Application of Symbolic Manipulation to the Hecke Transformations of Modular Forms in Two Variables, II†

HARVEY COHN‡ AND JESSE IRA DEUTSCH§

‡ Department of Mathematics, City College (CUNY), New York, N.Y. 10031; § Mathematics Program, CUNY, 33 West 42 St., New York, N.Y. 10036, U.S.A.

In the first part of this work (to appear in Mathematics of Computation, 1987) it was shown that the nonsymmetric modular forms for $Q(\sqrt{2})$ can be effectively generated by Hecke transformations on the symmetric forms, which are better known and more easily constructed. We show that the nonsymmetric forms need not be known in advance, indeed the polynomial equations defining them in terms of symmetric forms can be derived by symbolic manipulation. The computation was performed using MACSYMA on a DEC VAX computer and came close to the limit of the machine’s speed and capacity.

1. Resume of the Theory

We refer to the earlier work (Cohn & Deutsch, 1987) for the background of the present computation. There we considered Hilbert modular forms for $Q(\sqrt{2})$ represented for computational purposes by their power series in two variables $q$ and $r$ centered at 0.

$$f(z, z') = s(0) + \sum_{a} s(a)q^a r^a$$

where

$$\alpha = a + b\sqrt{2}, a, b \in \mathbb{Z}; \quad \alpha > 0 > \alpha'$$

$$q = \exp \pi i(\alpha + \alpha'), \quad r = \exp \frac{\pi i(z - z')}{\sqrt{2}}$$

$$\text{Im } z > 0, \quad \text{Im } z' > 0$$

and with coefficients $s(a)$ defined over integers of the field $Q(\sqrt{2})$. The modular forms are holomorphic and are called cusp forms if $s(0) = 0$. We consider only forms of even degree $2k$, so

$$f \left[ \frac{ax + \beta}{yz + \delta}, \frac{ax' + \beta'}{yz' + \delta'} \right] \cdot [(yz + \delta)(y'z' + \delta')]^{-2k} = f(z, z')$$

with conjugates as shown. Following the evolving notation (Gundlach, 1965; Cohn, 1982;...

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the Eisenstein series lead to convergent forms written as

\[ H_2(z, z') = 1 + q \left( \frac{48}{r} + 144 + 48r \right) + O(q^2) \]  
(1.4a)

\[ H_4(z, z') = q \left( \frac{1}{r} - 2 + r \right) + O(q^2) \]  
(1.4b)

\[ H_6(z, z') = q + O(q^2) \]  
(1.4c)

where subscripts of \( H \) indicate the degrees. These are symmetric forms satisfying \( f(z, z') = f(z', z) \), and serve as generators of the symmetric forms of even degree.

Now Gundlach's results show nonsymmetric forms to exist. The one of lowest even degree has degree 14. Its \( q-r \) expansion begins as follows:

\[ H_{14}(z, z') = q^2 \left( \frac{1}{r} - r \right) + \ldots \]  
(1.5)

Since the interchange \( z \leftrightarrow z' \) leads to \( r \leftrightarrow 1/r \), note that \( H_{14} \) satisfies the alternating condition \( H_{14}(z, z') = -H_{14}(z', z) \). An explicit formula for \( H_{14} \) remained elusive, however until more advanced methods were used such as the theory of Siegel modular functions and the theory of algebraic curves (see Nagaoka, 1983; Müller, 1983; Hirzebruch, 1977).

The net result was the discovery that

\[ H_{14}^2 = H_4 H_6 (H_2 H_4 + 4H_6) \cdot (H_6 H_2 - 1728 H_6^2 - 288 H_2 H_4 H_6 - 1024 H_4^2 + 4H_4^2 H_2^2). \]  
(1.6)

The last factor is the difficult one which was not explicitly given in Gundlach's paper. It is our present objective.

2. Use of the Hecke Transform

The general purpose of the past and present paper is to show the computational power of the Hecke transformation. We take the transformation, again, in the context of a prime \( \pi \in \mathbb{Z}[\sqrt{2}] \) which comes from the factorisation of a rational prime

\[ p = \pi \pi', \quad \pi > 0, \quad \pi' > 0 \]  
(2.1a)

\[ T_\pi f(z, z') = f(\pi z, \pi' z') p^{2k} + \sum_{j=0}^{p-1} f \left[ \frac{z+j}{\pi}, \frac{z'+j}{\pi'} \right]. \]  
(2.1b)

This operator transforms the space of modular (cusp) forms into itself and therefore is more powerful than the mere use of Fourier series. The action on the coefficients of the Hecke transform is

\[ T_\pi f(z, z') = (p^{2k} + p) s(0) + \sum \left[ p^{2k} s \left[ \frac{\pi}{\pi'} \right] + ps(\pi) \right] q^r a \]  
(2.2)

(see Cohn & Deutsch, 1987). For \( \pi = 2 + \sqrt{2}, \quad \pi' = 2 - \sqrt{2} (p = 2) \) the Hecke transformation is seen to split the symmetric from the alternating forms while for other \( p \) (with \( \pi/\pi' \neq \text{unit} \)) the forms intermix. Thus in the paper just cited we show how to determine the alternating form \( H_{14} \) from some symmetric forms, e.g.

\[ T_3 + \sqrt{2} H_2^2 H_2 - T_3 - \sqrt{2} H_6^2 H_2 = 92,160 \cdot H_{14}. \]  
(2.3)
In doing so we required the prior formula (1.6) from $H_{14}$ however. We shall now derive this formula for $H_{14}$ by the use of $T_n$ for $n = 2 + \sqrt{2}$ alone.

3. The Starting Form

Our starting point is a result following from Gundlach (op. cit. p. 119) that for some factor $G_{12}$ of degree 12 the alternating form $H_{14}$ satisfies the relation

$$H_{14}^2 = H_4 H_6 (H_2 H_4 + 4 H_6) G_{12}. \quad (3.1)$$

The form $G_{12}$ is really Gundlach's $G^2$. In principal, if MACSYMA had greater speed and capacity, we should not need to meet the theory halfway. We could write, in total ignorance of the form (3.1) the trial modular form

$$H_{14} \sim j k = E(A) k H_2 H_4 H_6 \quad (2l + 4j + 6k = 28). \quad (3.2)$$

The sum would have 24 terms and 24 unknown coefficients $A_{ijk}$ and the machine could hunt for several (actually 9) distinct combinations for which the square root of the right hand side is single-valued in $q$ and $r$, and the Hecke transform produces an eigenvector. One of these would be our $H_{14}$ of course; another would be the Eisenstein series of degree 14, etc.

Because of machine limitations, it becomes necessary to proceed from (3.1) with the unknown function $G_{12}$ given as the more restricted expression

$$G_{12} = a H_2 + b H_4 + c H_6 + d H_2^2 H_4 + d H_4^2 H_2 + C H_4^2. \quad (3.3)$$

This is the most general form of degree 12 with $A, B, C, a, b, c, d$ as unknown coefficients. The other factor, however in (3.1) is

$$H_4 H_6 (H_2 H_4 + 4 H_6) = \Sigma A_{ijk} H_i^j H_j^k \quad (2l + 4j + 6k = 28). \quad (3.4)$$

Hence $C = 0$, for otherwise $H_{14}$ would equal

$$\sqrt{C} q^{1/2} (r - \frac{1}{r}) + \ldots$$

and would not be single valued. So $G_{12}$ must likewise start with an odd power of $q$. If we examine the starting terms we see three levels for $q$ depending on which unknown coefficients vanish, namely

$$G_{12} = \begin{cases} 
A q + B q \left( \frac{1}{r} - 2 + r \right) + O(q^3) \\
+ a q^2 + b q^2 \left( r - 2 + \frac{1}{r} \right) + d q^2 \left( r - 2 + \frac{1}{r} \right)^2 + O(q^3) \\
+ c q^3 \left( r - 2 + \frac{1}{r} \right)^3 + O(q^4). 
\end{cases} \quad (3.5)$$

If $A = B = 0$ the odd exponent for $q$ requires $a = b = d = 0$. This leads to a contradiction.
since $H_{14}$ would then have the starting term
\[ \sqrt{c}q^{3}\left(\frac{1}{r}-r\right)\left(r-2+\frac{1}{r}\right)^{3/2}. \]
This would not be permissible since
\[ \left(r-2+\frac{1}{r}\right)^{3/2} \]
is not the square of a single valued polynomial in $r$ and $1/r$. (Note that an infinite power series in $r$ and $1/r$ would violate the condition $|a| < b\sqrt{2}$ in (1.2a)). For the same reason we cannot have $A = 0$ and $B \neq 0$. Hence with $A \neq 0$ we can adjust constants to set $A = 1$ and find that for $H_{14}$ to have a single valued starting term $B$ must equal 0. That is, if
\[ G_{12} = q + Bq\left(r-2+\frac{1}{r}\right) + \ldots \]
as before, no nonzero value of $B$ will make $G_{12}/q$ a perfect square. We thus obtain the starting value
\[ \begin{align*}
H_{14}^2 &= H_4 H_6(H_2 H_4 + 4H_6)G_{12} \\
G_{12} &= H_6 H_2^2 + bH_2 H_4 H_6 + cH_2^2 + dH_2^2 H_4^2
\end{align*} \]
with unknown coefficients $a$, $b$, $c$, $d$.

4. The Computation

We generate the Fourier series (see Cohn & Deutsch op. cit.) for $H_2$, $H_4$, $H_6$ up to $q^7$ and this yields the square root, $H_{14}$ up to $q^5$ in (3.6) as a single-valued power series with coefficients involving $a$, $b$, $c$, $d$. We use the Taylor series facility of MACSYMA extensively.

Since the alternating form $H_{14}$ splits off an eigenspace under $T_{2}^{+}\sqrt{2}$
\[ T_{2}^{+}\sqrt{2}H_{14} = \lambda H_{14}. \] (4.1)
Actually, $\lambda = 80$ as we saw before in Cohn and Deutsch (op. cit.) from the true value of $H_{14}$, but for now we must assume that $\lambda$ and the coefficients in $H_{14}$ are unknown. We match coefficients to check (4.1) as shown in the attached Table 1. We omit those $a$ for which the term $q^{\alpha}r^{\alpha}(\alpha = a + b\sqrt{2})$ has zero coefficient on both sides of (4.1). The expansion up to $q^5$ was necessary to make enough coefficients possible and this degree of accuracy, in essence, exhausted the capacity of MACSYMA!

There are six lines in Table 1, which seems to overdetermine the five unknowns $a$, $b$, $c$, $d$, $\lambda$ but it is necessary to avoid extraneous roots. We make the immediate observations from the indicated lines:

- (iv) $d = 4$
- (ii) $\lambda = -(b + 128)/2$
- (i) $c = 4b + 128$.

This eliminates $d$, $\lambda$, and $c$ and reduces the unknowns to $a$ and $b$. By further use of
Table 1. Coefficients used from equation (4.1)

<table>
<thead>
<tr>
<th></th>
<th>(a)</th>
<th>(a(2 + \sqrt{2}))</th>
<th>(a(2 + \sqrt{2}))</th>
<th>(s(a))</th>
<th>(\lambda a(a))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i)</td>
<td>(2\sqrt{2})</td>
<td>(-2 + 2\sqrt{2})</td>
<td>(4 + 4\sqrt{2})</td>
<td>0</td>
<td>(E_1)</td>
</tr>
<tr>
<td>(ii)</td>
<td>(-1 + 2\sqrt{2})</td>
<td>(2 + 3\sqrt{2})</td>
<td>1</td>
<td>((4d - b - 144)/2)</td>
<td></td>
</tr>
<tr>
<td>(iii)</td>
<td>(-2 + 3\sqrt{2})</td>
<td>(-5 + 4\sqrt{2})</td>
<td>(2 + 4\sqrt{2})</td>
<td>(-(4d - b - 144)/2)</td>
<td>(E_2)</td>
</tr>
<tr>
<td>(iv)</td>
<td>(-3 + 3\sqrt{2})</td>
<td>(3\sqrt{2})</td>
<td>((d - 4)/2)</td>
<td>0</td>
<td>(E_3)</td>
</tr>
<tr>
<td>(v)</td>
<td>(-1 + 3\sqrt{2})</td>
<td>(4 + 5\sqrt{2})</td>
<td>(E_4)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(vi)</td>
<td>(-3 + 4\sqrt{2})</td>
<td>(2 + 5\sqrt{2})</td>
<td>(E_5)</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[
E_1 = -[4d^2 + (-b + 64)d + 2e - 4b - 576]/4
\]
\[
E_2 = -[4120d^2 + (-14b + 4a + 512)d + 28c + 2b^2 - (a + 152)b - 176a - 18304]/4
\]
\[
E_3 = (5d - 2b + a + 348)/2
\]
\[
E_4 = [208d^2 + (-132b + 24a + 2624)d + (176c + 24b^2 + (-6 + 352)b - 480a + 22016)d
+ (-32b + 4a - 704)c - b^3 - 240b^2 - (16a + 5888)b - 2816a + 258048)/16
\]
\[
E_5 = -[27d^2 + (-12b + 2a + 576)d + 24c + b^3 + 96b + 16a - 6312]/8
\]
\[
E_6 = [572d^2 + (-495b + 144a + 18448)d^2
+ (660c + 144b^2 + (-84a - 12480)b + 12a^2 + 3904a - 67296)d + (-192b + 56a + 24960)c
- 14b^3 + (12a + 1952)b^2 + (-3a^2 - 704a + 17592)b - 496a^2 - 262144a + 3417984)/16
\]

Note \(s(a)\) is from \(H_{14}\) and \(\lambda a(a)\) is from \(T_{2,\sqrt{2}H_{14}}\).

Table 1, we determine these equations from the lines shown:

(iii) \(b^2 - (a + 352)b - 160a + 36864 = 0\)

(v) \(b^3 + 280b^2 + (20a + 10240)b + 3328a - 589824 = 0\)

(vi) \(15b^3 - (12a + 1488)b^2 + (3a^2 + 840a - 27648)b + 448a^2 + 240128a - 7077888 = 0\).

From (iii) MACSYMA finds
\[
a = (b^2 - 352b + 36864)/(b + 160)
\]
and substitutes it into (v) and (vi) to produce two polynomial equations in \(b\). These factor as
\[
(b + 288)(b^3 + 172b^2 + 1792b + 98304) = 0
\]
\[
(b + 288)(3b^4 + 1532b^3 - 206208b^2 - 28893184b + 3201302528) = 0.
\]

Hence \(b = -288\), with no extraneous roots since the accompanying factors are seen by MACSYMA to yield a greatest common divisor equal to one. The values are now
\[
a = -1728, \quad b = -288, \quad c = -1024, \quad d = 4, \quad \lambda = 80
\]
and the alternating form \(H_{14}\) is discovered.

5. Concluding Remarks

In using MACSYMA for this calculation we began to run into the 15 minute time limit for any foreground process on the CCNY VAX. To compute the Taylor series for \(G_{12}\) in
terms of $a$, $b$, $c$ and $d$ up to $q^9$ took 4.59 minutes on a MACSYMA with 1.5 megabytes of list space and an augmented "fixnum" region. We had seven "garbage collections" in the calculation with none of the utilisation ratios falling below 80%. To calculate the square of $H_{14}$ using the above $G_{12}$ took 4.91 minutes on the same MACSYMA with 11 "garbage collections" while the utilisation ratio fell as low as 71%. After opening a new MACSYMA session with the square of $H_{14}$ saved from before, we could not calculate the square root within the machine's time limit. Only by truncating the Taylor series for $H_{14}^2$ at $q^5$ could we then take the square root. This operation took 46.5 seconds on a MACSYMA with 1.75 megabytes of list space and only one "garbage collection", and gave the square root up to $q^5$, fortunately enough to nail down $b$ exactly. Indeed $H_{14}$ and its square, expressed in terms of $a$, $b$, $c$ and $d$ consume almost the whole of a MACSYMA file of approximately 56,800 bytes.

Actually, if we are ambitious about more extensive use of symbolic manipulation a more worthy goal would be to generate modular equations, say of norm seven. This has only been done for the case of $Q(\sqrt{2})$ and norm 2 (see Cohn, 1982). For functions of one variable such a goal is not unrealistic, for instance see Kaltofen & Yui (1984a) where the modular equation of order 7 is derived and a more detailed bibliography of the case of one variable is given. Also see Kaltofen & Yui (1984b) for the modular equation of order 11. An alternative p-adic method for obtaining the modular equation has been developed by A. O. L. Atkin. For two variables the corresponding modular equations could surely strain any symbolic system to its breaking point.

The methods of this paper can be extended to other real quadratic fields of class number one and fundamental unit of norm $-1$ for which the ring of modular forms is known, such as $\mathbb{Z}[\frac{1}{\sqrt{5}}]$ (see Gundlach, 1963).

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References


