Maximal Element Theorems of H-Majorized Correspondence and Existence of Equilibrium for Abstract Economies

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In this paper, two new existence theorems of maximal elements for H-majorized correspondences are established in a kind of nonparacompact H-spaces. As applications, the existence problems of equilibrium for abstract economies are studied. Our theorems improve and generalize some recent results in the literature greatly.

1. INTRODUCTION AND PRELIMINARIES

In recent years, many authors have proved the equilibrium existence theorems for abstract economies (or generalized games) by assuming that the strategy (choice) sets of agents are compact or paracompact topological vector spaces (e.g., see [3, 4, 6–8] and the references therein). In this paper, we first introduce the notions of H-majorized correspondences in H-spaces and establish new existence theorems of maximal elements for H-majorized correspondences in CH-spaces. Next, we shall prove some new existence theorems of equilibrium for abstract economies under the assumptions that the strategy set is a kind of nonparacompact CH-space, without any linear structure, and the preference correspondences are H-majorized correspondences. So, our theorems greatly improve and generalize the corresponding results of [3, 4, 6, 7, 9].

For the sake of convenience, we first give some concepts, notations, and terminologies.

Throughout the paper, all topological spaces are assumed to be Hausdorff. Let A be a subset of a topological space X. We shall denote by
the family of all subsets of $A$, by $\mathcal{H}(A)$ the family of all nonempty finite subsets of $A$, by $cI_X A$ the closure of $A$ in $X$ and by $\text{int}_X A$ the interior of $A$ in $X$. Also, $A$ is said to be compactly open in $X$ if for each nonempty compact subset $C$ of $X$, $A \cap C$ is open in $C$. Let $\{G_i\}_{i \in I}$ ($I$ is an index set) be a family of a subset of a topological space $X$; $\{G_i\}_{i \in I}$ is said to be transfer open if for each $i \in I$, $x \in G_i$ implies that there exists an $i' \in I$ such that $x \in \text{int}_X G_i$. Let $X$ and $Y$ be topological spaces and $T, S : X \rightarrow 2^Y$ be correspondences (or multivalued mapping), the graph of $T$, denoted by $\text{Gr}(T)$, is the set $\{(x, y) \in X \times Y : y \in T(x)\}$ and the correspondences $T \cap S, cI_T, \overline{T} : X \rightarrow 2^Y$ are defined by $(T \cap S)(x) := T(x) \cap S(x)$, $(cI_T)(x) := cI_y(T(x))$, and $\overline{T}(x) := \{y \in Y : (x, y) \in cI_X y \text{Gr}(T)\}$ for each $x \in X$, respectively. It is easy to see that $cI_T(x) \subset \overline{T}(x)$ for each $x \in X$.

The following notions were introduced by Bardaro and Ceppitelli [2]. A pair $(X, \{\Gamma_A\})$ is said to be an H-space if $X$ is a topological space and $\{\Gamma_A\}$ is a family of contractible subsets of $X$ indexed by $A \in \mathcal{H}(X)$ such that $\Gamma_A \subset \Gamma_{A'}$ whenever $A \subset A'$. A nonempty subset $D$ of H-space $(X, \{\Gamma_A\})$ is said to be H-convex if $\Gamma_A \subset D$ for each $A \in \mathcal{H}(D)$. For a nonempty subset $B$ of an H-space we define the H-convex hull of $B$, denoted by $\text{H-coB}$, as

$$\text{H-coB} := \bigcap \{D \subset X : D \text{ is H-convex and } D \supset B\}.$$ 

Let $(X, \{\Gamma_A\})$ be an H-space. A correspondence $T : X \rightarrow 2^X$ is said to be H-KKM mapping if $\Gamma_A \subset \bigcup_{x \in A} T(x)$, for each $A \in \mathcal{H}(X)$. Let $\{(X_i, \{\Gamma_{A_i}\})\}_{i \in I}$ be a family of H-spaces, where $I$ is a finite or an infinite index set. For each nonempty finite subset $A$ of $X := \prod_{i \in I} X_i$, we set $\Gamma_A := \prod_{i \in I} \Gamma_{A_i}$, where for each $i \in I, A_i = \pi_i(A)$ and $\pi_i : X \rightarrow X_i$ is the projection of $X$ onto $X_i$. By Lemma 1.1 of Tarafdar [8], the product space $(X, \{\Gamma_A\})$ is an H-space and the product of H-convex subsets in H-convex.

We now give the following concepts.

**Definition 1.1.** An H-space $(X, \{\Gamma_A\})$ is said to be a CH-space if for each $A \in \mathcal{H}(X)$ there exists a compact H-convex subset $D$ of $X$ containing $A$. For a nonempty finite subset $A$ of CH-space we define the compact H-convex hull of $A$, denoted by CH-coA, as

$$\text{CH-coA} := \bigcap \{D \subset X : D \text{ is compact H-convex and } D \supset A\}.$$ 

**Remark 1.1.** Clearly, any nonempty convex subset $X$ of topological vector space is a CH-space; any compact H-space is a CH-space and the product of any number of CH-spaces is a CH-space.
H-MAJORIZED CORRESPONDENCE

DEFINITION 1.2. Let \((X, \{\Gamma_A\})\) be an H-space, let \(Y\) be a nonempty set, let \(\theta: X \rightarrow Y\) be a single-valued map, and let \(\phi : X \rightarrow 2^Y\) be a correspondence. Then

(i) \(\phi\) is said to be an \(H_{\theta}\)-correspondence if 
\[
\Gamma_A \cap (\bigcap_{x \in A} \phi^{-1}(\theta(x))) = \emptyset
\]
for each \(A \in \mathfrak{H}(X)\) and if \(\phi\) has a compactly open lower section;

(ii) a correspondence \(\phi_x : X \rightarrow 2^Y\) is said to be an \(H_{\theta}\)-majorant of \(\phi\) at \(x \in X\) if \(\phi_x\) is an \(H_{\theta}\)-correspondence and there exists an open neighborhood \(N_x\) of \(x\) in \(X\) such that \(\phi(z) \subseteq \phi_x(z)\) for each \(z \in N_x\);

(iii) \(\phi\) is said to be \(H_{\theta}\)-majorized if for each \(x \in X\) with \(\phi(x) \neq \emptyset\), there exists an \(H_{\theta}\)-majorant of \(\phi\) at \(x\).

In this paper, we only deal with either the case (I) \(X = Y\) and \(\theta = I_X\), the identity map on \(X\), or the case (II) \(X = \prod_{j \in I} X_j\) and \(\theta = \pi_j: X \rightarrow X_j\) is the projection of \(X\) onto \(X_j\) and \(Y = X_j\). \((X_j, \{\Gamma_A\}) (j \in I)\) is an H-space. In both cases we write \(H\) in place of \(H_{\theta}\). It is easy to see that the notions of H-correspondence, \(H\)-majorant of \(\phi\) at \(x\), and \(H\)-majorized correspondence generalize the notions of L-correspondence, \(L\)-majorant of \(\phi\) at \(x\), and \(L\)-majorized correspondence (see \([4, 6]\)), respectively, from topological vector spaces to H-spaces. The following simple example shows that an H-correspondence may not be an L-correspondence.

Example. Let \(X = [0, +\infty)\) and \(\Gamma_A = \text{coA}\) for each \(A \in \mathfrak{H}(X)\); then \((X, \{\Gamma_A\})\) is an H-space. Define a correspondence \(T: X \rightarrow 2^X\) by
\[
T(x) := \begin{cases} 
[0, 2 - x], & \text{if } x \in [0, 2), \\
\{0\}, & \text{if } x \in [2, +\infty),
\end{cases}
\]
then
\[
T^{-1}(y) = \{x \in [0, 2) : y \in T(x)\} \cup \{x \in [2, +\infty) : y \in T(x)\}
\]
\[
= \begin{cases} 
[0, 2 - y], & \text{if } y \in [0, 2), \\
\emptyset, & \text{if } y \in [2, +\infty),
\end{cases}
\]
and then
\[
X \setminus T^{-1}(y) = \begin{cases} 
[2 - y, +\infty), & \text{if } y \in [0, 2), \\
X, & \text{if } y \in [2, +\infty),
\end{cases}
\]
for each \(y \in X\). Hence,
\[
\Gamma_A = \text{coA} \subset \bigcup_{y \in A} (X \setminus T^{-1}(y)) = X \setminus \bigcap_{y \in A} T^{-1}(y);
\]
i.e., \(\Gamma_A \cap (\bigcap_{y \in A} T^{-1}(y)) = \emptyset\) for each \(A \in \mathfrak{H}(X)\). But \(x \in \text{H-coT}(x)\) for each \(x \in [0, 1)\), and hence an H-correspondence is not an L-correspondence.
2. EXISTENCE OF MAXIMAL ELEMENTS

In this section, we shall establish existence theorems of maximal elements for H-correspondences and H-majorized correspondences.

**Theorem 2.1.** Let \((X, \{\Gamma_i\})\) be a CH-space, and let \(T : X \rightarrow 2^X\) be an H-correspondence. Suppose there exists a nonempty compact subset \(K\) of \(X\) and an \(M \in \mathfrak{M}(X)\) such that for each \(x \in X\), there is a \(y \in CH - co(M \cup \{x\})\) with \(y \in T(x)\). Then \(T\) has a maximal element in \(K\).

**Proof.** We define a correspondence \(G : X \rightarrow 2^K\) by

\[
G(x) := K \setminus T^{-1}(x), \quad \text{for each } x \in X.
\]

Then for each \(x \in X\), \(G(x) = K \setminus (K \cap T^{-1}(x))\) is closed in \(K\) since \(T\) has a compactly open lower section. Let us prove that the family \(\{G(x) : x \in X\}\) has a finite intersection property; i.e., for each \(A \in \mathfrak{M}(X)\), \(\bigcap_{x \in A} G(x) \neq \emptyset\).

For any \(A \in \mathfrak{M}(X)\), let \(D := CH - co(M \cup A)\), then \(D\) is a compact H-convex subset of \(X\). Now we define a correspondence \(G_0 : D \rightarrow 2^D\) by

\[
G_0(x) := D \setminus T^{-1}(x), \quad \text{for each } x \in D.
\]

Then for each \(x \in D\), \(G_0(x) = D \setminus (D \cap T^{-1}(x))\) is closed in \(D\).

Since \(D\) is H-convex and \(T\) is an H-correspondence, for each finite subset \(\{u_1, \ldots, u_n\}\) of \(D\) we have \(\Gamma_{\{u_1, \ldots, u_n\}} \subset D\) and

\[
\Gamma_{\{u_1, \ldots, u_n\}} \subset \bigcup_{i=1}^n (X \setminus T^{-1}(u_i)),
\]

and hence \(\Gamma_{\{u_1, \ldots, u_n\}} \subset \bigcap_{i=1}^n (D \setminus T^{-1}(u_i)) = \bigcup_{i=1}^n G_0(u_i)\); i.e., \(G_0\) is an H–KKM mapping. By the H–KKM theorem [2], \(\bigcap_{x \in D} G_0(x) = \bigcap_{x \in D} (D \setminus T^{-1}(x)) = \emptyset\). Taking \(y_0 \in \bigcap_{x \in D} (D \setminus T^{-1}(x))\), we claim that \(y_0 \in K\). Indeed, suppose that \(y_0 \in X\setminus K\) then there is a \(z_0 \in CH - co(M \cup \{y_0\})\) such that \(z_0 \in T(y_0)\). Note that \(M \subset D\), \(y_0 \in D\), and \(D\) is compact H-convex. We have \(z_0 \in D\) and hence \(y_0 \notin \bigcap_{x \in D} (D \setminus T^{-1}(x))\). It contradicts the choice of \(y_0\). Therefore,

\[
y_0 \in \bigcap_{x \in A} (K \setminus T^{-1}(x)) = \bigcap_{x \in A} G(x);
\]

i.e., a closed set family \(\{G(x) : x \in X\}\) has the finite intersection property. Hence \(\bigcap_{x \in X} G(x) \neq \emptyset\) since \(K\) is compact.

To summarize, we know that there exists a \(\tilde{y} \in K\) such that for all \(x \in X\), \(\tilde{y} \notin T^{-1}(x)\); i.e., \(T(\tilde{y}) = \emptyset\). This completes the proof.

The following result is a direct consequence of the above theorem.
Corollary 2.2. Let \((X, \{\Gamma_A\})\) be a CH-space, and let \(T : X \to 2^X\) be a correspondence with the following

(i) for each \(x \in X\), \(x \not\in H\text{-co}T(x)\);
(ii) \(T\) has a compactly open lower section,
(iii) there exists a nonempty compact subset \(K\) of \(X\) and a \(M \in \mathcal{R}(X)\) such that for each \(x \in X\setminus K\), \(\text{CH-co}(M \cup \{x\}) \cap T(x) \neq \emptyset\).

Then \(T\) has a maximal elements in \(K\).

Remark 2.1. If \(X\) is compact, then the assumptions in Theorem 2.1 are satisfied trivially. Hence Theorem 2.1 and Corollary 2.2 generalize Theorem 5.1 of Yannelis and Prabhakar [9] from the compact topological vector spaces to noncompact H-space and relax the conditions for correspondence \(T\).

We now give the existence theorem of maximal elements for H-majorized correspondence.

Theorem 2.3. Let \((X, \{\Gamma_A\})\) be a CH-space and let \(\phi : X \to 2^X\) be an H-majorized correspondence, such that

(i) there exists a paracompact subset \(E\) of \(X\) such that \(\{x \in X : \phi(x) \neq \emptyset\}\) \(\subset E\),
(ii) there exists a nonempty compact subset \(K\) of \(X\) and a \(M \in \mathcal{R}(X)\) such that for each \(x \in X\setminus K\), \(\text{CH-co}(M \cup \{x\}) \cap \phi(x) \neq \emptyset\).

Then \(\phi\) has maximal elements in \(K\).

Proof. We first prove that \(\phi\) has a maximal element in \(X\). If the conclusion is false, then for each \(x \in X\), \(\phi(x) \neq \emptyset\). Since \(\phi\) is an H-majorized correspondence, for each \(x \in X\) there exists a correspondence \(\phi_x : X \to 2^X\) and an open neighborhood \(N_x\) of \(x\) in \(X\) such that

1. for each \(z \in N_x\), \(\phi(z) \subset \phi_x(z)\);
2. for each \(A \in \mathcal{R}(X)\), \(\Gamma_A \cap (\bigcap_{x \in A} \phi_x^{-1}(x)) = \emptyset\);
3. the \(\phi_x\) has a compactly open lower section.

Since \(X = \{x \in X : \phi(x) \neq \emptyset\}\) = \(E\) is paracompact, by Theorem VIII 1.4 of Dugundji [5], the open covering \(\{N_x : x \in X\}\) of \(X\) has an open precise locally finite refinement \(\{\tilde{N}_x : x \in X\}\), and \(\text{cl}_X\tilde{N}_x \subset N_x\) for \(x \in X\) also since \(X\) is normal. For each \(x \in X\), define correspondence \(\tilde{\phi}_x : X \to 2^X\) by

\[
\tilde{\phi}_x(z) := \begin{cases} 
\phi_x(z), & \text{if } z \in \text{cl}_X\tilde{N}_x, \\
X, & \text{if } z \notin \text{cl}_X\tilde{N}_x.
\end{cases}
\]
Then for each \( y \in X \), we have

\[
\phi^{-1}_y(y) = \{ z \in cl_X \overline{N}_x : y \in \phi_y(z) \} \cup \{ z \in X \setminus cl_X \overline{N}_x : y \in \phi_y(z) \}
\]

\[
= \{ z \in cl_X \overline{N}_x : y \in \phi_y(z) \} \cup (X \setminus cl_X \overline{N}_x)
\]

\[
= \phi^{-1}_x(y) \cup (X \setminus cl_X \overline{N}_x),
\]

which shows that \( \phi_y \) has a compactly open lower section by (3).

Let \( \Phi : X \rightarrow 2^X \) be defined by

\[
\Phi(z) := \bigcap_{x \in X} \phi_y(z), \quad \text{for each } z \in X.
\]

We claim that \( \Phi \) is an H-correspondence and for each \( z \in X \), \( \phi(z) \subset \Phi(z) \). To this purpose, for each \( y \in X \) and for each nonempty compact subset \( C \) of \( X \), let \( t \in \Phi^{-1}(y) \cap C \) be arbitrarily fixed. Since \( \{ \overline{N}_x : x \in X \} \) is locally finite, there exists an open neighborhood \( V_t \) of \( t \) in \( X \) such that \( \{ x \in X : V_t \cap \overline{N}_x \neq \emptyset \} = \{ x_1, \ldots, x_n \} \) is a finite set. If \( x \notin \{ x_1, \ldots, x_n \} \), then \( \emptyset = V_t \cap \overline{N}_x = V_t \cap cl_X \overline{N}_x \) and hence \( \phi_y(z) = X \) for all \( z \in V_t \), implying that \( \Phi(z) = \bigcap_{x \in X} \phi_y(z) = \bigcap_{i=1}^n \phi_y(z) \) for all \( z \in V_t \). Therefore,

\[
\Phi^{-1}(y) = \{ z \in X : y \in \Phi(z) \} \supset \{ z \in V_t : y \in \Phi(z) \}
\]

\[
= \left\{ z \in V_t : y \in \bigcap_{i=1}^n \phi_y(z) \right\} = V_t \cap \bigcap_{i=1}^n (\phi_y^{-1}(y)).
\]

Let \( \widetilde{V}_t := (V_t \cap C) \cap \bigcap_{i=1}^n (\phi_y^{-1}(y)) \cap C \), then \( \widetilde{V}_t \) is an open neighborhood of \( t \) in \( C \) such that \( \widetilde{V}_t \subset \Phi^{-1}(y) \cap C \); i.e., \( \Phi \) has a compactly open lower section.

On the other hand, for each \( A \in \mathcal{H}(X) \), if \( t \in \bigcap_{x \in A} \Phi^{-1}(z) \), then \( A \subset \Phi(t) \). Since there exists an \( x_0 \in X \) such that \( t \in cl_X \overline{N}_{x_0} \), \( A \subset \Phi(t) \subset \phi_{x_0}(t) \); i.e., \( t \in \bigcap_{x \in A} \phi_y^{-1}(z) \) and hence \( t \notin \overline{\Gamma_A} \) by (2). Hence \( \overline{\Gamma_A} \cap (\bigcap_{x \in A} \Phi^{-1}(z)) = \emptyset \) for each \( A \in \mathcal{H}(X) \), which shows that \( \Phi \) is an H-correspondence.

For each \( z \in X \), if \( t \notin \Phi(z) \), then there exists an \( x_0 \in X \) such that \( t \notin \phi_{x_0}(z) \) and hence \( t \notin \phi_y(z) \) and \( z \in cl_X \overline{N}_{x_0} \subset N_{x_0} \). It follows that \( t \notin \phi(z) \) by (1). Hence \( \phi(z) \subset \Phi(z) \) for all \( z \in X \).

Since for each \( x \in X \setminus \mathcal{K} \) there exists a \( y \in \text{CH-co}(M \cup \{ x \}) \) with \( y \in \phi(x) \subset \Phi(x) \), by virtue of Theorem 2.1, there exists an \( x^* \in K \) such that \( \Phi(x^*) = \emptyset \) and hence \( \phi(x^*) = \emptyset \), which contradicts the assumption that \( \phi(x) \) is nonempty for all \( x \in X \). Therefore, there exists an \( x \in X \) such that \( \phi(x) = \emptyset \); by assumption (ii), \( x \) must be in \( K \). This completes the proof.
Remark 2.2. If \( X \) is compact, then the assumptions (i) and (ii) in Theorem 2.3 are satisfied trivially. Hence Theorem 2.3 generalizes Corollary 5.1 of Yannelis and Prabhakar [9] from the compact topological vector spaces to noncompact H-spaces and also improves and extends Theorem 1 of Ding and Tan [3] from the paracompact topological vector spaces to nonparacompact H-spaces.

3. EXISTENCE OF EQUILIBRIUM POINTS

In this section, we apply Theorem 2.3 to obtain the existence of equilibrium for an abstract economy with an H-majorized preference correspondence in a CH-space. Let \( I \) be a (finite or infinite) set of agents (players). An abstract economy (generalized game) \( \Gamma := (X_i; A_i, B_i; P_i)_{i \in I} \) is defined as a family of ordered quadruples \((X_i; A_i, B_i; P_i)\), where for each \( i \in I, X_i \) is a nonempty topological space (a choice set or strategy set), \( A_i, B_i : X := \Pi_{i \in I} X_i \rightarrow 2^{X_i} \) are constraint correspondences, and \( P_i : X \rightarrow 2^{X_i} \) is a preference correspondence. An equilibrium for \( \Gamma \) is a point \( \hat{x} \in X = \Pi_{i \in I} X_i \) such that for each \( i \in I, \hat{x}_i := \pi_i(\hat{x}) \in cl_{X_i} B_i(\hat{x}) \) and \( A_i(\hat{x}_i) \cap P_i(\hat{x}_i) = \emptyset \). A qualitative game is a family \( \Gamma := (X_i, P_i)_{i \in I} \) of ordered pairs \((X_i, P_i)\), where for each \( i \in I, X_i \) is a nonempty topological space (strategy set) and \( P_i : X = \Pi_{i \in I} X_i \rightarrow 2^{X_i} \) is a preference correspondence. A point \( \hat{x} \in X \) is said to be an equilibrium of the qualitative game \( \Gamma \) if \( P_i(\hat{x}) = \emptyset \) for all \( i \in I \).

**Theorem 3.1.** Let \( (X, \{\Gamma_i\}) \) be a CH-space. Let \( A, B : X \rightarrow 2^X \) be constraint correspondences and let \( P : X \rightarrow 2^X \) be a preference correspondence satisfying the following conditions:

(i) \( A \cap P \) is an H-majorized correspondence,

(ii) for each \( x \in X \), \( A(x) \) is nonempty and H-co\( A(x) \subseteq B(x) \),

(iii) \( A \) has a compactly open lower section,

(iv) for each \( x \in X \), \( B(x) = cl_X B(x) \),

(v) there exists a paracompact subset \( E \) of \( X \) such that \( \{x \in X : x \notin cl_X B(x)\} \subseteq E \) and \( \{x \in X : (A \cap P)(x) \neq \emptyset\} \subseteq E \),

(vi) there exists a nonempty compact subset \( K \) of \( X \) and a \( M \in \mathcal{H}(X) \) such that for each \( x \in X \setminus K \), H-co\( M \cup \{x\} \) \( A \cap P)(x) \neq \emptyset \).

Then \( \Gamma = (X; A, B; P) \) has an equilibrium \( \hat{x} \in K \).

**Proof.** Let \( F := \{x \in X : x \notin cl_X B(x)\} \). Then \( F \) is open in \( X \) by (iv). Define \( \Phi : X \rightarrow 2^X \) by

\[
\Phi(x) := \begin{cases} 
A(x) \cap P(x), & \text{if } x \notin F, \\
A(x), & \text{if } x \in F.
\end{cases}
\]
Case 1. \( x \in F \). Let
\[
\Phi_x(z) := \begin{cases} 
A(z), & \text{if } z \in F, \\
\emptyset, & \text{if } z \notin F,
\end{cases}
\]
and \( N_x = F \). Then \( N_x \) is an open neighborhood of \( x \) in \( X \) such that

(1) \( \Phi_x \) has a compactly open lower section by (iii),
(2) for each \( z \in X \), \( z \notin \text{H-co} \Phi_x(z) \) by (ii), and hence for each \( D \in \mathcal{R}(X) \), \( \Gamma_D \cap \bigcap_{z \in D} \Phi_x^{-1}(z) = \emptyset \),
(3) for each \( z \in N_x \), \( \Phi(z) = A(z) = \Phi_x(z) \).

Hence \( \Phi_x \) is an H-majorant of \( \Phi \) at \( x \).

Case 2. \( x \notin F \) and \( \Phi(x) = A(x) \cap P(x) \neq \emptyset \). By (i), there exists an open neighborhood \( N_x \) of \( x \) in \( X \) and a correspondence \( \tilde{\Phi}_x : X \rightarrow 2^X \) such that

(A) \( \tilde{\Phi}_x \) has a compactly open lower section,
(B) for each \( D \in \mathcal{R}(X) \), \( \Gamma_D \cap \bigcap_{z \in D} \Phi_x^{-1}(z) = \emptyset \),
(C) \( \Phi(z) \subseteq \Phi_x(z) \) for all \( z \in N_x \).

Now we define \( \Phi_x : X \rightarrow 2^X \) by
\[
\Phi_x(z) := \begin{cases} 
A(z) \cap \tilde{\Phi}_x(z), & \text{if } z \notin F, \\
A(z), & \text{if } z \in F.
\end{cases}
\]

Then it is easy to see that

(a) For each \( y \in X \), \( \Phi_x^{-1}(y) = [F \cup \tilde{\Phi}_x^{-1}(y)] \cap A^{-1}(y) \), and hence \( \Phi_x \) has a compactly open lower section by (iii) and (A).
(b) For each \( D \in \mathcal{R}(X) \), \( t \in \bigcap_{z \in D} \Phi_x^{-1}(z) \), then \( D \subseteq \Phi_x(t) \). We have the following two subcases: if \( t \notin F \), then \( D \subseteq \Phi(t) = A(t) \) and hence \( \Gamma_D \subseteq \text{H-co} A(t) \subseteq \text{cl} B(t) \) by (ii), so that \( t \notin \Gamma_D \), \( \Gamma_D \cap \bigcap_{z \in D} \Phi_x^{-1}(z) = \emptyset \); and if \( t \notin F \), then \( D \subseteq \Phi_x(t) = A(t) \cap \Phi_x(t) \subset \Phi_x(t) \), and hence \( t \in \bigcap_{z \in D} \tilde{\Phi}_x^{-1}(z) \). By (B), \( t \notin \Gamma_D \); i.e., \( \Gamma_D \cap \bigcap_{z \in D} \Phi_x^{-1}(z) = \emptyset \).
(c) For each \( z \in N_x \), \( \Phi(z) \subseteq \Phi_x(z) \) by (C). Another two subcases should be considered: if \( z \in F \), then \( \Phi(z) = A(z) = \Phi_x(z) \); and if \( z \notin F \), then \( \Phi(z) = A(z) \cap P(z) \subseteq A(z) \), and hence \( \Phi(z) \subseteq A(z) \cap \Phi_x(z) = \Phi_x(z) \).

We conclude that \( \Phi_x \) is an H-majorant of \( \Phi \) at \( x \). Hence we know that \( \Phi \) is an H-majorized correspondence.
By condition (vi) and the relations
\[\{x \in X : \Phi(x) \neq \emptyset\} = \{x \in F : \Phi(x) \neq \emptyset\} \cup \{x \in X \setminus F : \Phi(x) \neq \emptyset\}\]
\[= F \cup \{x \in X : x \in cl_X B(x), A(x) \cap P(x) \neq \emptyset\}\]
\[= F \cup \{x \in X : A(x) \cap P(x) \neq \emptyset\} \subset E\]
(by (v)), we know that \(\Phi\) satisfies all the conditions of Theorem 2.3. And, there exists an \(\hat{x} \in K\) such that \(\Phi(\hat{x}) = \emptyset\). Since \(A(\hat{x}) \neq \emptyset\) by (ii), we must have \(\hat{x} \in cl_X B(\hat{x})\) and \(A(\hat{x}) \cap P(\hat{x}) = \emptyset\). This completes the proof.

Remark 3.1. If \(X\) is compact, then the conditions (v) and (vi) of Theorem 3.1 are satisfied trivially. By Aubin and Cellina [1], the map \(\pi_x\) is upper semicontinuous, implying that condition (iv) of Theorem 3.1 is fulfilled, and if \(P\) is H-majorized then condition (i) of Theorem 3.1 holds. Hence Theorem 3.1 generalizes Theorem 2 of Ding et al. [4] from the compact topological vector spaces to noncompact H-space. Also, Theorem 3.1 improves and extends Theorem 2 of Ding and Tan [3] from the paracompact topological vector spaces to nonparacompact H-space.

Theorem 3.2. Let \(\Gamma = (X_i, P_i)_{i \in I}\) be a qualitative game and \(X = \Pi_{i \in I} X_i\), such that for each \(i \in I:\)

(i) \((X_i, \{\Gamma_{D_i}\})\) is a CH-space,
(ii) \(P_i : X \rightarrow 2^X\) is an H-majorized correspondence,
(iii) the set family \(\{G_i\}_{i \in I}\) is transfer open, where \(G_i = \{x \in X : P_i(x) \neq \emptyset\}\),
(iv) there exists a paracompact subset \(E_i\) of \(X\) such that \(G_i \subset E_i\),
(v) there exists a nonempty compact subset \(K\) of \(X\) and an \(M \in \mathcal{H}(X)\) such that for each \(x \in X \setminus K\), there is a \(y \in CH-co(M \cup \{x\})\) with \(\pi_i(y) \in P_i(x)\).

Then \(\Gamma\) has an equilibrium \(\hat{x}\) in \(K\).

Proof. First we note that \((X, \{\Gamma_D\})\) is a CH-space, where \(\Gamma_D = \Pi_{i \in I} \Gamma_{D_i}\), \(D_i = \pi_i(D)\), and \(D \in \mathcal{H}(X)\).

For each \(x \in X\), let \(I(x) = \{i \in I : P_i(x) \neq \emptyset\}\). Define a correspondence \(P : X \rightarrow 2^X\) by
\[P(x) := \begin{cases} \bigcap_{i \in I(x)} \tilde{P}_i(x), & \text{if } I(x) \neq \emptyset, \\ \emptyset, & \text{if } I(x) = \emptyset, \end{cases}\]
where \(\tilde{P}_i(x) := \pi_i^{-1}(P_i(x))\) for each \(x \in X\). Then for each \(x \in X\), \(P(x) \neq \emptyset\) if and only if \(I(x) \neq \emptyset, \tilde{P}_i(x) \neq \emptyset\). Let \(x \in X\) with \(P(x) \neq \emptyset\). Then there exists an \(j_0 \in I(x)\); i.e., \(P_{j_0}(x) \neq \emptyset\). Consequently, there exists an \(i_0 \in I(x)\) such that \(x \in \text{int}_X \{x \in X : P_{i_0}(x) \neq \emptyset\}\) by (iii). By (ii), there exists an H-majorant
of \( P_{b_i} \) at \( x \); i.e., there exists an open neighborhood \( N_x \) of \( x \) in \( X \) and a correspondence \( \phi_{x, b_i} : X \rightarrow 2^{X_{b_i}} \) such that

1. \( \phi_{x, b_i} \) has a compactly open lower section,
2. for each \( D \in \mathcal{H}(X), \Gamma_D \cap (\cap_{z \in D} \phi_{x, b_i}^{-1}(\pi_{b_i}(z))) = \emptyset, \)
3. \( P_{b_i}(z) \subset \phi_{x, b_i}(z) \) for all \( z \in N_x. \)

Without loss of generality, we assume that \( N_x \subset \text{int}_X \{x \in X : P_{b_i}(x) \neq \emptyset\} \), so that \( P_{b_i}(z) \neq \emptyset \) for all \( z \in N_x. \) Now we define \( \Phi_{x, b_i} : X \rightarrow 2^X \) by

\[ \Phi_{x, b_i}(z) := \pi_{b_i}^{-1}((\phi_{x, b_i}(z)), \quad \text{for each } Z \in X. \]

We prove that \( \Phi_{x, b_i} \) is an H-majorant of \( P \) at \( x \). Indeed, for each \( y \in X, \Phi_{x, b_i}(y) = \phi_{x, b_i}(y) \) and hence \( \Phi_{x, b_i} \) has a compactly open lower section by (1). For each \( D \in \mathcal{H}(X), \) if \( t \in \cap_{z \in D} \Phi_{x, b_i}^{-1}(\pi_{b_i}(z)) \), then \( D \subset \Phi_{x, b_i}(t) \).

So that \( \pi_{b_i}(D) \subset \phi_{x, b_i}(t) \), i.e., \( t \in \cap_{z \in D} \Phi_{x, b_i}^{-1}(\pi_{b_i}(z)) \) and hence \( \Gamma_D \cap (\cap_{z \in D} \Phi_{x, b_i}^{-1}(\pi_{b_i}(z))) = \emptyset \) by (2). By (3), for each \( z \in N_x, P(z) = \cap_{j \in I(z)} \tilde{P}_j(z) \subset \tilde{P}_b(z) \subset \Phi_{x, b_i}(z). \) Hence \( \Phi_{x, b_i} \) is an H-majorant of \( P \) at \( x. \)

Therefore, \( P \) is an H-majorized correspondence. By (iv), the following relations hold:

\[ \{x \in X : P(x) \neq \emptyset\} \subset \{x \in X : P_{b_i}(x) \neq \emptyset\} = G_{b_i} \subset E_{b_i}. \]

And, since there exists a nonempty compact subset \( K \) of \( X \) and an \( M \in \mathcal{H}(X) \) such that for each \( x \in X \setminus K \), there exists a \( y \in \text{CH-co}(M \cup \{x\}) \) with \( \pi_{y}(y) \in P_{i}(x) \) for all \( i \in I \), by (v), and hence we have \( y \in P(x). \) So, \( P \) satisfies all hypotheses of Theorem 2.3. By Theorem 2.3, there exists an \( \hat{x} \in K \) such that \( P(\hat{x}) = \emptyset \) which implies that \( I(\hat{x}) = \emptyset \) which in turn implies that \( P_i(\hat{x}) = \emptyset \) for all \( i \in I. \) This completes the proof.

**Remark 3.2.** If \( X_i \) is compact, then the conditions (iv) and (v) of Theorem 3.2 are satisfied trivially. Clearly, if \( G_i = \{x \in X : P_i(x) \neq \emptyset\} \) is open in \( X \) for all \( i \in I, \) then \( \{G_i\}_{i \in I} \) is transfer open. Therefore, Theorem 3.2 generalizes Theorem 3.1 of Tan and Yuan [6] from the compact topological vector spaces to noncompact H-space. Since the condition (iii) of Theorem 3 of Ding and Tan [3] implies condition (iii) of Theorem 3.2 here, Theorem 3.2 improves and extends Theorem 3 of Ding and Tan [3] from the paracompact topological vector spaces to nonparacompact H-space.

**Theorem 3.3.** Let \( \Gamma := (X_i; A_i, B_i, P_i)_{i \in I} \) be an abstract economy and let \( X := \Pi_{i \in I} X_i \), such that for each \( i \in I: \)

1. \( (X_i, (\Gamma^*_p)) \) is a CH-space,
2. \( A_i(x) \) is nonempty and H-co \( A_i(x) \subset B_i(x) \) for each \( x \in X. \),
(iii) $A_i$ has a compactly open lower section,

(iv) $\bar{B}_i(x) = cl_{X_i}(x)$ for each $x \in X$,

(v) $A_i \cap P_i$ is an $H$-majorized correspondence,

(vi) the set $G_i := \{x \in X : (A_i \cap P_i)(x) \neq \emptyset\}$ is open in $X$,

(vii) there exists a nonempty compact subset $K$ of $X$ and an $M \in \mathcal{S}(X)$ such that for each $x \in X \setminus K$, there is a $y \in CH \cdot \text{co}(M \cup \{x\})$ with $\pi_i(y) \in (A_i \cap P_i)(x)$.

Then $\Gamma$ has an equilibrium $\hat{x}$ in $K$.

Proof. By (iv), $F_i$ is open in $X$ for all $i \in I$. We now define $Q_i : X \to 2^{X_i}$ by

$$Q_i(x) := \begin{cases} (A_i \cap P_i)(x), & \text{if } x \notin F_i, \\ A_i(x), & \text{if } x \in F_i. \end{cases}$$

We shall show that the qualitative game $\tilde{\Gamma} = (X_i, Q_i)_{i \in I}$ satisfies all hypotheses of Theorem 3.2. First, we have that for each $i \in I$, the set

$$\{x \in X : Q_i(x) \neq \emptyset\} = \{x \in F_i : Q_i(x) \neq \emptyset\} \cup \{x \in X \setminus F_i : Q_i(x) \neq \emptyset\}$$

is open in $X$ by (vi), and $\{x \in X : Q_i(x) \neq \emptyset\} \subset E_i$ by (vii). Hence the conditions (iii) and (iv) of Theorem 3.2 are satisfied.

Let $x \in X$ be such that $Q_i(x) \neq \emptyset$. We consider the following two cases.

Case 1. $x \in F_i$. Define $\Phi_{x,i} : X \to 2^{X_i}$ by

$$\Phi_{x,i}(z) := \begin{cases} A_i(z), & \text{if } z \in F_i, \\ \emptyset, & \text{if } z \notin F_i, \end{cases}$$

and let $N_x = F_i$; then $N_x$ is an open neighborhood of $x$ in $X$. Thus the correspondence $\Phi_{x,i}$ has the following properties:

(1) for each $y \in X_i$, $\Phi_{x,i}^{-1}(y) = A_i^{-1}(y) \cap F_i$ and hence $\Phi_{x,i}$ has a compactly open lower section by (iii),

(2) for each $z \in X$, $\pi_i(z) \notin H \cdot \text{co} \Phi_{x,i}(z)$ by (ii), and hence

$$\Gamma_D \cap \left( \bigcap_{z \in D} \Phi_{x,i}^{-1}(\pi_i(z)) \right) = \emptyset$$

for each $D \in \mathcal{S}(X)$,

(3) for each $z \in N_x$, $Q_i(z) = A_i(z) = \Phi_{x,i}(z)$.

Therefore, $\Phi_x$ is a $H$-majorant of $Q_i$ at $x$. 
Case 2. \( x \notin F_i \). Since \( Q_i(x) = (A_i \cap P_i)(x) \neq \emptyset \) and by (v), there exists an open neighborhood \( N_x \) of \( x \) in \( X \) and a correspondence \( \phi_{x,i} : X \rightarrow 2^{X_i} \) such that the correspondence \( \phi_{x,i} \) satisfies the following properties:

(A) \( \phi_{x,i} \) has a compactly open lower section,
(B) for each \( D \in \mathcal{H}(X) \), \( \Gamma_D \cap (\bigcap_{x \in D} \phi_{x,i}^{-1}(\pi_i(\mu))) = \emptyset \),
(C) for each \( z \in N_x \), \( (A_i \cap P_i)(z) \subset \phi_{x,i}(z) \).

We now define \( \Phi_{x,i} : X \rightarrow 2^{X_i} \) by

\[
\Phi_{x,i}(z) = \begin{cases} 
(A_i \cap \phi_{x,i})(z), & \text{if } z \notin F_i, \\
A_i(z), & \text{if } z \in F_i.
\end{cases}
\]

Noting that for each \( y \in X \), the set \( \Phi_{x,i}^{-1}(y) \) has the property

\[
\Phi_{x,i}^{-1}(y) = \{ z \in F_i : y \in \Phi_{x,i}(z) \} \cup \{ z \in X \setminus F_i : y \in \Phi_{x,i}(z) \} = (F_i \cap A_i^{-1}(y)) \cup ((X \setminus F_i) \cap A_i^{-1}(y) \cap \phi_{x,i}^{-1}(y)) = A_i^{-1}(y) \cap (\phi_{x,i}^{-1} \cup F_i),
\]

\( \Phi_{x,i} \) has a compactly open lower section by (iii) and (A). Moreover, for each \( D \in \mathcal{H}(X) \), if \( t \in \bigcap_{x \in D} \Phi_{x,i}^{-1}(\pi_i(\mu)) \), then \( \pi_i(D) \subset \phi_{x,i}(t) \). We again consider two subcases: if \( t \notin F_i \), then \( \pi_i(D) \subset \phi_{x,i}(t) \), i.e., \( t \in \bigcap_{x \in D} \Phi_{x,i}^{-1}(\pi_i(\mu)) \) and hence \( t \notin \Gamma_D \) by (B); and if \( t \in F_i \), then \( \pi_i(D) \subset A_i(t) \) and hence \( \Gamma_{t}^i \subset \mathcal{H}(A_i(t) \subset B_i(t) \mid \text{by (ii)} \). So that \( \pi_i(t) \notin \Gamma_{t}^i \) and \( t \notin \Gamma_D \).

Therefore, \( \Gamma_D \cap (\bigcap_{x \in D} \Phi_{x,i}^{-1}(\pi_i(\mu))) = \emptyset \). By (C), for each \( z \in N_x \) we have \( Q_i(z) \subset \Phi_{x,i}(z) \). Hence \( \Phi_{x,i} \) is an H-majorant of \( Q_i \) at \( x \). We now know that \( Q_i \) is an H-majorized correspondence.

By (viii), there exists a nonempty compact subset \( K \) of \( X \) and an \( M \in \mathcal{H}(X) \) such that for each \( x \in X \setminus K \), there is a \( y \in \mathcal{H}(M \cup \{ x \}) \) with \( \pi_i(y) \in (A_i \cap P_i)(x) \subset Q_i(x) \) for all \( i \in I \). Therefore, the qualitative game \( \Gamma = (X_i, Q_i)_{i \in I} \) satisfies all hypotheses of Theorem 3.2. By Theorem 3.2, there exists an \( \hat{x} \in K \) such that \( Q_i(\hat{x}) = \emptyset \) for all \( i \in I \); by (ii), this implies that for each \( i \in I \), \( \pi_i(\hat{x}) \in cI_{X_i}B_i(\hat{x}) \) and \( A_i(\hat{x}) \cap P_i(\hat{x}) = \emptyset \). This completes the proof.

Remark 3.3. If \( X_i \) is compact, then the conditions (vii) and (viii) of Theorem 3.3 are satisfied trivially; if \( X_i \) is paracompact, then condition (vii) of Theorem 3.3 is also satisfied trivially. Therefore, Theorem 3.3 improves and generalizes Theorem 3 of Ding et al. [4], Corollary 3.4 of Tan and Yuan [6], Theorem 4 of Ding and Tan [3], and Corollary 3.4 of Tan and Yuan [7] from the compact or paracompact topological vector spaces to noncompact or nonparacompact H-space, respectively. And, Theorem 3.3 relaxes the conditions of constraint correspondence and preference correspondence included in the above-mentioned references.
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