(\xi) & a_{22}(\xi) \end{pmatrix} \begin{pmatrix} w \\ z \end{pmatrix} + \frac{w(\ell)}{\beta_2} \begin{pmatrix} h_1(\xi) \\ h_2(\xi) \end{pmatrix}$$

with domain boundary defined by conditions of the form (2.39), (2.40). Substituting (2.42) and (2.43) into (2.41) and recalling that we are taking $u(\cdot, 0) = v(\cdot, 0) = 0$, we have

$$\left(\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix}, \begin{pmatrix} w(\cdot, 2\ell) \\ z(\cdot, 2\ell) \end{pmatrix} \right) = \int_0^{2\ell} \frac{\bar{w}_j(\ell)}{\beta_2} e^{\lambda_j(2\ell - t)} g(t)\, dt,$$

$$\left(\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix}, \begin{pmatrix} \hat{w}(\cdot, 2\ell) \\ \hat{z}(\cdot, 2\ell) \end{pmatrix} \right) = \int_0^{2\ell} \frac{\bar{\hat{w}}_j(\ell)}{\beta_2} e^{\lambda_j(2\ell - t)} g(t)\, dt.$$

Then for all states $\binom{u(\cdot, 2\ell)}{v(\cdot, 2\ell)}$ reachable from zero via (2.35)–(2.37) with controls $g \in L^2[0, 2\ell]$ we have

$$\left(\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix}, \frac{\beta_2}{\bar{\hat{w}}_j(\ell)} \begin{pmatrix} w(\cdot, 2\ell) \\ z(\cdot, 2\ell) \end{pmatrix} - \frac{\beta_2}{\bar{w}_j(\ell)} \begin{pmatrix} \hat{w}(\cdot, 2\ell) \\ \hat{z}(\cdot, 2\ell) \end{pmatrix} \right) = 0.$$

But this cannot be so if, as we claim, the answer to (a) is "yes." Thus, assuming the positive answer to (a), L_1^*, and hence L_1, has simple eigenvalues.

If some number $\rho \pm i(j\pi/\ell)$ [or $\rho \pm (j - \frac{1}{2})\pi/\ell$] is missing from the spectrum of L_1, assume it is $\rho + i(j\pi/\ell)$ for definiteness, then we note that

$$g_j(t) = e^{[-\rho + i(j\pi/\ell)]t}$$

has the property that

$$\int_0^{2\ell} e^{\bar{\lambda}_k t} g_j(t)\, dt = \int_0^{2\ell} e^{-i(k\pi/\ell)t} e^{i(j\pi/\ell)t}\, dt = 0$$

for all λ_k which are eigenvalues of L_1. Letting $\binom{w_k}{z_k}$ be the eigenvector of $L_1{}^*$ corresponding to its eigenvalue $\bar{\lambda}_k$ and setting

$$\begin{pmatrix} w_k(\xi, t) \\ z_k(\xi, t) \end{pmatrix} = e^{\bar{\lambda}_k(2\ell - t)} \begin{pmatrix} w_k(\xi) \\ z_k(\xi) \end{pmatrix},$$

we find, after substitution in (2.41), again with $(u(\cdot, 0), v(\cdot, 0)) = (0, 0)$, that

$$\left(\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix}, \begin{pmatrix} w_k \\ z_k \end{pmatrix} \right) = 0$$

for all k. Since the eigenvectors of $L_1{}^*$ span $L^2[0, \ell] \oplus L^2[0, \ell]$, we conclude that

$$\begin{pmatrix} u(\cdot, 2\ell) \\ v(\cdot, 2\ell) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

and thus $g_j(t)$ is a nonzero control taking (0, 0) into (0, 0). Hence if, as we will show, the answer to (b) is "no," we conclude that each of the numbers (2.28) is an eigenvalue of L_1 when $\gamma > 0$ and each of the numbers (2.29) is an eigenvalue of L_1 when $\gamma < 0$.

Now to complete the proof of Theorem 5, we take up questions (a) and (b). Let initial and terminal states (u_0, v_0), (u_1, v_1) be given, u_0, v_0, u_1, v_1 all in $L^2[0, \ell]$. By Theorem 1 there is a unique f in $L^2[0, 2\ell]$ such that if (u, v) solves (2.35), (2.36), (2.1) with

$$\alpha_2 u(\ell, t) + \beta_2 v(\ell, t) = f(t), \tag{2.44}$$

then $(u(\xi, 2\ell), v(\xi, 2\ell)) = (u_1(\xi), v_1(\xi))$, a.e. Then let

$$g(t) = f(t) - \int_0^{\ell} [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, d\xi$$

and we have

$$\alpha_2 u(\ell, t) + \beta_2 v(\ell, t) - \int_0^{\ell} [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, d\xi = g(t)$$

so g, which clearly lies in $L^2[0, 2\ell]$, steers (2.35)–(2.37) from (u_0, v_0) to (u_1, v_1). Thus the answer to (a) is, indeed, "yes."

Passing to question (b), if g steers a solution of (1.10), (2.36), (2.37) from (0, 0) to (0, 0) then

$$f(t) = g(t) = \int_0^{\ell} [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, dt$$

steers a solution of (2.35), (2.36), (2.44) from (0, 0) to (0, 0). Then Theorem 1 shows that $f(t) = 0$ a.e. in $[0, 2\ell]$ so that

$$g(t) = -\int_0^{\ell} [h_1(\xi)\, u(\xi, t) + h_2(\xi)\, v(\xi, t)]\, dt \tag{2.45}$$

a.e. in $L^2[0, \ell]$. Then the solution (u, v) satisfies

$$\alpha_2 u(\ell, t) + \beta_2 v(\ell, t) = 0 \qquad \text{a.e. in } [0, 2\ell]$$

which implies $(u(\xi, t), v(\xi, t)) = (0, 0)$ a.e. and we have, from (2.45),

$$g(t) \equiv 0 \qquad \text{a.e. in } L^2[0, 2\ell],$$

showing that the answer to (b) is "no." With this the proof of Theorem 5 is complete.

3. Proof of Theorem 1

The theory of hyperbolic partial differential equations is discussed in detail in [6] and [4], to which we refer the reader if a treatment of basic material is desired. The characteristics for the system (1.10) are families $\mathscr{C}^+$ and $\mathscr{C}^-$ of straight lines with slopes 1 and -1. A member of $\mathscr{C}^{+(-)}$ will be denoted by $c^{+(-)}(\xi, t)$, (ξ, t) being a point on the line in question which serves to specify that line. The quantities

$$\theta^+ = u + v, \qquad \theta^- = u - v$$

satisfy linear ordinary differential equations along characteristics in the families $\mathscr{C}^-$, $\mathscr{C}^+$, respectively. We may parametrize characteristics $c^+(0, t_0)$, $c^-(0, t_0)$ by arc length σ, τ:

$$c^+(0, t_0) = \left\{(\xi, t) \mid \xi = \frac{\sigma}{\sqrt{2}},\, t = t_0 + \frac{\sigma}{\sqrt{2}},\, 0 \leqslant \sigma \leqslant \sqrt{2}\ell\right\},$$

$$c^-(0, t_0) = \left\{(\xi, t) \mid \xi = \frac{\tau}{\sqrt{2}},\, t = t_0 - \frac{\tau}{\sqrt{2}},\, 0 \leqslant \tau \leqslant \sqrt{2}\ell\right\}.$$

Then we compute without difficulty

$$\frac{d\theta^+}{d\tau}\left(\frac{\tau}{\sqrt{2}}, t_0 - \frac{\tau}{\sqrt{2}}\right) + a_+^+\left(\frac{\tau}{\sqrt{2}}\right)\theta^+\left(\frac{\tau}{\sqrt{2}}, t_0 - \frac{\tau}{\sqrt{2}}\right) + a_-^+\left(\frac{\tau}{\sqrt{2}}\right)\theta^-\left(\frac{\tau}{\sqrt{2}}, t_0 - \frac{\tau}{\sqrt{2}}\right) = 0, \tag{3.1}$$

$$\frac{d\theta^-}{d\sigma}\left(\frac{\sigma}{\sqrt{2}}, t_0 + \frac{\sigma}{\sqrt{2}}\right) + a_+^-\left(\frac{\sigma}{\sqrt{2}}\right)\theta^+\left(\frac{\sigma}{\sqrt{2}}, t_0 + \frac{\sigma}{\sqrt{2}}\right) + a_-^-\left(\frac{\sigma}{\sqrt{2}}\right)\theta^-\left(\frac{\sigma}{\sqrt{2}}, t_0 + \frac{\sigma}{\sqrt{2}}\right) = 0, \tag{3.2}$$

where

$$a_+^+(\xi) = \tfrac{1}{2}\left(a_{11}(\xi) + a_{21}(\xi) + a_{12}(\xi) + a_{22}(\xi)\right),$$
$$a_-^+(\xi) = \tfrac{1}{2}\left(a_{11}(\xi) + a_{21}(\xi) - a_{12}(\xi) - a_{22}(\xi)\right),$$
$$a_+^-(\xi) = \tfrac{1}{2}\left(-a_{11}(\xi) + a_{21}(\xi) - a_{12}(\xi) + a_{22}(\xi)\right),$$
$$a_-^-(\xi) = \tfrac{1}{2}\left(-a_{11}(\xi) + a_{21}(\xi) + a_{12}(\xi) - a_{22}(\xi)\right),$$

are continuously differentiable. Because the equations (3.1) and (3.2) are valid on different characteristic lines the coupling between them is more complicated than that usually encountered in the theory of ordinary differential equations.

The construction of the control f of Theorem 1 was first described in [15]. Assuming for the moment that u_0, v_0, u_1, v_1 are functions in $C^1[0, \ell]$, we direct the reader's attention to Fig. 1. The basic domain $D : 0 \leqslant \xi \leqslant \ell$,

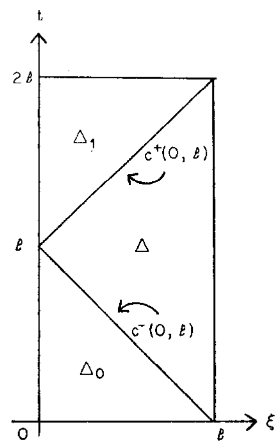

FIGURE 1

$0 \leqslant t \leqslant 2\ell$ is divided by the characteristics $C^+(0, \ell)$ and $C^-(0, \ell)$ into three closed triangular domains which we have denoted by Δ_0, Δ_1 and Δ as indicated in the diagram. The differential equations (3.1) and (3.2) can be used to prove certain existence and uniqueness theorems. In particular, the initial data u_0, v_0 together with the boundary condition (1.7) satisfying $\alpha_0/\beta_0 \neq 1$ [cf. (1.9)] uniquely determine a solution $(u(\xi, t), v(\xi, t))$ of (1.10) which lies in $C^1(\Delta_0)$. In the same way the terminal data u_1, v_1 given at $t = 2\ell$ together with (1.7) and the restriction $\alpha_0/\beta_0 \neq -1$ determine $(u(\xi, t), v(\xi, t))$ as a solution of (1.10) in $C^1(\Delta_1)$. If $u_1(\xi) \equiv v_1(\xi) \equiv 0$ we just set $u(\xi, t) \equiv v(\xi, t) \equiv 0$ in Δ_1 and the condition $\alpha_0/\beta_0 \neq -1$ can be dispensed with.

The next step is the extension of the solution into Δ. The portions of the solution already constructed in Δ_0 and Δ_1 determine θ^+ and θ^- on $C^+(0, \ell)$ and $C^-(0, \ell)$. The problem of constructing a solution of (1.10) in Δ agreeing with these data on $C^+(0, \ell)$ and $C^-(0, \ell)$ is the Goursat, or characteristic initial-value, problem. Again the equations (3.1) and (3.2) can be used to establish the existence and uniqueness of a solution $(u(\xi, t), v(\xi, t))$ of (1.10) in Δ. See [6, 4] for details. Then $u(\ell, t)$ and $v(\ell, t)$ determine the control $f(t)$ via (1.8). Standard uniqueness results show that $(u(\xi, t), v(\xi, t))$ as now constructed in D is the unique generalized solution of (1.10), (1.7), (1.8), (2.1) in D and it clearly has the desired terminal values at $t = 2\ell$. We say "generalized" solution because the limiting values at $(0, \ell)$ of $(u(\xi, t), v(\xi, t))$ as defined in Δ_0 and Δ_1 may not agree, resulting in θ^+ and θ^- having jump discontinuities across $C^+(0, \ell)$ and $C^-(0, \ell)$, respectively. The solution is of class C^1 in each of Δ_0, Δ_1 and Δ separately.

It remains only to prove the inequalities (2.3), (2.4). This is done with the aid of two lemmas.

LEMMA 2.1. *Let (u, v) be a solution of* (1.10), (1.7) *lying in $C^1(\Delta)$. Then there are positive constants P_1, P_2 such that*

$$\begin{aligned} P_1 \int_0^{2\ell} [| u(\ell, t)|^2 + | v(\ell, t)|^2]\, dt &\leqslant \int_0^{\sqrt{2}\ell} \left| \theta^+\left(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}}\right)\right|^2 d\sigma + \int_0^{\sqrt{2}\ell} \left| \theta^-\left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right)\right|^2 d\tau \\ &\leqslant P_2 \int_0^{2\ell} [| u(\ell, t)|^2 + | v(\ell, t)|^2]\, dt. \end{aligned} \tag{3.3}$$

Proof. We begin with the first inequality of (3.3). Let $\Delta(\zeta)$ denote that portion of Δ lying to the left of the line $\xi = \zeta$, $0 \leqslant \zeta \leqslant \ell$. Since u and v satisfy (1.10) in $\Delta(\zeta)$ they also satisfy

$$\frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} a_{21}(\xi) & a_{22}(\xi) \\ a_{11}(\xi) & a_{12}(\xi) \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = 0.$$

Therefore, with

$$A = \begin{pmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{pmatrix},$$

$$\begin{aligned} 0 = \iint_{\Delta(\zeta)} \Big[\Big(\begin{pmatrix} u \\ v \end{pmatrix}, \frac{\partial}{\partial \xi} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} u \\ v \end{pmatrix} \Big) \\ + \Big(\frac{\partial}{\partial \xi} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} - A \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \Big) \Big] \, d\xi \, dt \qquad (3.4) \\ = \iint_{\Delta(\zeta)} \Big\{ \frac{\partial}{\partial \xi} [| u |^2 + | v |^2] - \frac{\partial}{\partial t} [u\bar{v} + v\bar{u}] - \Big(\begin{pmatrix} u \\ v \end{pmatrix}, (A + A^*) \begin{pmatrix} u \\ v \end{pmatrix} \Big) \Big\} \, d\xi \, dt. \end{aligned}$$

Using the divergence theorem,

$$\begin{aligned} \iint_{\Delta(\zeta)} \Big\{ \frac{\partial}{\partial \xi} [| u |^2 + | v |^2] - \frac{\partial}{\partial t} [u\bar{v} + v\bar{u}] \Big\} \, d\xi \, dt \\ = - \int_0^{\sqrt{2}\zeta} \Big| \theta^+ \Big(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}} \Big) \Big|^2 d\sigma - \int_0^{\sqrt{2}\zeta} \Big| \theta^- \Big(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}} \Big) \Big|^2 d\tau \\ + \int_{\ell-\zeta}^{\ell+\zeta} [| u(\zeta, t)|^2 + | v(\zeta, t)|^2] \, dt. \qquad (3.5) \end{aligned}$$

Substituting (3.5) into (3.4), setting

$$E(\zeta) = \int_{\ell-\zeta}^{\ell+\zeta} [| u(\zeta, t)|^2 + | v(\zeta, t)|^2] \, dt$$

and differentiating with respect to ζ, we obtain

$$\begin{aligned} E'(\zeta) = \int_{\ell-\zeta}^{\ell+\zeta} \Big(\begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix}, (A(\zeta) + A(\zeta)^*) \begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix} \Big) \, dt \\ + \sqrt{2} \, | \theta^+(\zeta, \ell + \zeta)|^2 + \sqrt{2} \, | \theta^-(\zeta, \ell - \zeta)|^2. \end{aligned} \qquad (3.6)$$

Since the a_{ij} are in $C^1[0, \ell]$ there is a positive number M_0 such that

$$\Big| \int_{\ell-\zeta}^{\ell+\zeta} \Big(\begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix}, (A(\zeta) + A(\zeta)^*) \begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix} \Big) \, dt \Big| \leqslant M_0 E(\zeta)$$

for $0 \leqslant \zeta \leqslant \ell$. Thus, since $E(0) = 0$

$$E(\zeta) \leqslant \sqrt{2} \int_0^{\zeta} e^{M_0(\zeta - \xi)} [| \theta^+(\xi, \ell + \xi)|^2 + | \theta^-(\xi, \ell - \xi)|^2] \, d\xi$$

and, setting $\zeta = \ell$ we have the first inequality in (3.3) with

$$P_1 = e^{-M_0 \ell}.$$

To get the second inequality in (3.3) we note that (3.6) implies

$$-E'(\zeta) \leqslant -\int_{\ell-\zeta}^{\ell+\zeta} \left(\begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix}, (A(\zeta) + A(\zeta)^*) \begin{pmatrix} u(\zeta, t) \\ v(\zeta, t) \end{pmatrix} \right) dt \leqslant M_0 E(\zeta)$$

so that

$$E(\zeta) \leqslant e^{M_0(\ell-\zeta)} E(\ell).$$

Then from (3.4), (3.5) we have

$$\int_0^{\sqrt{2}\zeta} \left| \theta^+ \left(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}} \right) \right|^2 d\sigma + \int_0^{\sqrt{2}\zeta} \left| \theta^- \left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}} \right) \right|^2 d\tau$$

$$= E(\zeta) - \iint_{\Delta(\zeta)} \left(\begin{pmatrix} u \\ v \end{pmatrix}, (A + A^*) \begin{pmatrix} u \\ v \end{pmatrix} \right) d\xi \, dt$$

$$\leqslant e^{M_0(\ell-\zeta)} E(\ell) + \int_0^{\zeta} M_0 E(\xi) \, d\xi$$

$$\leqslant \left[e^{M_0(\ell-\zeta)} + M_0 \int_0^{\zeta} e^{M_0(\ell-\xi)} \, d\xi \right] E(\ell).$$

Setting $\zeta = \ell$ we have the second inequality in (3.3) with

$$P_2 = e^{M_0 \ell}.$$

Thus the proof of Lemma 2.1 is complete.

Actually, Lemma 2.1 is a rather standard estimate for hyperbolic equations and can be found, in some form, in good texts. The next lemma is no harder to prove but somewhat harder to find in the literature.

LEMMA 2.2. *Let* (u, v) *be a solution of* (1.10), (1.8) *in* $C^1(\Delta)$. *Then there are positive constants* P_3 *and* P_4 *such that*

$$\int_0^{\sqrt{2}\ell} \left| \theta^+ \left(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}} \right) \right|^2 d\sigma$$

$$\leqslant P_3 \int_0^{\sqrt{2}\ell} \left| \theta^- \left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}} \right) \right|^2 d\tau + P_4 \int_0^{2\ell} |f(t)|^2 \, dt.$$

Proof. In order to do this it will be convenient to employ a different representation of the differential equations (3.1), (3.2), which are given there in parametric form. If we introduce coordinates

$$\eta = \frac{t + \xi - \ell}{\sqrt{2}}, \qquad \zeta = \frac{t - \xi + \ell}{\sqrt{2}}$$

the characteristics of the partial differential system (1.10) become lines parallel to the η and ζ axes. If we put

$$\psi^+(\eta, \zeta) = \theta^+ \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell, \frac{1}{\sqrt{2}}(\eta + \zeta)\right),$$

$$\psi^-(\eta, \zeta) = \theta^- \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell, \frac{1}{\sqrt{2}}(\eta + \zeta)\right),$$

the equations (3.1), (3.2) are equivalent to

$$\begin{aligned} \frac{\partial\psi^+}{\partial\zeta}(\eta, \zeta) &+ a_+^+ \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell\right) \psi^+(\eta, \zeta) \\ &+ a_-^+ \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell\right) \psi^-(\eta, \zeta) = 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} \frac{\partial\psi^-}{\partial\eta}(\eta, \zeta) &+ a_+^- \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell\right) \psi^+(\eta, \zeta) \\ &+ a_-^- \left(\frac{1}{\sqrt{2}}(\eta - \zeta) + \ell\right) \psi^-(\eta, \zeta) = 0. \end{aligned} \tag{3.8}$$

The domain Δ now becomes the region $0 \leqslant \eta \leqslant \zeta$, $0 \leqslant \zeta \leqslant \ell$, and we are asked to show that

$$\int_0^\ell |\psi^+(\eta, \ell)|^2 \, d\eta \leqslant P_3 \int_0^\ell |\psi^-(0, \zeta)|^2 \, d\zeta + \sqrt{2}\, P_4 \int_0^\ell |f(2\zeta)|^2 \, d\zeta. \tag{3.9}$$

The boundary condition (1.8) now becomes

$$\begin{aligned} \psi^+(\zeta, \zeta) &= \left(\frac{\beta_1 - \alpha_1}{\alpha_1 + \beta_1}\right) \psi^-(\zeta, \zeta) + \frac{2}{\alpha_1 + \beta_1} f(2\zeta) \\ &\equiv c_1\psi^-(\zeta, \zeta) + c_2 f(2\zeta). \end{aligned} \tag{3.10}$$

Let us set

$$F(\zeta) = \int_0^\zeta |\psi^+(\eta, \zeta)|^2 \, d\eta$$

and compute

$$F'(\zeta) = \int_0^\zeta \left(\frac{\partial\psi^+(\eta, \zeta)}{\partial\zeta} \overline{\psi^+(\eta, \zeta)} + \psi^+(\eta, \zeta) \overline{\frac{\partial\psi^+(\eta, \zeta)}{\partial\zeta}}\right) d\eta + |\psi^+(\zeta, \zeta)|^2.$$

Using (3.7) and (3.10) we have

$$F'(\zeta) = \int_0^\zeta [-2a_+^+ |\psi^+|^2 - a_-^+(\psi^-\bar{\psi}^+ + \psi^+\bar{\psi}^-)] \, d\eta + |c_1\psi^-(\zeta, \zeta + c_2 f(2\zeta)|^2.$$

Letting M be a common bound for the absolute values of all of the coefficients $a_{+(-)}^{+(-)}(\xi)$ in $0 \leqslant \xi \leqslant \ell$ we have

$$\begin{aligned} | F'(\zeta) | \leqslant 3MF(\zeta) + M \int_0^\zeta | \psi^-(\eta, \zeta) |^2 \, d\eta + 2 \, | c_1 |^2 \, | \psi^-(\zeta, \zeta) |^2 \\ + 2 \, | c_2 |^2 \, | f(2\zeta) |^2. \end{aligned} \tag{3.11}$$

Now we make use of (3.8) to obtain the estimate

$$| \psi^-(\eta, \zeta) | \leqslant e^{M\eta} \, | \psi^-(0, \zeta) | + M \int_0^\eta e^{M(\eta - s)} \, | \psi^+(s, \zeta) | \, ds$$

from which it follows that

$$\begin{aligned} | \psi^-(\eta, \zeta) |^2 &\leqslant 2e^{2M\eta} \, | \psi^-(0, \zeta) |^2 + 2M^2 \left(\int_0^\eta e^{2M(\eta - s)} \, ds \right) \left(\int_0^\eta | \psi^+(s, \zeta) |^2 \, ds \right) \\ &\leqslant 2Me^{2M\ell} \, | \psi^-(0, \zeta) |^2 + M(e^{2M\ell} - 1) \, F(\zeta) \\ &\equiv M_1 \, | \psi^-(0, \zeta) |^2 + M_2 F(\zeta) \end{aligned}$$

uniformly for $0 \leqslant \eta \leqslant \zeta$, $0 \leqslant \zeta \leqslant \ell$. Then going back to (3.11) and substituting,

$$\begin{aligned} | F'(\zeta) | &\leqslant 3MF(\zeta) + M \int_0^\zeta [M_1 \, | \psi^-(0, \zeta) |^2 + M_2 F(\zeta)] \, d\eta \\ &\qquad + 2 \, | c_1 |^2 \, [M_1 \, | \psi^-(0, \zeta) |^2 + M_0 F(\zeta)] + 2 \, | c_2 |^2 \, | f(2\zeta) |^2 \\ &\leqslant [3M + MM_2 \ell + 2 \, | c_1 |^2 \, M_2] \, F(\zeta) \\ &\qquad + [MM_1 \ell + 2 \, | c_1 |^2 \, M_1] \, | \psi^-(0, \zeta) |^2 + 2 \, | c_2 |^2 \, | f(2\zeta) |^2 \\ &\equiv M_3 F(\zeta) + M_4 \, | \psi^-(0, \zeta) |^2 + M_5 \, | f(2\zeta) |^2 \end{aligned}$$

uniformly for $0 \leqslant \zeta \leqslant \ell$. This implies, since $F(0) = 0$,

$$F(\ell) \leqslant \int_0^\ell e^{M_3(\ell - \zeta)} [M_4 \, | \psi^-(0, \zeta) |^2 + M_5 \, | f(2\zeta) |^2] \, d\zeta$$

which proves Lemma 2.2 with

$$P_3 = M_4 e^{M_3 \ell}, \qquad \sqrt{2} \, P_4 = M_5 e^{M_3 \ell}.$$

We can proceed now to complete the proof of Theorem 1. As shown in Fig. 2, we divide the basic rectangle into five triangular subregions. In Δ_{01} we

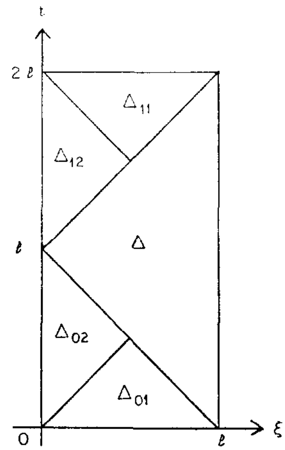

FIGURE 2

use, essentially, Lemma 2.1, but with the roles of t and ξ interchanged, to show that

$$\int_0^{\ell/\sqrt{2}} \left| \theta^+ \left(\frac{\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right) \right|^2 d\sigma + \int_{\ell/\sqrt{2}}^{\sqrt{2}\ell} \left| \theta^- \left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 d\tau$$

$$\leqslant P_5 \int_0^{\ell} (| u(\xi, 0)|^2 + | v(\xi, 0)|^2)\, d\xi.$$

Then we apply, essentially, Lemma 2.2 to the region Δ_{02} (instead of Δ) with (u, v) satisfying $\alpha_0 u(0, t) + \beta_0 v(0, t) = 0$ in place of (1.8) (here f is zero) to show that

$$\int_0^{\ell/\sqrt{2}} \left| \theta^- \left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 d\tau \leqslant P_6 \int_0^{\ell/\sqrt{2}} \left| \theta^+ \left(\frac{\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right) \right|^2 d\sigma.$$

Therefore,

$$\int_0^{\sqrt{2}\ell} \left| \theta^- \left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 d\tau \leqslant P_7 \int_0^{\ell} (| u(\xi, 0)|^2 + | v(\xi, 0)|^2)\, d\xi.$$

Arguing similarly in Δ_{11} and Δ_{12},

$$\int_0^{\sqrt{2}\ell} \left| \theta^+ \left(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}}\right) \right|^2 d\sigma \leqslant P_8 \int_0^{\ell} (| u(\xi, 2\ell)|^2 + | v(\xi, 2\ell)|^2)\, d\xi.$$

We combine these two inequalities with precisely the first inequality of Lemma 2.1 to get inequality (2.3) of Theorem 1.

To get inequality (2.4) of Theorem 1 we note that $u_1 = v_1 = 0$ implies, together with the boundary condition (1.7), that

$$\theta^+\left(\frac{\sigma}{\sqrt{2}}, \ell + \frac{\sigma}{\sqrt{2}}\right) = 0 \qquad \text{a.e.,} \qquad \sigma \in [0, \sqrt{2}\ell].$$

Then essentially the same argument as used in Lemma 2.2 shows that

$$\int_0^{\sqrt{2}\ell} \left| \theta^-\left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 d\tau \leqslant \tilde{P}_4 \int_0^{2\ell} |f(t)|^2 \, dt.$$

A similar argument using the boundary condition $\alpha_0 u(0, t) + \beta_0 v(0, t) = 0$ in Δ_{02} shows

$$\int_0^{\ell/\sqrt{2}} \left| \theta^+\left(\frac{\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right) \right|^2 d\sigma \leqslant \tilde{P}_3 \int_0^{\ell/\sqrt{2}} \left| \theta^-\left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 d\tau,$$

so that

$$\int_0^{\ell/\sqrt{2}} \left| \theta^+\left(\frac{\sigma}{\sqrt{2}}, \frac{\sigma}{\sqrt{2}}\right) \right|^2 d\sigma + \int_{\ell/\sqrt{2}}^{\sqrt{2}\ell} \left| \theta^-\left(\frac{\tau}{\sqrt{2}}, \ell - \frac{\tau}{\sqrt{2}}\right) \right|^2 dt$$
$$\leqslant P_9 \int_0^{2\ell} |f(t)|^2 \, dt.$$

Then we use Lemma 2.1 again with the roles of t and ξ reversed to get (2.4).

In the above work we have assumed u_0, v_0, u_1, v_1 all in $C^1[0, \ell]$. To get the result for u_0, v_0, u_1, v_1 in $L^2[0, \ell]$ it is sufficient to consider sequences $\{u_{0k}\}$, $\{v_{0k}\}$, $\{u_{1k}\}$, $\{v_{1k}\}$ in $C^1[0, \ell]$ converging to u_0, v_0, u_1, v_1, respectively, in $L^2[0, \ell]$.

That (2.4) cannot be proved if we require only $\alpha_0/\beta_0 \neq 1$, $\alpha_1/\beta_1 \neq -1$ is easily illustrated by taking all $a_{ij} = 0$ in (1.10) and letting $\alpha_1 = \beta_1 = 1$. The solutions (u, v) of

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi}\begin{pmatrix} u \\ v \end{pmatrix} = 0,$$

$$\alpha_0 u(0, t) + \beta_0 v(0, t) = 0, \qquad u(\ell, t) + v(\ell, t) = 0$$

vanish at $t = 2\ell$ no matter what initial conditions are prescribed. But here $f(t) \equiv 0$ so (2.4) could not hold.

4. Proof of Theorem 3

We shall again assume that u_0, v_0 lie in $C^1[0, \ell]$ so that (u, v), the corresponding solution of (1.10), (1.7), (1.8) is likewise of class C^1 in each of the domains Δ_0, Δ_1, Δ as described in the proof of Theorem 1. We define domains

$$\Delta(\tau) = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, \ell + \tau - \xi \leqslant t \leqslant \ell + \tau + \xi\}, \qquad \tau \geqslant 0,$$

observing that $\Delta(0) = \Delta$. The domain $\Delta(\tau)$ is the domain of determinacy for data given along the line $\xi = \ell$, $\tau \leqslant t \leqslant 2\ell + \tau$, as far as solutions of (2.10) [or (1.10)] are concerned.

We let (w_τ, z_τ) be the unique solution in $\Delta(\tau)$ of (2.10) correponding to constant real data

$$w_\tau(\ell, t) \equiv \hat{w}, \qquad z_\tau(\ell, t) \equiv \hat{z}, \qquad \tau \leqslant t \leqslant 2\ell + \tau. \tag{4.1}$$

Because these are constant data and the coefficients of the partial differential system (2.10) are functions of ξ only, w_τ and z_τ are functions of ξ only:

$$w_\tau(\xi, t) \equiv w(\xi), \qquad z_\tau(\xi, t) \equiv z(\xi).$$

Thus, from (2.10),

$$w'(\xi) = -a_{11}(\xi)\, w(\xi) - a_{21}(\xi)\, z(\xi),$$
$$z'(\xi) = -a_{12}(\xi)\, w(\xi) - a_{22}(\xi)\, z(\xi),$$

which implies that $w(\ell) = \hat{w}$ and $z(\ell) = \hat{z}$ can be chosen so that

$$\alpha_0 w(0) - \beta_0 z(0) = 1. \tag{4.2}$$

We now extend the real solution (w_τ, z_τ) from $\Delta(\tau)$ into the rest of

$$D(\tau) = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, \tau \leqslant t \leqslant 2\ell + \tau\}$$

by solving two characteristic-boundary-value problems [6] for (2.10) in the domains

$$\Delta_0(\tau) = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, \tau \leqslant t \leqslant \tau + \ell - \xi\},$$
$$\Delta_1(\tau) = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, \tau + \ell + \xi \leqslant t \leqslant 2\ell + \tau\},$$

corresponding to the boundary condition

$$\alpha_0 w_\tau(0, t) - \beta_0 z_\tau(0, t) \equiv 0 \tag{4.3}$$

and the values of

$$\hat{\theta}^+ = w + z, \qquad \hat{\theta}^- = w - z \tag{4.4}$$

already provided on $C^+(0, \tau + \ell)$, $C^-(0, \tau + \ell)$, respectively, by the solution (w_τ, z_τ) already constructed in $\Delta(\tau)$. Because the system (2.10) has C^1 coefficients, the complete solution (w_τ, z_τ), now constructed in all of $D(\tau)$, is of class C^1 in $\Delta(\tau)$, $\Delta_0(\tau)$ and $\Delta_1(\tau)$ individually, but $\hat{\theta}^+$ and $\hat{\theta}^-$ will suffer discontinuities across $C^-(0, \tau + \ell)$, $C^+(0, \tau + \ell)$, respectively, because (4.1) does not agree with the boundary condition (4.2) which we imposed on (w_τ, z_τ) when we extended the solution into $\Delta_0(\tau)$ and $\Delta_1(\tau)$. These discontinuities are essential to our proof.

Because we have related the terminal state to the inital state by means of the constant γ, we may set

$$u(\xi, t + 2k\ell) = \gamma^k u(\xi, t), \qquad v(\xi, t + 2k\ell) = \gamma^k v(\xi, t) \tag{4.5}$$

and obtain a solution of (1.10), (1.7) for $0 \leqslant \xi \leqslant \ell$, $0 \leqslant t < \infty$ ($-\infty < t < \infty$ if $\gamma \neq 0$). Of course the control function f appearing in the boundary condition (1.8) is extended similarly:

$$f(t + 2k\ell) = \gamma^k f(t).$$

Now consider Fig. 3 where we have shown $D(\tau)$, $\Delta(\tau)$, $\Delta_0(\tau)$ and $\Delta_1(\tau)$ for $\tau > 0$ as subregions of $0 \leqslant \xi \leqslant \ell$, $0 \leqslant t < \infty$. The discontinuities of (w_τ, z_τ) lie along $C^+(0, \tau + \ell)$, $C^-(0, \tau + \ell)$, whereas those of (u, v) lie along $C^+(0, (2k - 1)\ell)$, $C^-(0, (2k - 1)\ell)$. In the polygonal regions bounded by these characteristics and the lines $t = 0$, $\xi = 0$ and $\xi = \ell$ both solutions are of class C^1.

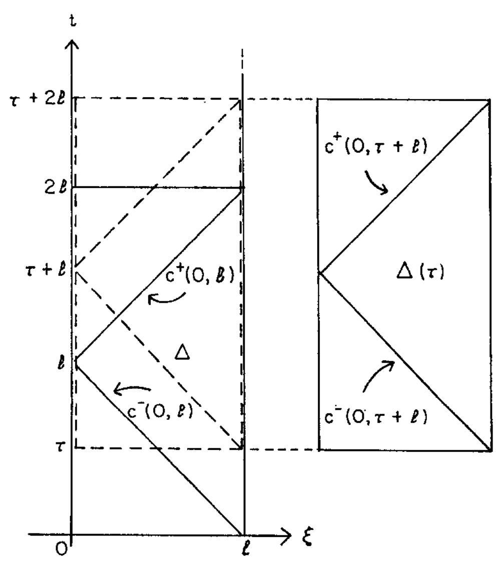

FIGURE 3

We proceed now as in (2.11) with (w, z) replaced by (w_τ, z_τ) and the region D replaced by $D(\tau)$. [Strictly speaking, the integration should be done individually over each of the polygonal regions which make up $D(\tau)$ (see Fig. 3) followed by cancellation of boundary terms. We leave this detail to the reader.]

$$0 = \iint_{D(\tau)} \left\{ \left(\begin{pmatrix} u \\ v \end{pmatrix}, \left[\frac{\partial}{\partial t} \begin{pmatrix} w_\tau \\ z_\tau \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} w_\tau \\ z_\tau \end{pmatrix} - \begin{pmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{pmatrix} \begin{pmatrix} w_\tau \\ z_\tau \end{pmatrix} \right] \right) \right.$$
$$\left. + \left(\left[\frac{\partial}{\partial t} \begin{pmatrix} u \\ v \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \frac{\partial}{\partial \xi} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right], \begin{pmatrix} w_\tau \\ z_\tau \end{pmatrix} \right) \right\} d\xi \, dt \tag{4.6}$$
$$= \iint_{D(\tau)} \left\{ \frac{\partial}{\partial t} (u w_\tau + v z_\tau) - \frac{\partial}{\partial \xi} (u z_\tau + v w_\tau) \right\} d\xi \, dt.$$

Using the boundary conditions (1.7), (4.3) and (4.1) together with the fact that

$$u(\xi, \tau + 2\ell) = \gamma u(\xi, \tau), \qquad v(\xi, \tau + 2\ell) = \gamma v(\xi, \tau),$$

(4.6) is seen to imply

$$\begin{aligned} 0 = \int_0^\ell &\{[\gamma w_\tau(\xi, \tau + 2\ell) - w_\tau(\xi, \tau)] \, u(\xi, \tau) \\ &+ [\gamma z_\tau(\xi, \tau + 2\ell) - z_\tau(\xi, \tau)] \, v(\xi, \tau)\} \, d\xi \\ &- \int_\tau^{\tau + 2\ell} [\hat{z} u(\ell, t) + \hat{w} v(\ell, t)] \, dt. \end{aligned} \tag{4.7}$$

Because the initial data for a solution (w_{τ_1}, z_{τ_1}) are the same as those for (w_{τ_2}, z_{τ_2}) but given on $[\tau_1, \tau_1 + 2\ell]$ instead of on $[\tau_2, \tau_2 + 2\ell]$, it is clear that

$$w_\tau(\xi, \tau + 2\ell) \equiv \tilde{w}_1(\xi), \qquad w_\tau(\xi, \tau) \equiv \tilde{w}_0(\xi),$$
$$z_\tau(\xi, \tau + 2\ell) \equiv \tilde{z}_1(\xi), \qquad z_\tau(\xi, \tau) \equiv \tilde{z}_0(\xi),$$

are C^1 functions of ξ alone and do not depend upon τ.

We now differentiate (4.7) with respect to τ. In doing so we must take account of the discontinuities of u and v. In our work below we assume $0 < \tau < \ell$. Other cases are handled in the same way. We obtain, noting (4.5),

$$\begin{aligned} &(\gamma - 1) \, \hat{z} u(\ell, \tau) + (\gamma - 1) \, \hat{w} v(\ell, \tau) \\ &\quad = \int_0^\ell \left\{ [\gamma \tilde{w}_1(\xi) - \tilde{w}_0(\xi)] \frac{\partial u}{\partial t}(\xi, \tau) + [\gamma \tilde{z}_1(\xi) - \tilde{z}_0(\xi)] \frac{\partial v}{\partial t}(\xi, \tau) \right\} d\xi \\ &\quad + [\gamma \tilde{w}_1(\ell - \tau) - \tilde{w}_0(\ell - \tau)] \, [u((\ell - \tau) +, \tau) - u((\ell - \tau) -, \tau)] \\ &\quad + [\gamma \tilde{z}_1(\ell - \tau) - \tilde{z}_0(\ell - \tau)] \, [v((\ell - \tau) +, \tau) - v((\ell - \tau) -, \tau)]. \end{aligned} \tag{4.8}$$

Using (1.10) and then integrating by parts, the integral in (4.8) becomes

$$\int_0^{\ell} \Big\{[\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)] \left(\frac{\partial v}{\partial \xi}(\xi, \tau) - a_{11}(\xi)\, u(\xi, \tau) - a_{12}(\xi)\, v(\xi, \tau)\right)$$

$$+ [\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)] \left(\frac{\partial u}{\partial \xi}(\xi, \tau) - a_{21}(\xi)\, u(\xi, \tau) - a_{22}(\xi)\, v(\xi, \tau)\right)\Big\}\, d\xi$$

$$\begin{aligned}
= &- \int_0^{\ell} \{[a_{11}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{21}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\
&+ \gamma\tilde{z}_1{}'(\xi) - \tilde{z}_0{}'(\xi)]\, u(\xi, \tau) \\
&+ [a_{12}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{22}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\
&+ \gamma\tilde{w}_1{}'(\xi) - \tilde{w}_0{}'(\xi)]\, v(\xi, \tau)\}\, d\xi \\
&- [\gamma\tilde{w}_1(\ell - \tau) - \tilde{w}_0(\ell - \tau)]\, [v((\ell - \tau)+, \tau) - v((\ell - \tau)-, \tau)] \\
&- [\gamma\tilde{z}_1(\ell - \tau) - \tilde{z}_0(\ell - \tau)]\, [u((\ell - \tau)+, \tau) - u((\ell - \tau)-, \tau)] \\
&+ [\gamma\tilde{w}_1(\ell) - \tilde{w}_0(\ell)]\, v(\ell, \tau) + [\gamma\tilde{z}_1(\ell) - \tilde{z}_0(\ell)]\, u(\ell, \tau).
\end{aligned} \tag{4.9}$$

In (4.9) we have also used the boundary conditions satisfied by u, v, $\tilde{w}_0$, $\tilde{w}_1$, $\tilde{z}_0$, $\tilde{z}_1$ at $\xi = 0$. Taking account of the fact that $\hat{\theta}^- = u - v$ is continuous across $c^-(0, \ell)$, the substitution of (4.9) into (4.8) yields

$$\begin{aligned}
&(\gamma - 1)\, \hat{z} u(\ell, \tau) + (\gamma - 1)\, \hat{w} v(\ell, \tau) \\
&\quad = - \int_0^{\ell} \{[a_{11}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{21}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\
&\quad\quad + \gamma\tilde{z}_1{}'(\xi) - \tilde{z}_0{}'(\xi)]\, u(\xi, t) \\
&\quad\quad + [a_{12}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{22}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\
&\quad\quad + \gamma\tilde{w}_1{}'(\xi) - \tilde{w}_0{}'(\xi)]\, v(\xi, t)\}\, d\xi \\
&\quad\quad + [\gamma\tilde{w}_1(\ell) - \tilde{w}_0(\ell)]\, v(\ell, \tau) + [\gamma\tilde{z}_1(\ell) - \tilde{z}_0(\ell)]\, u(\ell, \tau).
\end{aligned} \tag{4.10}$$

Consider now the quantities

$$(\gamma - 1)\, \hat{z} - [\gamma\tilde{z}_1(\ell) - \tilde{z}_0(\ell)] \equiv \alpha_2\,,$$
$$(\gamma - 1)\, \hat{w} - [\gamma\tilde{w}_1(\ell) - \tilde{w}_0(\ell)] \equiv \beta_2\,.$$

These can be written in another way, namely,

$$\gamma(\hat{z} - z_\tau(\ell -, \tau + 2\ell)) - (\hat{z} - z_\tau(\ell -, \tau)) = \alpha_2\,,$$
$$\gamma(\hat{w} - w_\tau(\ell -, \tau + 2\ell)) - (\hat{w} - w_\tau(\ell -, \tau)) = \beta_2\,.$$

If (w_τ, z_τ) were a continuous solution of (2.10) throughout $D(\tau)$ these quantities would be zero. However, because of the discontinuities of (w_τ, z_τ) propagating along $c^+(0, \tau + \ell)$, $c^-(0, \tau + \ell)$, this is not so. The quantities $\hat{\theta}^+$ and $\hat{\theta}^-$ satisfy coupled linear first order differential equations similar to the equations (3.1), (3.2) satisfied by θ^+ and θ^- along characteristics c^- and c^+, respectively. Using the continuity of $\hat{\theta}^+$ and $\hat{\theta}^-$ across c^+ and c^-, respectively, one can show that the jump discontinuities exhibited by $\hat{\theta}^+$ and $\hat{\theta}^-$ across $c^-(0, \tau + \ell)$ and $c^+(0, \tau + \ell)$, respectively, i.e.,

$$\Delta\hat{\theta}^+(\xi, \tau + \ell - \xi) = \hat{\theta}^+(\xi +, \tau + \ell - \xi) - \hat{\theta}^+(\xi -, \tau + \ell - \xi), \qquad 0 < \xi < \ell,$$

$$\Delta\hat{\theta}^-(\xi, \tau + \ell - \xi) = \hat{\theta}^-(\xi +, \tau + \ell + \xi) - \hat{\theta}^-(\xi -, \tau + \ell + \xi), \qquad 0 < \xi < \ell$$

(and defined by continuity at $\xi = 0$ and $\xi = \ell$), satisfy uncoupled linear first order homogeneous differential equations

$$\frac{d}{d\xi}(\Delta\hat{\theta}^+(\xi, \tau + \ell - \xi)) = -\tfrac{1}{2}(a_{11}(\xi) + a_{12}(\xi) + a_{21}(\xi) + a_{22}(\xi))\,\Delta\hat{\theta}^+(\xi, \tau + \ell - \xi),$$

$$\frac{d}{d\xi}(\Delta\hat{\theta}^-(\xi, \tau + \ell + \xi)) = -\tfrac{1}{2}(-a_{11}(\xi) + a_{12}(\xi) + a_{21}(\xi) - a_{22}(\xi)\,\Delta\hat{\theta}^+(\xi, \tau + \ell + \xi).$$

Thus, with σ^+ and σ^- defined by (2.18) and (2.19), respectively,

$$\alpha_2 + \beta_2 = -\Delta\hat{\theta}^+(\ell, \tau) = -\sigma^+\Delta\hat{\theta}^+(0, \tau + \ell), \tag{4.11}$$

$$\beta_2 - \alpha_2 = \gamma\Delta\hat{\theta}^-(\ell, \tau + 2\ell) = \gamma\sigma^-\Delta\hat{\theta}^-(0, \tau + \ell). \tag{4.12}$$

Using the boundary conditions (4.3) we can readily calculate

$$\Delta\hat{\theta}^+(0, \tau + \ell) = \frac{2(\alpha_0 w(0) - \beta_0 z(0))}{\alpha_0 - \beta_0},$$

$$\Delta\hat{\theta}^-(0, \tau + \ell) = \frac{2(\alpha_0 w(0) - \beta_0 z(0))}{\alpha_0 + \beta_0}.$$

Hence, from (4.11), (4.12), (4.2),

$$\alpha_2 + \beta_2 = \frac{2\sigma^+}{\beta_0 - \alpha_0}, \qquad \beta_2 - \alpha_2 = \frac{2\gamma\sigma^-}{\alpha_0 + \beta_0},$$

so that

$$\alpha_2 = \left(\frac{\sigma^+}{\beta_0 - \alpha_0} - \frac{\gamma\sigma^-}{\alpha_0 + \beta_0}\right), \tag{4.13}$$

$$\beta_2 = \left(\frac{\sigma^+}{\beta_0 - \alpha_0} + \frac{\gamma\sigma^-}{\alpha_0 + \beta_0}\right). \tag{4.14}$$

If we put

$$\begin{aligned} h_1(\xi) &= -\,[a_{11}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{21}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\ &\quad + \gamma\tilde{z}_1{}'(\xi) - \tilde{z}_0{}'(\xi)], \\ h_2(\xi) &= -\,[a_{12}(\xi)\,(\gamma\tilde{w}_1(\xi) - \tilde{w}_0(\xi)) + a_{22}(\xi)\,(\gamma\tilde{z}_1(\xi) - \tilde{z}_0(\xi)) \\ &\quad + \gamma\tilde{w}_1{}'(\xi) - \tilde{w}_0{}'(\xi)], \end{aligned}$$

then (4.10) has the form

$$\alpha_2 u(\ell, \tau) + \beta_2 v(\ell, \tau) = \int_0^{\ell} [h_1(\xi)\, u(\xi, \tau) + h_2(\xi)\, v(\xi, \tau)]\, d\xi. \tag{4.15}$$

This takes care of the "only if" point of Theorem 3. To prove the "if" part we note that not both of the coefficients

$$\frac{\sigma^+}{\beta_0 - \alpha_0} - \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}, \qquad \frac{\sigma^+}{\beta_0 - \alpha_0} + \frac{\gamma\sigma^-}{\beta_0 + \alpha_0}$$

are zero. For definiteness, assume the first is not zero. Then consider the control problem (1.10), (1.7), (2.1), (2.20) with control

$$u(\ell, t) \equiv \tilde{f}(t).$$

By Theorem 1 there is a solution to this control problem and, by the "only if" part of the present theorem, $u(\ell, t)$ must satisfy (4.15).

Next we consider the mixed initial-boundary-value problem (1.10), (1.7), (4.15), (2.1). Existence and uniqueness are as easy to prove here as in the case of a standard boundary condition of the form $\alpha_1 u(\ell, t) + \beta_1 v(\ell, t) = 0$. Therefore the solution of this initial-boundary-value problem coincides with the solution of the control problem posed in the preceding paragraph. Hence (2.20) must be satisfied and we have taken care of the "if" part of Theorem 3.

When $\gamma = 0$ the rectangle $D(\tau)$ used in (4.6) can be replaced by the region

$$R(\tau) = \{(\xi, t) \mid 0 \leqslant \xi \leqslant \ell, \tau \leqslant t \leqslant \tau + \ell + \xi\}.$$

The analysis proceeds as above, provided we note that $\theta^+ = u + v$ vanishes along $t = \tau + \ell + \xi$ for $\tau \geqslant 0$. The solution (w_τ, z_τ) of (2.10) only needs to be constructed in $R(\tau)$ rather than in $D(\tau)$ in this case and the condition $\alpha_0/\beta_0 \neq 1$ is sufficient to carry out this construction.

Thus the proof of Theorem 3 is complete, provided we recognize that the inequalities of Theorem 1 imply that we can readily extend the results proved above for u_0, $v_0 \in C^1[0, \ell]$ to the general case u_0, $v_0 \in L^2[0, \ell]$.

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