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Simple Purely Infinite C^*-Algebras and n-Filling Actions[1]

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Let n be a positive integer. We introduce a concept, which we call the n-filling property, for an action of a group on a separable unital C^*-algebra A. If $A = C(\Omega)$ is a commutative unital C^*-algebra and the action is induced by a group of homeomorphisms of Ω then the n-filling property reduces to a weak version of hyperbolicity. The n-filling property is used to prove that certain crossed product C^*-algebras are purely infinite and simple. A variety of group actions on boundaries of symmetric spaces and buildings have the n-filling property. An explicit example is the action of $\Gamma = SL_n(\mathbb{Z})$ on the projective n-space. © 2000 Academic Press
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INTRODUCTION

Consider a C^*-dynamical system (A, α, Γ) where A is a separable unital C^*-algebra on which a discrete group Γ acts by $*$-automorphisms.

Definition 0.1. Let $n \geqslant 2$ be a positive integer. We say that an action $\alpha: g \mapsto \alpha_g$ of Γ on A is *n-filling* if, for all $b_1, b_2, ..., b_n \in A^+$, with $\|b_j\| = 1$, $1 \leqslant j \leqslant n$, and for all $\varepsilon > 0$, there exist $g_1, g_2, ..., g_n \in \Gamma$ such that $\sum_{j=1}^n \alpha_{g_j}(b_j) \geqslant 1 - \varepsilon$.

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If A is a commutative unital C^*-algebra and α is induced by a group of homeomorphisms of the spectrum Ω of A, then the n-filling property is equivalent to a generalized global version of hyperbolicity (Proposition 0.3 below). In this setting, the definition was motivated by ideas from [A-D, LS, BCH]. The present article applies the n-filling property to give a proof that certain crossed product C^*-algebras are purely infinite and simple (Theorem 1.2). In the commutative case, similar results were obtained in [A-D, LS] using local properties of the action. Simple crossed product algebras have been constructed using the related concept of a strongly hyperbolic action in [H, Appendix 2].

Remark 0.2. In order to prove the n-filling condition as stated in Definition 0.1 it is sufficient to verify it for all $b_1, b_2, ..., b_n$ in a dense subset C of A^+. For then if $b_1, b_2, ..., b_n \in A^+$, with $\|b_j\| = 1$, $1 \leqslant j \leqslant n$, and if $\varepsilon > 0$, choose $c_1, c_2, ..., c_n \in C$ such that $\|b_j - c_j\| < \frac{\varepsilon}{2n}$ for all j and $\sum_{j=1}^n \alpha_{g_j}(c_j) \geqslant 1 - \varepsilon/2$. Write

$$\sum_{j=1}^{n} \alpha_{g_j}(b_j - c_j) = x = x_+ - x_-,$$

where $x_+, x_- \in A^+$ and $x_+ x_- = 0$. We have $x \geqslant -\varepsilon/2$ and therefore

$$\sum_{j=1}^{n} \alpha_{g_j}(b_j) = \sum_{j=1}^{n} \alpha_{g_j}(c_j) + x \geqslant 1 - \varepsilon/2 - \varepsilon/2 = 1 - \varepsilon.$$

Suppose that $A = C(\Omega)$, the algebra of continuous complex valued functions on a compact Hausdorff space Ω. If the action arises from an action of Γ on Ω by homeomorphisms, then the n-filling condition can be expressed in the following way, which explains its name.

PROPOSITION 0.3. *Let Ω be an infinite compact Hausdorff space and let Γ be a group which acts on Ω by homeomorphisms. The induced action α of Γ on $C(\Omega)$ is n-filling if and only if the following condition is satisfied: for any nonempty open subsets $U_1, ..., U_n$ of Ω, there exist $g_1, ..., g_n \in \Gamma$ such that $g_1 U_1 \cup \cdots \cup g_n U_n = \Omega$.*

Proof. If the action is n-filling, let $U_1, ..., U_n$ be nonempty open subsets of Ω. There exist elements $b_1, b_2, ..., b_n \in A^+$, with $\|b_j\| = 1$, such that $\operatorname{supp}(b_j) \subset U_j$, $1 \leqslant j \leqslant n$. By hypothesis there exist $g_1, g_2, ..., g_n \in \Gamma$ such that $\sum_{j=1}^n \alpha_{g_j}(b_j) \geqslant 1/2$. Then if $\omega \in \Omega$ there exists $i \in \{1, 2, ..., n\}$ such that $\alpha_{g_i}(b_i)(\omega) > 0$. Therefore $g_i^{-1}\omega \in U_i$, i.e. $\omega \in g_i U_i$. Thus $g_1 U_1 \cup \cdots \cup g_n U_n = \Omega$.

Conversely, suppose the stated assertion holds. Fix $b_1, b_2, ..., b_n \in A^+$, with $\|b_j\| = 1$, $1 \leqslant j \leqslant n$, let $\varepsilon > 0$. For each j, the set $U_j = \{\omega \in \Omega; b_j(\omega) > 1 - \varepsilon\}$ is a nonempty and open. Choose $g_1, ..., g_n \in \Gamma$ such that $g_1 U_1 \cup \cdots \cup g_n U_n = \Omega$. If $\omega \in \Omega$, then $g_i^{-1} \omega \in U_i$ for some i and so $\alpha_{g_i}(b_i)(\omega) > 1 - \varepsilon$. Therefore $\sum_{j=1}^n \alpha_{g_j}(b_j) \geqslant 1 - \varepsilon$. ∎

Remark 0.4. If the action of the group Γ on the space Ω is topologically transitive (in particular, if it is minimal) then the n-filling condition is equivalent to the following apparently weaker condition: for each nonempty open subset U of Ω there exist $t_1, ..., t_n \in \Gamma$ such that $t_1 U \cup \cdots \cup t_n U = \Omega$.

In order to see this, suppose that $U_1, ..., U_n$ are nonempty open subsets of Ω. There exists $g_2 \in \Gamma$ such that $U_1 \cap g_2 U_2 \neq \emptyset$. Then there exists $g_3 \in \Gamma$ such that $U_1 \cap g_2 U_2 \cap g_3 U_3 \neq \emptyset$. Finally, there exists $g_n \in \Gamma$ such that $U = U_1 \cap g_2 U_2 \cdots \cap g_n U_n \neq \emptyset$. Then there exist $t_1, ..., t_n \in \Gamma$ such that $t_1 U \cup \cdots \cup t_n U = \Omega$ and so $t_1 U_1 \cup t_2 g_2 U_2 \cdots \cup t_n g_n U_n = \Omega$.

Definition 0.5. Let $\phi(\Gamma, \Omega)$ be the smallest integer n for which the conclusion of Proposition 0.3 holds. Set $\phi(\Gamma, \Omega) = \infty$ if no such n exists, that is, if the action is not n-filling for any integer n.

Topologically conjugate actions have the same value of $\phi(\Gamma, \Omega)$. It is easy to see that the notion of a 2-filling action is equivalent to what is called a *strong boundary action* in [LS] and an *extremely proximal flow* in [G]. The action of a word hyperbolic group on its Gromov boundary is 2-filling [LS, Example 2.1]. In our first example below (Example 2.1) we show that the canonical action of $\Gamma = SL_n(\mathbb{Z})$ on the projective space $\Pi = \mathbb{P}^{n-1}(\mathbb{R})$ satisfies $\phi(\Gamma, \Pi) = n$.

The final part of the paper is devoted to estimating $\phi(\Gamma, \Omega)$ for some group actions on the boundaries of affine buildings. These estimates show that $\phi(\Gamma, \Omega)$ is not a stable isomorphism invariant for the algebra $C(\Omega) \rtimes_r \Gamma$ (Example 4.3).

1. PURELY INFINITE C^*-ALGEBRAS FROM N-FILLING ACTIONS

Definition 1.1. An automorphism α of a C^*-algebra A is said to be *properly outer* if for each nonzero α-invariant ideal I of A and for each inner automorphism β of I we have $\|\alpha \mid I - \beta\| = 2$.

We shall say that an action $\alpha: g \mapsto \alpha_g$ is properly outer if for all $g \in \Gamma \backslash \{e\}$, α_g is properly outer.

The purpose of this section is to prove the following result.

THEOREM 1.2. *Let (A, α, Γ) be a C^*-dynamical system, where Γ is a discrete group and A is a separable unital C^*-algebra. Suppose that for every nonzero projection $e \in A$ the hereditary C^*-subalgebra eAe is infinite dimensional. Suppose also that the action α is n-filling and properly outer. Then the reduced crossed product algebra $B = A \rtimes_{\alpha, r} \Gamma$ is a purely infinite simple C^*-algebra.*

Remark 1.3. If $A = C(\Omega)$, with Ω a compact Hausdorff space, the condition that eAe is infinite dimensional for every nonzero projection $e \in A$ says simply that the space Ω has no isolated points.

It was shown in [AS, Proposition 1] that if the action α is *topologically free* then α is properly outer.

Proof (inspired by [LS, Theorem 5]). Denote by $E: B \to A$ the canonical conditional expectation. Fix $x \in B$, $x \neq 0$. In order to prove the result it is enough to show that there exist $y, z \in B$ such that $yxz = 1$. Put $a = x^*x/\|E(x^*x)\|$. Let $0 < \varepsilon < 1/(2(2n+1))$. There exists $b \in C_c(\Gamma, A)^+$ such that $\|a - b\| < \varepsilon$. Write $b = b_e + \sum_{g \in F} b_g u_g$, where $b_e = E(b) \geqslant 0$ and $F \subset \Gamma \backslash \{e\}$ is finite. Note that $\varepsilon > \|E(a-b)\| = \|E(a) - b_e\| \geqslant |1 - \|b_e\||$, and so $\|b_e\|^{-1} < 1 + 2\varepsilon$. It follows that

$$\left\| a - \frac{b}{\|b_e\|} \right\| = \|b_e\|^{-1} \, \|(\|b_e\| - 1)\, a + a - b\|$$

$$< (1+2\varepsilon)(\varepsilon \, \|a\| + \varepsilon) = \varepsilon(1+2\varepsilon)(1 + \|a\|).$$

Choosing b so that $\|a - b\| < \varepsilon/(3(1 + \|a\|))$ then replacing b by $b/\|b_e\|$ shows that we can assume that $\|b_e\| = 1$.

Since α_g is properly outer for each $g \in F$, it follows from [OP, Lemma 7.1] that there exists $y \in A^+$, $\|y\| = 1$ such that $\|b_e\| \geqslant \|yb_e y\| > \|b_e\| - \varepsilon/|F|$ and $\|yb_g \alpha_g(y)\| < \varepsilon/|F|$ for all $g \in F$. Using Lemma 1.5 below, we see that there exists $c \in B$ such that $\|c\| \leqslant \sqrt{n}$ and $c^* y b_e y c \geqslant 1 - 3\varepsilon$.

Then

$$\|c^*yayc - c^*yb_e yc\| \leqslant \|c^*yayc - c^*ybyc\| + \|c^*ybyc - c^*yb_e yc\|$$

$$\leqslant n \, \|a - b\| + n \, \|yby - yb_e y\|$$

$$\leqslant n\varepsilon + n \sum_{g \in F} \|yb_g u_g y u_g^{-1} u_g\| \leqslant 2n\varepsilon.$$

Therefore c^*yayc is invertible since $(\|c^*yb_e yc)^{-1}\| \leqslant \frac{1}{1-3\varepsilon}$ and

$$\|1-(c^*yb_e yc)^{-1}(c^*yayc)\| \leqslant \frac{2n\varepsilon}{1-3\varepsilon} < \frac{n}{2n-1} < 1.$$

Setting $z=(c^*yayc)^{-1}$ we have $\|E(x^*x)\|^{-1} c^*yx^* \cdot x \cdot ycz = 1$. ▮

It remains to prove Lemma 1.5. A preliminary observation is necessary.

Lemma 1.4. *Let A be a unital C^*-algebra such that for every nonzero projection $e \in A$ the hereditary C^*-subalgebra eAe is infinite dimensional. Let $b \in A^+$, $\|b\|=1$ and $\varepsilon > 0$. For every integer $n \geqslant 1$ there exist elements b_1, b_2, ..., $b_n \in A^+$, with $\|b_j\|=1$, $bb_j=b_jb$, $\|bb_j\| \geqslant 1-\varepsilon$ and $b_ib_j=0$, for $i \neq j$.*

Proof. There are two cases to consider.

Case 1. Suppose that 1 is not an isolated point of $\mathrm{Sp}(b)$. Then there exist pairwise disjoint nonempty open sets $U_1, ..., U_n$ contained in $\mathrm{Sp}(b) \cap [1-\varepsilon, 1]$. Let C be the C^*-subalgebra of A generated by $\{b, 1\}$. By functional calculus, there exist b_1, b_2, ..., $b_n \in C^+$, $\|b_j\|=1$ $(1 \leqslant j \leqslant n)$ with $\|bb_j\| \geqslant 1-\varepsilon$ and $b_ib_j=0$, $i \neq j$.

Case 2. Suppose that 1 is an isolated point of $\mathrm{Sp}(b)$. Then there exists a nonzero projection $e \in A$ such that $be=eb=e$. By hypothesis the hereditary C^*-subalgebra eAe is infinite dimensional. Therefore every masa of eAe is infinite dimensional [KR, p. 288]. Inside such an infinite dimensional masa of eAe we can find positive elements b_1, b_2, ..., b_n, $\|b_j\|=1$ $(1 \leqslant j \leqslant n)$ with $b_ib_j=0$, $i \neq j$. Then $bb_j = b(eb_j) = eb_j = b_j = b_jb$ and $\|bb_j\| = \|b_j\| = 1$ for $1 \leqslant j \leqslant n$. ▮

Lemma 1.5. *Let (A, α, Γ) be as in the statement of Theorem* 1.2, *let $0 < \varepsilon < 1/3$, and let $b \in A^+$, with $1-\varepsilon \leqslant \|b\| \leqslant 1$. Then there exists $c \in B$ such that $\|c\| \leqslant \sqrt{n}$ and $c^*bc \geqslant 1-3\varepsilon$.*

Proof. By Lemma 1.4, there exist $b_1, b_2, ..., b_n \in A^+$, with $\|b_j\|=1$, $bb_j=b_jb$, $b_ib_j=0$ for $i \neq j$, and $\|bb_j\| \geqslant 1-2\varepsilon$. Since the action is n-filling, there exist $g_1, g_2, ..., g_n \in \Gamma$ such that $\sum_{i=1}^n (1/\|bb_i\|)\, \alpha_{g_i}(bb_i) \geqslant 1-\varepsilon$. Therefore $\sum_{i=1}^n \alpha_{g_i}(bb_i) \geqslant (1-\varepsilon)(1-2\varepsilon) \geqslant 1-3\varepsilon$. Put $c = \sum_{j=1}^n \sqrt{b_j}\, u_{g_j}^{-1} \in B$.

Now $c^*c = \sum_{i,j} u_{g_i} \sqrt{b_i} \sqrt{b_j}\, u_{g_j}^{-1} = \sum_{i=1}^n \alpha_{g_i}(b_i) \leqslant n$ and so $\|c\| \leqslant \sqrt{n}$. Finally, we have $c^*bc = \sum_{i,j} u_{g_i} \sqrt{b_i}\, b \sqrt{b_j}\, u_{g_j}^{-1} = \sum_{i=1}^n \alpha_{g_i}(bb_i) \geqslant 1-3\varepsilon$. ▮

2. EXAMPLES

We now give some explicit examples of n-filling actions.

EXAMPLE 2.1. For the canonical action of $\Gamma = SL_n(\mathbb{Z})$ on the projective space $\Pi = \mathbb{P}^{n-1}(\mathbb{R})$, we have $\phi(\Gamma, \Pi) = n$.

Proof. Denote by $u \mapsto [u]$ the canonical map from $\mathbb{R}^n$ onto Π.

We first show that the action of Γ on Π is not $(n-1)$-filling. Choose a linear subspace E of $\mathbb{R}^n$ of dimension $n-1$. Let $U = \Pi \setminus [E]$, which is a nonempty open subset of Π. If $t_j \in \Gamma$ $(1 \leqslant j \leqslant n-1)$ then $t_1 U \cup \cdots \cup t_{n-1} U \neq \Pi$. For the subspace $t_1 E \cap \cdots \cap t_{n-1} E$ of $\mathbb{R}^n$ has dimension at least one, and so contains a nonzero vector v. Then $[v] \notin \bigcup_{j=1}^{n-1} t_j U$. Thus the action (Γ, Π) is not $(n-1)$-filling. It remains to show that it is n-filling. For this we use ideas from [BCH, Example 1].

We claim that there exists a basis $\{u_1, u_2, ..., u_n\}$ for $\mathbb{R}^n$, elements $g_1, g_2, ..., g_n \in \Gamma$, and (compact) sets $K_1, K_2, ..., K_n \subset \Pi$ with $K_1 \cup K_2 \cup \cdots \cup K_n = \Pi$, and with the following property: for any open neighbourhood U_j of $[u_j]$ $(1 \leqslant j \leqslant n)$ there exists a positive integer N_j such that $g_j^n K_j \subset U_j$ for all $n \geqslant N_j$. It follows that the action is n-filling. For let $U_1, ..., U_n$ be nonempty open subsets of Π. Since the action of Γ on Π is minimal, we may assume that $[u_j] \in U_j$ $(1 \leqslant j \leqslant n)$. Let $t_j = g_j^{-N_j}$, so that $K_j \subset t_j U_j$ $(1 \leqslant j \leqslant n)$. Then $t_1 U_1 \cup \cdots \cup t_n U_n = \Pi$.

It remains to verify our claim. Fix a positive integer $k \geqslant 4$ and let $a = 2/(\sqrt{k^2+4k}+k)$, $b = (\sqrt{k^2+4k}-k)/2$. Consider the matrices $A = \begin{pmatrix} k+1 & k \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 1 \\ k & k+1 \end{pmatrix}$ in $SL_2(\mathbb{Z})$. These matrices have eigenvalues $\lambda_+ = 1 + \frac{1}{a}$, $\lambda_- = 1 - b$, which satisfy $0 < \lambda_- < 1 < \lambda_+$. The corresponding eigenvectors for A are $\begin{pmatrix} 1 \\ a \end{pmatrix}$ and $\begin{pmatrix} -b \\ 1 \end{pmatrix}$, for B they are $\begin{pmatrix} a \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -b \end{pmatrix}$. If $1 \leqslant j \leqslant n-1$ let

$$g_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & A & \\ 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad u_j = \begin{pmatrix} 0 \\ 0 \\ 1 \\ a \\ 0 \end{pmatrix}, \qquad v_j = \begin{pmatrix} 0 \\ 0 \\ -b \\ 1 \\ 0 \end{pmatrix},$$

where A occupies the j and $j+1$ rows and columns and the nonzero entries of the vectors are in rows j and $j+1$. Also let

$$g_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & & \\ & & & B \end{pmatrix}, \qquad u_n = \begin{pmatrix} 0 \\ 0 \\ a \\ 1 \end{pmatrix}, \qquad v_n = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -b \end{pmatrix},$$

Let $R=\max(\frac{1+a}{1-b}, \frac{1+ab}{1-b})=\frac{1+a}{1-b}$. For $1\leqslant j\leqslant n-1$ let

$$K_j=\left\{\left[\xi_j u_j+\eta_j v_j+\sum_{l\neq j,\, j+1}\xi_l e_l\right];\ \xi_j\neq 0;\right.$$
$$\left.\left|\frac{\eta_j}{\xi_j}\right|\leqslant R,\ \left|\frac{\xi_l}{\xi_j}\right|\leqslant R,\ l\neq j,\ j+1\right\},$$

$$K_n=\left\{\left[\xi_n u_n+\eta_n v_n+\sum_{l\neq n-1,\, n}\xi_l e_l\right];\ \xi_n\neq 0,\right.$$
$$\left.\left|\frac{\eta_n}{\xi_n}\right|\leqslant R,\ \left|\frac{\xi_l}{\xi_n}\right|\leqslant R,\ l\neq n-1,\ n\right\}.$$

Direct computation shows if $[x]\in\Pi$ then $[x]\in K_j$, where $|x_j|=\max_{1\leqslant l\leqslant n}|x_l|$. Therefore $\Pi=\bigcup_{j=1}^{n}K_j$.

Let $\varepsilon>0$ and consider the basic open neighborhood U_j of $[u_j]$ defined by

$$U_j=\left\{\left[\xi_j u_j+\eta_j v_j+\sum_{l\neq j,\, j+1}\xi_l e_l\right];\ \xi_j\neq 0,\ \left|\frac{\eta_j}{\xi_j}\right|<\varepsilon,\ \left|\frac{\xi_l}{\xi_j}\right|<\varepsilon,\ l\neq j,\ j+1\right\}.$$

Let $N>\frac{\log(R/\varepsilon)}{\log(\lambda_+)}$. Recall that $0<\lambda_-<1<\lambda_+$. Therefore $R/\lambda_+^N<\varepsilon$.

For $m\geqslant N$ and $[\xi_j u_j+\eta_j v_j+\sum_{l\neq j,\, j+1}\xi_l e_l]\in K_j$, we have

$$g^m\left[\xi_j u_j+\eta_j v_j+\sum_{l\neq j,\, j+1}\xi_l e_l\right]=\left[\lambda_+^m\xi_j u_j+\lambda_-^m\eta_j v_j+\sum_{l\neq j,\, j+1}\xi_l e_l\right].$$

Now $|\lambda_-^m\eta_j/\lambda_+^m\xi_j|\leqslant(1/\lambda_+^m)\,|\eta_j/\xi_j|\leqslant R/\lambda_+^m<\varepsilon$, and for $l\neq j,\ j+1$,

$$|\xi_l/\lambda_+^m\xi_j|\leqslant(1/\lambda_+^m)\,|\xi_l/\xi_j|\leqslant R/\lambda_+^m<\varepsilon.$$

This means that $g_j^m K_j\subset U_j$ for all $m\geqslant N$. ∎

Remark 2.2. The fact that the action of $SL_3(\mathbb{Z})$ on the projective plane $\mathbb{P}^2(\mathbb{R})$ is not 2-filling can also be seen in a different way. More generally the action of a group Γ on a non-orientable compact surface Ω cannot be 2-filling. For let M be a closed subset of Ω homeomorphic to a Möbius band, let $U_1=M^c$ and let $U_2\subset\Omega$ be homeomorphic to an open disc in $\mathbb{R}^2$. Then it is impossible to have $t_1U_1\cup t_2U_2=\Omega$ for $t_1,\ t_2\in\Gamma$. For $t_2^{-1}t_1(M)$ would be a homeomorphic copy of a Möbius band embedded in the disc U_2. To see that this is impossible note that a Mobius band is not disconnected by its centre circle, and apply the Jordan curve theorem.

Definition 2.3. Let the group Γ act on the topological space Ω. An element $g\in\Gamma$ is said to have an attracting fixed point $x\in\Omega$ if $gx=x$ and there exists a neighbourhood V_x of x such that $\lim_{n\to\infty}g^n(V_x)=\{x\}$.

Remark 2.4. Let G be a noncompact semisimple real algebraic group and let Γ be a Zariski-dense subgroup of G. Consider the action of G on its Furstenberg boundary G/P, where P is a minimal parabolic subgroup of G. It follows from [BeL, Appendice] that there exist elements $g \in \Gamma$ which have attracting fixed points in G/P. In fact the set H of all such elements $g \in \Gamma$ is Zariski-dense in G: the elements of H are called *h-regular* in [BeL] and *maximally hyperbolic* in [BCH].

It follows from a result of H. Furstenberg [Fur, Theorem 5.5, Corollary] that if G is a semisimple group with finite centre which acts minimally on a locally compact Hausdorff space Ω with an attracting fixed point, then Ω is necessarily a compact homogeneous space of G.

The following result shows that many of the actions considered in [A-D, LS] are n-filling for some integer n.

PROPOSITION 2.5. *Let Ω be a compact Hausdorff space and let (Ω, Γ) be a minimal action. Suppose that there exists an element $g \in \Gamma$ which has an attracting fixed point in Ω. Then the action (Ω, Γ) is n-filling for some integer n.*

Proof. Choose $x \in \Omega$ with $gx = x$ and an open neighbourhood V_x of x such that $\lim_{n \to \infty} g^n(V_x) = \{x\}$. Since the action is minimal, the family $\{hV_x ; h \in \Gamma\}$ forms an open covering of Ω. By compactness, there exists a finite subcovering $\{h_1 V_x, h_2 V_x, ..., h_n V_x\}$.

Let $U_1, ..., U_n$ be nonempty open subsets of Ω. Since the action of Γ on Ω is minimal, we may choose elements $s_j \in \Gamma$ such that $h_j x \in s_j U_j$ $(1 \leqslant j \leqslant n)$. For $1 \leqslant j \leqslant n$, choose an integer N_j such that $g^{N_j} V_x \subset h_j^{-1} s_j U_j$. Then $h_j V_x \subset t_j U_j$, where $t_j = h_j g^{-N_j} h_j^{-1} s_j$. Therefore $t_1 U_1 \cup \cdots \cup t_n U_n = \Omega$. ∎

Remark 2.6. Consider the action of a noncompact semisimple real algebraic group G on its Furstenberg boundary G/P. Let Γ be a Zariski-dense subgroup of G and let $n(W)$ be the order of the Weyl group. In this case one can be more precise: the action $(G/P, \Gamma)$ is $n(W)$-filling. The proof follows from the remarks in [BCH, p. 127]. In the next section we prove an analogue of this result for groups acting on affine buildings.

Recall that an action (Ω', Γ) is said to be a *factor* of the action (Ω, Γ) if there is a continuous equivariant surjection from Ω onto Ω'.

PROPOSITION 2.7. *Suppose that the action (Ω, Γ) is n-filling and that (Ω', Γ) is a factor of (Ω, Γ). Then (Ω', Γ) is an n-filling action.*

Proof. This is an easy consequence of the definitions. ∎

3. GROUP ACTIONS ON BOUNDARIES OF AFFINE BUILDINGS

We now turn to some examples which motivated our definition of an n-filling action. They are discrete analogues of those referred to Remark 2.6. We show that if a group Γ acts properly and cocompactly on an affine building Δ with boundary Ω, then the induced action on Ω is a n-filling, where n is the number of boundary points of an apartment in Δ. If Δ is the affine Bruhat–Tits building of a linear group then n is the order of the associated spherical Weyl group.

An *apartment* in Δ is a subcomplex of Δ isomorphic to an affine Coxeter complex. Each apartment inherits a natural metric from the Coxeter complex, which gives rise to a well-defined metric on the whole building [Br, Chap. IV.3]. Every geodesic of Δ is a straight line in some apartment. A *sector* (or *Weyl chamber*) is a sector based at a special vertex in some apartment [Ron]. Two sectors are *equivalent* (or parallel) if their intersection contains a sector. The boundary Ω is defined to be the set of equivalence classes of sectors in Δ. Fix a special vertex x. For any $\omega \in \Omega$ there is a unique sector $[x, \omega)$ in the class ω having base vertex x [Ron, Theorem 9.6, Lemma 9.7]. In the terminology of [Br, Chap. VI.9] Ω is the set of chambers of the building at infinity Δ^∞. Topologically, Ω is a totally disconnected compact Hausdorff space and a basis for the topology is given by sets of the form

$$\Omega_x(v) = \{\omega \in \Omega : [x, \omega) \text{ contains } v\},$$

where v is a vertex of Δ. See [CMS, Sect. 2] for the $\tilde{A}_2$ case, which generalizes directly.

We will need to use the fact that Ω also has the structure of a spherical building [Ron, Theorem 9.6], and its apartments are topological spheres.

DEFINITION 3.1. Two boundary points ω, ϖ in Ω are said to be *opposite* [Br, IV.5] if the distance between them is the diameter of the spherical building Ω. Opposite boundary points are opposite in a spherical apartment of Ω which contains them; this apartment is necessarily unique. Two subsets of Ω are opposite if each point in one set is opposite each point in the other.

We define $\mathcal{O}(\omega)$ to be the set of all $\omega' \in \Omega$ such that ω' is opposite to ω. It is easy to see that $\mathcal{O}(\omega)$ is an open set.

LEMMA 3.2. *If $\omega \in \Omega$ and $\mathcal{A}$ is an apartment in Δ, then there exists a boundary point ϖ of $\mathcal{A}$ such that ϖ is opposite ω.*

Proof. Consider the geometric realization of the spherical building Ω. By [Ron, Theorem (A.19)], the subcomplex Ω' obtained from Ω by deleting all chambers opposite ω is geodesically contractible. However, this is impossible if Ω' contains the spherical apartment of Ω made up of the boundary points of $\mathscr{A}$. ▮

COROLLARY 3.3. *If $\omega_1, ..., \omega_n$ are the boundary points of an apartment then*

$$\Omega = \mathcal{O}(\omega_1) \cup \cdots \cup \mathcal{O}(\omega_n).$$

Remark 3.4. The union is not disjoint in general, as is seen by considering the example of a tree.

LEMMA 3.5. *Two chambers ω_1, ω_2 in Ω are opposite if and only if they are represented by opposite sectors S_1, S_2 with the same base vertex in some apartment of Δ. Moreover if two sectors S_1, S_2 in an apartment $\mathscr{A}$ with the same base vertex represent opposite elements ω_1, ω_2 in Ω, then S_1, S_2 are opposite sectors and $\mathscr{A}$ is the unique apartment containing them.*

Proof. Suppose that ω_1, ω_2 in Ω are opposite. There exists an apartment $\mathscr{A}$ containing sectors S_1, S_2 representing ω_1, ω_2, respectively [Ron, Proposition 9.5, Br, VI.8, Theorem]. By taking parallel sectors, we may assume that S_1, S_2 have the same base vertex $x \in \mathscr{A}$. The sectors of $\mathscr{A}$ based at x correspond to the chambers of an apartment in Ω containing ω_1, ω_2 [Ron, Theorem 9.8]. Therefore S_1, S_2 are opposite sectors. The converse is clear.

The final assertion follows from [Br, VI.9, Lemma 2 and IV.5, Theorem 1]. ▮

Remark 3.6. (a) It is not necessarily true that if ω_1, ω_2 in Ω are opposite then the sectors $[z, \omega_1)$, $[z, \omega_2)$ based at any vertex z are opposite sectors in some apartment.

(b) If C_1, C_2 are opposite chambers with a common vertex x in an apartment, then $\Omega_x(C_1)$ and $\Omega_x(C_2)$ are opposite sets in Ω.

Suppose that a group Γ acts properly and cocompactly on an affine building Δ of dimension n. An apartment $\mathscr{A}$ in Δ is said to be *periodic* if there is a subgroup $\Gamma_0 < \Gamma$ preserving $\mathscr{A}$ such that $\Gamma_0 \backslash \mathscr{A}$ is compact [Gr, 6.B_3]. Note that Γ_0 is commensurable with $\mathbb{Z}^n$, and this concept coincides with the notion of periodicity described in [MZ, RR] for buildings of type $\tilde{A}_2$. In [BB], a periodic apartment is called Γ-closed. This terminology makes it clear that periodicity depends upon the choice of the group Γ acting on the building.

It is important to observe that there are many periodic apartments. In fact, according to [BB, Theorem 8.9], any compact subset of an apartment is contained in some periodic apartment.

Now let $\mathscr{A}_0$ be a periodic apartment, and fix a special vertex z in $\mathscr{A}_0$. Choose a pair of opposite sectors W^+, W^- in $\mathscr{A}_0$ based at z. Denote by $\omega^\pm$ the boundary points represented by $W^\pm$, respectively. By periodicity of the apartment there is a periodic direction represented by a line L in any of the sector directions of $\mathscr{A}_0$. For definiteness choose this direction to be that of the sector W^+. This means that there is an element $u \in \Gamma$ which leaves L invariant and translates the apartment $\mathscr{A}_0$ in the direction of L. (In the terminology of [BB, Moz], L is said to be an *axis* of u.) Then $u^n\omega^+ = \omega^+$, $u^n\omega^- = \omega^-$ for all $n \in \mathbb{Z}$. Moreover $u^n z$ is in the interior of W^+ for $n > 0$ and in the interior of W^- for $n < 0$. (See Fig. 1. Here and in what follows, the figures illustrate the case of a building Δ of type $\tilde{A}_2$, where each apartment contains precisely six sectors based at a given vertex.) The element u above is the analogue of the maximally hyperbolic elements in [BCH].

The following crucial result shows that ω^- is an attracting fixed point for u^{-1}.

PROPOSITION 3.7. *Let $\mathscr{A}_0$ be a periodic apartment and choose a pair of opposite boundary points $\omega^\pm$. Let $u \in \Gamma$ be an element which translates the apartment $\mathscr{A}_0$ in the direction of ω^+. Then u^{-1} attracts $\mathcal{O}(\omega^+)$ towards ω^-; that is, for each compact subset G of $\mathcal{O}(\omega^+)$ we have* $\lim_{n\to\infty} u^{-n}(G) = \{\omega^-\}$.

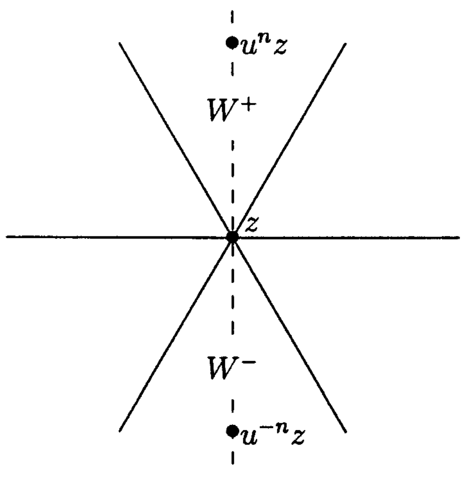

FIG. 1. The periodic apartment A_0.

Proof. We use the notation introduced above. Let $\omega \in \mathcal{O}(\omega^+)$. By considering a retraction of Δ centered at ω^+ [Br, p. 170, VI.8, Theorem], we see that Δ is a union of apartments which contain a subsector of W^+. Moreover for any sector W representing ω there are subsectors $V^+ \subset W^+$ and $V \subset W$ which lie in a common apartment $\mathscr{A}$. Replacing V^+ by a subsector, we may assume that V^+ has base vertex $u^N z$ for some N, that is $V^+ = [u^N z, \omega^+)$. Replacing V by a parallel sector in $\mathscr{A}$ we may also assume that V has base vertex $u^N z$. By Lemma 3.5, V lies in the apartment $\mathscr{A}$ as shown in Fig. 2.

For each $N \geqslant 0$ let G_N denote the set of all boundary points $\omega \in \mathcal{O}(\omega^+)$ such that $[u^N z, \omega)$ and $[u^N z, \omega^+)$ are opposite sectors in some apartment $\mathscr{A}^{(N)}$. Then $G_0 \subset G_1 \subset G_2 \subset \cdots$ is an increasing family of compact open sets and we have observed above that $\bigcup_{N=0}^{\infty} G_N = \mathcal{O}(\omega^+)$. The result will follow if we show that $\lim_{n \to \infty} u^{-n}(G_N) = \{\omega^-\}$ for each $N \geqslant 0$. It is clearly enough to consider the case $N = 0$.

Consider a basic open neighbourhood of ω^- of the form $\Omega_z(v)$, where $v \in [z, \omega^-) \subset \mathscr{A}_0$. Choose an integer $p \geqslant 0$ such that $u^n v \in [z, \omega^+)$ for all $n \geqslant p$. If $\omega \in G_0$ then $u^n v \in [u^n z, \omega)$ (that is $v \in [z, u^{-n}\omega)$) for all $n \geqslant p$. (See Fig. 3.) This means that $u^{-n}\omega \in \Omega_z(v)$ for all $n \geqslant p$. Thus $u^{-n}(G_0) \subset \Omega_z(v)$ for all $n \geqslant p$. This proves the result. ∎

THEOREM 3.8. *Suppose that a group Γ acts properly and cocompactly on the vertices of an affine building Δ with boundary Ω. Let k denote the number of boundary points of an apartment of Δ. Then the action (Ω, Γ) is k-filling.*

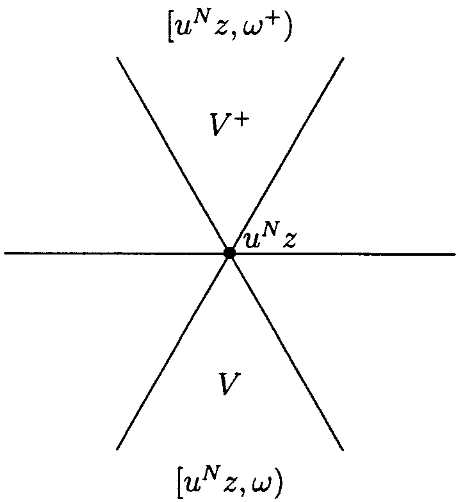

FIG. 2. The apartment $\mathscr{A}$.

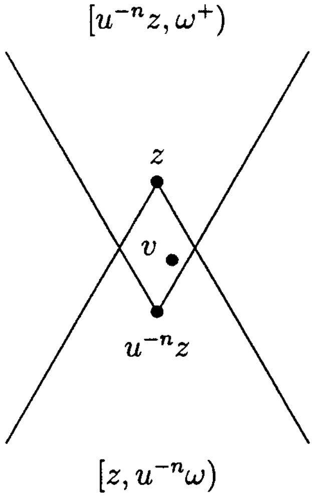

FIG. 3. Sectors in the apartment $u^{-n}\mathscr{A}^{(0)}$.

Proof. Let $U_1, ..., U_k$ be nonempty open subsets of Ω. Let $\mathscr{A}_0$ be a periodic apartment with boundary points ω_j, $1 \leqslant j \leqslant k$. By minimality of the action we can assume that $\omega_j \in U_j$, $1 \leqslant j \leqslant k$. By Corollary 3.3, we have $\Omega = \mathcal{O}(\omega_1) \cup \cdots \cup \mathcal{O}(\omega_n)$. It follows from the existence of a partition of unity that there exist compact sets $K_j \subset \mathcal{O}(\omega_j)$, $1 \leqslant j \leqslant k$ such that $\Omega = K_1 \cup \cdots \cup K_k$.

Let $u_j \in \Gamma$ translate the apartment $\mathscr{A}_0$ in the direction of ω_j, $1 \leqslant j \leqslant k$. Then by Proposition 3.7, there exists $N_j \geqslant 0$ such that $u_j^{-n} K_j \subset U_j$ whenever $n \geqslant N_j$, $1 \leqslant j \leqslant k$. In other words, $K_j \subset u_j^n U_j$ whenever $n \geqslant N_j$, $1 \leqslant j \leqslant k$. Let $t_j = u_j^{N_j}$. Then

$$\Omega = K_1 \cup \cdots \cup K_k \subset t_1 U_1 \cup \cdots \cup t_k U_k$$

as required. ∎

Remark 3.9. The action of an $\tilde{A}_2$ group Γ on the boundary Ω of the associated building is 6-filling. We do not know the precise value of $\phi(\Gamma, \Omega)$, but it is certainly greater than 2. To see this, fix a point $\omega_0 \in \Omega$ and choose U to be a nonempty open set opposite ω_0. If $t_1, t_2 \in \Gamma$ then $t_1 U$ and $t_2 U$ are opposite the boundary points $t_1 \omega_0$ and $t_2 \omega_0$ respectively and therefore cannot cover Ω. To see this, choose a hexagonal apartment of Ω which contains $t_1 \omega_0$ and $t_2 \omega_0$ and choose a chamber ϖ in this apartment which is not opposite $t_1 \omega_0$ or $t_2 \omega_0$. Then ϖ cannot lie in $t_1 U \cup t_2 U$. Therefore $2 < \phi(\Gamma, \Omega) \leqslant 6$.

4. PURELY INFINITE SIMPLE C^*-ALGEBRAS

Throughout this section we consider only affine buildings of type $\tilde{A}_2$. The $\tilde{A}_2$ buildings are a particularly natural setting for our investigation. They are the simplest two-dimensional buildings but they do not necessarily arise from linear groups. Crossed product C^*-algebras associated with them have been studied in [RS1, RS2]. In this case the building Δ is a simplicial complex whose maximal simplices (*chambers*) are triangles. An apartment of Δ is a subcomplex isomorphic to the Euclidean plane tessellated by equilateral triangles.

The boundary Ω may be identified with the flag complex of a projective plane (P, L) [Br, p. 81]. Flags will be denoted (x_1, x_2) where $x_1 \in x_2$. If we identify chambers of Ω with sectors based at a fixed vertex v_0 of type 0, then a sector wall whose base panel is of type 1 corresponds to an element of P and a sector wall whose base panel is of type 2 corresponds to an element of L [Ron, Sect. 9.3]. P is the *minimal boundary* of Δ and has been studied in [CMS], where it is denoted Ω^l. The topology on P comes from the natural quotient map $\Omega \to P$. Moreover the action of Γ on Ω induces an action on P. Similar statements apply to L, and there is a homeomorphism $P \cong L$.

From now on assume that the group Γ is an $\tilde{A}_2$ group; that is, Γ acts simply transitively in a type rotating manner on the vertices of an affine building Δ of type $\tilde{A}_2$.

PROPOSITION 4.1. *The actions (Ω, Γ), (P, Γ) are topologically free. That is, if $g \in \Gamma \backslash \{e\}$ then*

$$\operatorname{Int}\{\omega \in \Omega : g\omega = \omega\} = \emptyset$$

$$\operatorname{Int}\{w \in P : gw = w\} = \emptyset.$$

Proof. The statement for the action on Ω is proved in [RS1, Theorem 4.3.2].

Suppose that the result fails for the action on P. Then there exists an open set $V \subset P$ such that $gw = w$ for all $w \in V$. Let $\tilde{V} = \pi^{-1}(V)$, where $\pi: \Omega \to P$ is the quotient map. Then $\tilde{V}$ is a nonempty open subset of Ω. By [RS1, Proposition 4.3.1], $\tilde{V}$ contains all six boundary points of some apartment $\mathscr{A}$ of Δ. These boundary points are the six chambers of an apartment $\mathscr{A}_0$ in Ω, as illustrated in Fig. 5. The apartment $\mathscr{A}_0$ contains three points w_1, w_2, $w_3 \in P$ (Fig. 4). These three points lie in V and hence are fixed by g. It follows that the lines l_1, l_2, $l_3 \in L$ are also fixed by g. Therefore each boundary point of $\mathscr{A}_0$ is fixed by g. By the proof of [RS1, Theorem 4.3.2], it follows that $g\mathscr{A} = \mathscr{A}$ and g acts by translation on $\mathscr{A}$. The same is true for all nearby apartments $\mathscr{A}'$, since the corresponding

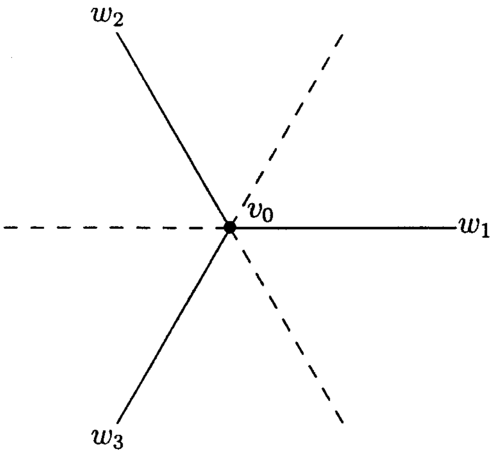

FIG. 4. Sector walls w_1, w_2, w_3 corresponding to points in P.

walls w'_1, w'_2, $w'_3 \in P$ will also be fixed by g, if they belong to V. The argument of [RS1, Theorem 4.3.2] now gives a contradiction. ∎

PROPOSITION 4.2. *If Γ is an $\tilde{A}_2$ group, then the algebras $C(\Omega) \rtimes \Gamma$, $C(P) \rtimes \Gamma$ are simple purely infinite C^*-algebras.*

Proof. The actions are topologically free by Proposition 4.1 and hence properly outer [AS, Proposition 1]. Moreover they are 6-filling by Theorem 3.8. The result follows from Theorem 1.2. ∎

We now give examples of properly outer actions (Ω_i, Γ_i), $i = 1, 2$, with $\phi(\Gamma_1, \Omega_1) = 2$ and $\phi(\Gamma_2, \Omega_2) > 2$ but for which $C(\Omega_1) \rtimes \Gamma_1$ is stably isomorphic to $C(\Omega_2) \rtimes \Gamma_2$.

EXAMPLE 4.3. Let $\Gamma_1 \subset \mathrm{PSL}(2, \mathbb{R})$ be a non-cocompact Fuchsian group isomorphic to $\mathbb{F}_3$, the free group on three generators. Consider the action of Γ_1 on the boundary S^1 of the Poincaré disc. This action is 2-filling and

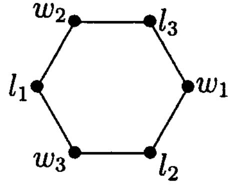

FIG. 5. The apartment $\mathscr{A}_0$.

the algebra $\mathscr{A}_1 = C(S^1) \rtimes \Gamma_1$ is p.i.s.u.n. with K-theory given by $K_0(\mathscr{A}_1) = K_1(\mathscr{A}_1) = \mathbb{Z}^4$, $[\mathbf{1}] = (1, 0, 0, 0,)$ [A-D]. (The K-theory is independent of the embedding $\Gamma_1 \subset \mathrm{PSL}(2, \mathbb{R})$.)

Let Γ_2 be the $\tilde{A}_2$ group B.3 of [CMSZ]. This group is a lattice subgroup of $\mathrm{PGL}_3(\mathbb{Q}_2)$ and acts naturally on the corresponding building of type $\tilde{A}_2$ and its boundary Ω. By Remark 3.9, $2 < \phi(\Gamma, \Omega) \leqslant 6$. By [RS2], the algebra $\mathscr{A}_2 = C(\Omega) \rtimes \Gamma_2$ is p.i.s.u.n. and satisfies the Universal Coefficient Theorem. By [RS3] the K-theory of $\mathscr{A}_2$ is given by $K_0(\mathscr{A}_2) = K_1(\mathscr{A}_2) = \mathbb{Z}^4$, $[\mathbf{1}] = 0$.

It follows from the classification theorem of [Kir] that $\mathscr{A}_1$, $\mathscr{A}_2$ are stably isomorphic (but not isomorphic, since the classes $[\mathbf{1}]$ do not correspond).

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