Totally Positive Matrices

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ABSTRACT

Though total positivity appears in various branches of mathematics, it is rather unfamiliar even to linear algebraists, when compared with positivity. With some unified methods we present a concise survey on totally positive matrices and related topics.

INTRODUCTION

This paper is based on the short lecture, delivered at Hokkaido University, as a complement to the earlier one, Ando (1986).

The importance of positivity for matrices is now widely recognized even outside the mathematical community. For instance, positive matrices play a decisive role in theoretical economics. On the contrary, total positivity is not very familiar even to linear algebraists, though this concept has strong power in various branches of mathematics.

This work is planned as an invitation to total positivity as a chapter of the theory of linear and multilinear algebra. The theory of totally positive matrices originated from the pioneering work of Gantmacher and Krein (1937) and was brought together in their monograph (1960). On the other hand, under the influence of I. Schoenberg, Karlin published the monumental monograph on total positivity, Karlin (1968), which mostly concerns totally positive kernels but also treats the discrete version, totally positive matrices.

Most of the materials of the present paper is taken from these two monographs, but some recent contributions are also incorporated. The novelty

* The research in this paper was supported by a Grant-in-Aid for Scientific Research.

52 Vanderbilt Ave., New York, NY 10017 0024-3795/87/$3.50

is in the systematic use of skew-symmetric products of vectors and Schur complements of matrices as the key tools to derive the results in a transparent way.

The paper is divided into seven sections. In Section 1 classical determinantal identities are proved for later use. The notions of total positivity and sign regularity are introduced in Section 2, and effective criteria for total positivity are presented. Section 3 is devoted to the study of various methods of production of new totally positive matrices from given ones. In Section 4 a simple criterion for a totally positive matrix to have a strictly totally positive power is given. Section 5 is devoted to the study of the relationship between the sign regularity of a matrix and the variation-diminishing property of the linear map it induces. In Section 6 the refined spectral theorems of Perron-Frobenius type are established for totally positive matrices. Examples of totally positive matrices are collected in Section 7. But the most significant results, concerning the total positivity of Toeplitz matrices and translation kernels, are only mentioned without proof.

1. DETERMINANTAL IDENTITIES

This section is devoted to the derivation of classical spectral and determinantal identities, which are used in the subsequent sections. The use of skew-symmetric products of vectors and Schur complements of matrices will unify and simplify the proofs.

For each $n \geqslant 1$, let $\mathscr{H}_n$ stand for the (real or complex) linear spaces of (column) n-vectors $\vec{x} = (x_i)$, equipped with inner product

$$\langle \vec{x}, \vec{y} \rangle := \sum_{i=1}^{n} x_i \bar{y}_i. \tag{1.1}$$

The canonical orthonormal basis of $\mathscr{H}_n$ consists of the vectors $\vec{e}_i$ $(=\vec{e}_i^{(n)})$, $i = 1,2,\dots,n$, with 1 as its ith component, 0 otherwise. A vector $\vec{x} = (x_i)$ is *positive* (respectively, *strictly positive*), in symbols $\vec{x} \geqslant 0$ $(\gg 0)$, if $x_i \geqslant 0$ (>0) for $i = 1,2,\dots,n$.

A linear map from $\mathscr{H}_m$ to $\mathscr{H}_n$ is identified with its $n \times m$ matrix $A = [a_{ij}]$, relative to the canonical basis of $\mathscr{H}_m$ and $\mathscr{H}_n$:

$$a_{ij} = \langle A\vec{e}_j^{(m)}, \vec{e}_i^{(n)} \rangle, \qquad i = 1,2,\dots,n, \quad j = 1,2,\dots,m. \tag{1.2}$$

The linear map A is also identified with the ordered m-tuple of n-vectors:

$A = [\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m]$, where

$$\vec{a}_j = A\vec{e}_j^{(m)}, \qquad j = 1,2,\dots,m. \tag{1.3}$$

A is called *positive* (*strictly positive*), in symbols $A \geqslant 0$ ($A \gg 0$), if it transforms every nonzero positive vector to a positive (strictly positive) vector. Obviously A is positive (strictly positive) if and only if $\vec{a}_i \geqslant 0$ ($\gg 0$), $i = 1,2,\dots,m$, or equivalently, if and only if $a_{ij} \geqslant 0$ (> 0), $i = 1,2,\dots,n$, $j = 1,2,\dots,m$.

For each $k \geqslant 1$, let $\overset{k}{\otimes} \mathscr{H}_n$ denote the *k-tensor space* over $\mathscr{H}_n$. The inner product in $\overset{k}{\otimes} \mathscr{H}_n$ is determined by

$$\langle \vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_k, \vec{y}_1 \otimes \vec{y}_2 \otimes \cdots \otimes \vec{y}_k \rangle = \prod_{i=1}^{k} \langle \vec{x}_i, \vec{y}_i \rangle. \tag{1.4}$$

The canonical orthonormal basis of $\overset{k}{\otimes} \mathscr{H}_n$ is by definition $\{\vec{e}_{i_1}^{(n)} \otimes \vec{e}_{i_2}^{(n)} \otimes \cdots \otimes \vec{e}_{i_k}^{(n)} : 1 \leqslant i_j \leqslant n,\ j = 1,2,\dots,k\}$.

Each linear map A from $\mathscr{H}_m$ to $\mathscr{H}_n$ induces a linear map from $\overset{k}{\otimes} \mathscr{H}_m$ to $\overset{k}{\otimes} \mathscr{H}_n$, called the *k-tensor power* and denoted by $\overset{k}{\otimes} A$:

$$\left(\overset{k}{\otimes} A \right) (\vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_k) = (A\vec{x}_1) \otimes (A\vec{x}_2) \otimes \cdots \otimes (A\vec{x}_k). \tag{1.5}$$

If B is a linear map from $\mathscr{H}_l$ to $\mathscr{H}_m$, then it follows from (1.5) that

$$\overset{k}{\otimes} (AB) = \left(\overset{k}{\otimes} A \right) \cdot \left(\overset{k}{\otimes} B \right). \tag{1.6}$$

Let S_k denote the *symmetric group* of degree k, that is, the group of all permutations of $\{1,2,\dots,k\}$. Each $\pi \in \mathrm{S}_k$ gives rise to a linear map $P_\pi^{(n)}$ of $\overset{k}{\otimes} \mathscr{H}_n$, determined by

$$P_\pi^{(n)}(\vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_k) = \vec{x}_{\pi^{-1}(1)} \otimes \vec{x}_{\pi^{-1}(2)} \otimes \cdots \otimes \vec{x}_{\pi^{-1}(k)}. \tag{1.7}$$

A k-tensor $\vec{\mathbf{x}}$ is called *skew-symmetric* if

$$P_\pi^{(n)}\vec{\mathbf{x}} = \operatorname{sgn}\pi \cdot \vec{\mathbf{x}} \qquad \text{for any} \quad \pi \in \mathbf{S}_k, \tag{1.8}$$

where $\operatorname{sgn}\pi = 1$ or -1 according as π is an even or odd permutation. The subspace of all skew-symmetric k-tensors over $\mathscr{H}_n$ is called the kth *skew-symmetric* (or kth *Grassmann*) *space* over $\mathscr{H}_n$, and denoted by $\overset{k}{\bigwedge}\mathscr{H}_n$. The orthogonal projection $\mathbf{P}_k^{(n)}$ to $\overset{k}{\bigwedge}\mathscr{H}_n$ is given by

$$\mathbf{P}_k^{(n)} = \frac{1}{k!} \sum_{\pi \in \mathbf{S}_k} \operatorname{sgn}\pi \cdot P_\pi^{(n)}. \tag{1.9}$$

The k-tensor

$$\vec{x}_1 \wedge \vec{x}_2 \wedge \cdots \wedge \vec{x}_k := \mathbf{P}_k^{(n)}(\vec{x}_1 \otimes \vec{x}_2 \otimes \cdots \otimes \vec{x}_k) \tag{1.10}$$

is called the kth *skew-symmetric product* of the ordered k-tuple $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$. Then it follows from (1.8) and (1.10) that

$$\vec{x}_{\pi^{-1}(1)} \wedge \vec{x}_{\pi^{-1}(2)} \wedge \cdots \wedge \vec{x}_{\pi^{-1}(k)} = \operatorname{sgn}\pi \cdot \vec{x}_1 \wedge \vec{x}_2 \wedge \cdots \wedge \vec{x}_k. \tag{1.11}$$

Further, it follows from (1.4), via the definition of determinant, that

$$\langle \vec{x}_1 \wedge \vec{x}_2 \wedge \cdots \wedge \vec{x}_k, \vec{y}_1 \wedge \vec{y}_2 \wedge \cdots \wedge \vec{y}_k \rangle = \frac{1}{k!} \det\left[\langle \vec{x}_i, \vec{y}_j \rangle\right], \tag{1.12}$$

where det means determinant. A consequence is that $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_k\}$ is linearly dependent if and only if $\vec{x}_1 \wedge \vec{x}_2 \wedge \cdots \wedge \vec{x}_k = 0$.

It follows from (1.5) and (1.7) that, for each linear map A from $\mathscr{H}_m$ to $\mathscr{H}_n$, its k-tensor power $\overset{k}{\otimes} A$ *intertwines* $P_\pi^{(n)}$ and $P_\pi^{(m)}$ in the sense that

$$P_\pi^{(n)} \cdot \left(\overset{k}{\otimes} A \right) = \left(\overset{k}{\otimes} A \right) \cdot P_\pi^{(m)} \qquad \text{for} \quad \pi \in \mathbf{S}_k. \tag{1.13}$$

Therefore $\overset{k}{\otimes} A$ intertwines the projections $\mathbf{P}_k^{(n)}$ and $\mathbf{P}_k^{(m)}$:

$$\mathbf{P}_k^{(n)} \cdot \left(\overset{k}{\otimes} A \right) = \left(\overset{k}{\otimes} A \right) \cdot \mathbf{P}_k^{(m)}. \tag{1.14}$$

The restriction of $\overset{k}{\otimes} A$ to the skew-symmetric space is called the *k-exterior power* of A, and denoted by $\overset{k}{\bigwedge} A$. In view of (1.5), the exterior power $\overset{k}{\bigwedge} A$ is determined by the formula

$$\left(\overset{k}{\bigwedge} A\right)(\vec{x}_1 \wedge \vec{x}_2 \wedge \cdots \wedge \vec{x}_k) = (A\vec{x}_1) \wedge (A\vec{x}_2) \wedge \cdots \wedge (A\vec{x}_k). \quad (1.15)$$

If I_n stands for the identity map of $\mathscr{H}_n$, then

$$\overset{k}{\bigwedge} I_n = I_{\overset{k}{\wedge}\mathscr{H}_n}, \quad (1.16)$$

the right-hand side being the identity map of $\overset{k}{\bigwedge}\mathscr{H}_n$. It follows from (1.5) or (1.15) that if B is a linear map from $\mathscr{H}_l$ to $\mathscr{H}_m$,

$$\overset{k}{\bigwedge}(AB) = \left(\overset{k}{\bigwedge} A\right) \cdot \left(\overset{k}{\bigwedge} B\right). \quad (1.17)$$

A consequence of (1.16) and (1.17) is that if A is an invertible map of $\mathscr{H}_n$, then $\overset{k}{\bigwedge} A$ is invertible, and

$$\left(\overset{k}{\bigwedge} A\right)^{-1} = \overset{k}{\bigwedge} A^{-1}. \quad (1.18)$$

When $1 \leqslant k \leqslant n$, $Q_{k,n}$ will denote the totality of strictly increasing sequences of k integers chosen from $\{1,2,\ldots,n\}$:

$$Q_{k,n} \in \alpha = (\alpha_i), \qquad (1 \leqslant)\ \alpha_1 < \alpha_2 < \cdots < \alpha_k\ (\leqslant n). \quad (1.19)$$

The order relation $\alpha \leqslant \beta$ for $\alpha, \beta \in Q_{k,n}$ means by definition that $\alpha_i \leqslant \beta_i$, $i = 1, 2, \ldots, k$. The *complement* α' is the increasingly rearranged $\{1,2,\ldots,n\} \setminus \alpha$, so that α' is an element of $Q_{n-k,n}$. When $\alpha \in Q_{k,n}$, $\beta \in Q_{l,n}$, and $\alpha \cap \beta = \varnothing$, their union $\alpha \cup \beta$ should be always rearranged increasingly to become an element of $Q_{k+l,n}$.

For each $\alpha \in Q_{k,n}$, its *dispersion number* $d(\alpha)$ is defined by

$$d(\alpha) := \sum_{i=1}^{k-1} (\alpha_{i+1} - \alpha_i - 1) = \alpha_k - \alpha_1 - (k-1), \tag{1.20}$$

with the convention $d(\alpha) = 0$ for $\alpha \in Q_{1,n}$. Then $d(\alpha) = 0$ means that α consists of k consecutive integers. For $\alpha \in Q_{k,n}$ the *α-projection* of an n-vector $\vec{x} = (x_i)$ is the k-vector with components $x_{\alpha_1}, x_{\alpha_2}, \ldots, x_{\alpha_k}$. The space of all α-projections is denoted by $\mathscr{H}_\alpha$, that is, $\mathscr{H}_\alpha$ is $\mathscr{H}_k$ indexed by $(\alpha_1, \alpha_2, \ldots, \alpha_k)$.

Let A be an $n \times m$ matrix, $\alpha \in Q_{k,n}$, and $\beta \in Q_{l,m}$. Then $A[\alpha|\beta]$ is by definition the $k \times l$ submatrix of A using rows numbered by α and columns numbered by β. If A is considered a linear map from $\mathscr{H}_m$ to $\mathscr{H}_n$, then $A[\alpha|\beta]$ is one from $\mathscr{H}_\beta$ to $\mathscr{H}_\alpha$. When $\alpha = \beta$, $A[\alpha|\alpha]$ is simply denoted by $A[\alpha]$. Further we shall use the following notation:

$$A[\alpha|\beta) := A[\alpha|\beta'], \qquad A(\alpha|\beta] := A[\alpha'|\beta],$$
$$A(\alpha|\beta) := A[\alpha'|\beta'], \qquad A(\alpha) := A[\alpha'|\alpha'],$$

and

$$A[\text{-}|\beta] := A[1,2,\ldots,n|\beta], \qquad A[\alpha|\text{-}] := A[\alpha|1,2,\ldots,m],$$
$$A[\text{-}|\beta) := A[1,2,\ldots,n|\beta'], \qquad A(\alpha|\text{-}] := A[\alpha'|1,2,\ldots,m].$$

Given $\alpha \in Q_{k,n}$, let us use the abbreviation

$$\vec{\mathbf{e}}_\alpha^{\wedge} \left(= \vec{\mathbf{e}}^{(n)\wedge}_{\alpha} \right) := \vec{e}_{\alpha_1}^{(n)} \wedge \vec{e}_{\alpha_2}^{(n)} \wedge \cdots \wedge \vec{e}_{\alpha_k}^{(n)}. \tag{1.21}$$

Then by (1.12), $\{\sqrt{k!}\,\vec{\mathbf{e}}_\alpha^{\wedge} : \alpha \in Q_{k,n}\}$ becomes a complete orthonormal system of the k-skew-symmetric space over $\mathscr{H}_n$, and is taken as the canonical orthonormal basis of $\bigwedge^k \mathscr{H}_n$. Therefore the notions of positivity for a k-skew-symmetric tensor and a linear map between skew-symmetric spaces always refer to these canonical basis. According to (1.12) and (1.21), for a linear map A from $\mathscr{H}_m$ to $\mathscr{H}_n$, the (α, β) entry of the matrix of $\bigwedge^k A$ is determined by

$$k! \left\langle \left(\bigwedge^k A \right) \vec{\mathbf{e}}^{(m)\wedge}_{\beta}, \vec{\mathbf{e}}^{(n)\wedge}_{\alpha} \right\rangle = \det A[\alpha|\beta]. \tag{1.22}$$

Therefore $\bigwedge^k A$ is positive (strictly positive), in symbols $\bigwedge^k A \geqslant 0$ ($\gg 0$) if and only if $\det A[\alpha|\beta] \geqslant 0$ (> 0) for any $\alpha \in Q_{k,n}$ and $\beta \in Q_{k,m}$, or equivalently if and only if $\vec{a}_{\beta_1} \wedge \vec{a}_{\beta_2} \wedge \cdots \wedge \vec{a}_{\beta_k} \geqslant 0$ ($\gg 0$) for any $\beta \in Q_{k,n}$.

In the rest of this section, we assume $n = m$, so that A, B are n-square matrices. First of all, the multiplication law (1.17) produces, via (1.22), the following determinantal identity:

$$\det(AB)[\alpha|\beta] = \sum_{\omega \in Q_{k,n}} \det A[\alpha|\omega] \cdot \det B[\omega|\beta] \qquad \text{for} \quad \alpha, \beta \in Q_{k,n}. \tag{1.23}$$

Given an n-square matrix, it is sometimes convenient to consider its *adjoint* A^* and its *conversion* $A^{\#}$, whose (i, j) entries are given by $\bar{a}_{ji}$ and $a_{n-i+1,n-j+1}$, respectively. Then it is immediate from the definition that for $\alpha, \beta \in Q_{k,n}$ one has $\det A^*[\alpha|\beta] = \overline{\det A[\alpha|\beta]}$ and $\det A^{\#}[\alpha|\beta] = \det A[\alpha^{\#}|\beta^{\#}]$, where $(\alpha^{\#})_i = n - \alpha_i + 1$, $i = 1,2,\ldots, k$, and similarly for $\beta^{\#}$.

For a linear map A on $\mathscr{H}_n$ we can speak about its *spectrum*, the set of complex numbers λ for which $\lambda I_n - A$ is not invertible, or equivalently $A\vec{x} = \lambda\vec{x}$ has a nonzero solution $\vec{x}$. When $A\vec{x} = \lambda\vec{x}$, then λ is usually called an *eigenvalue* of A, and $\vec{x}$ an *eigenvector* corresponding to λ. Therefore the spectrum consists of all eigenvalues. Since the noninvertibility of an n-square matrix is equivalent to the linear dependence of its n column (or row) vectors, λ is an eigenvalue of A if and only if it is a root of the polynomial $\det(\lambda I_n - A)$ of degree n. The *multiplicity* of an eigenvalue λ is by definition the multiplicity of λ as a root of $\det(\lambda I_n - A)$.

Let $\vec{\lambda}(A) = (\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A))$ stand for the eigenvalues of an n-square matrix A, arranged in modulus-decreasing order:

$$|\lambda_1(A)| \geqslant |\lambda_2(A)| \geqslant \cdots \geqslant |\lambda_n(A)|,$$

with multiplicities counted. For each n-tuple of complex numbers $\{\xi_1, \xi_2, \ldots, \xi_n\}$, let $\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ denote the diagonal matrix with diagonal entries $\{\xi_1, \xi_2, \ldots, \xi_n\}$. Obviously the eigenvalues of this diagonal matrix coincide with $\xi_1, \xi_2, \ldots, \xi_n$, with multiplicities counted. A matrix A is called *diagonalizable* if it is similar to a diagonal matrix, that is, there are a diagonal matrix $\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ and an invertible matrix T such that $A = T \cdot \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n) \cdot T^{-1}$. In this case the eigenvalues of A are just $\{\xi_1, \xi_2, \ldots, \xi_n\}$ with multiplicities counted.

THEOREM 1.1. *Let A be an n-square matrix. Then for each $1 \leqslant k \leqslant n$, the eigenvalues of the $\binom{n}{k}$-square matrix $\bigwedge^k A$ are given by $\prod_{i=1}^k \lambda_{\alpha_i}(A)$, $\alpha \in Q_{k,n}$, with multiplicities counted.*

Proof. Since the set of diagonalizable matrices is dense in the space of n-square matrices, and the spectrum depends continuously on matrix entries, we may assume that A is diagonalizable. Therefore with $\lambda_i = \lambda_i(A)$, $i = 1,2,\dots,n$, we have $A = T \cdot \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \cdot T^{-1}$ for some invertible T. Then according to (1.17) and (1.18)

$$\bigwedge^k A = \left(\bigwedge^k T\right) \cdot \bigwedge^k \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n) \cdot \left(\bigwedge^k T\right)^{-1}.$$

But it is readily seen that $\bigwedge^k \mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$ has eigenvalue $\prod_{i=1}^k \lambda_{\alpha_i}$, $\alpha \in Q_{k,n}$. ■

Let A be an n-square matrix, $1 \leqslant k \leqslant n$, and $\alpha, \beta \in Q_{k,n}$. When $A[\alpha|\beta]$ is invertible, the *Schur complement* of $A[\alpha|\beta]$ in A, in symbols $A/[\alpha|\beta]$, is defined as the following $(n-k)$-square matrix indexed by α', β':

$$A/[\alpha|\beta] = A(\alpha|\beta) - A(\alpha|\beta] \cdot A[\alpha|\beta]^{-1} \cdot A[\alpha|\beta). \tag{1.24}$$

When $\alpha = \beta$, we shall use A/α for $A/[\alpha|\beta]$.

For $\alpha \in Q_{k,n}$, $\mathrm{sgn}(\alpha)$ is defined as $\mathrm{sgn}(\pi)$ of the permutation $\pi \in S_n$ that assigns α_i to i for $i = 1,2,\dots,k$ and α'_j to $k+j$ for $j = 1,2,\dots,n-k$, so that

$$\mathrm{sgn}(\alpha) = (-1)^{\sum_1^k \alpha_i - k(k+1)/2}. \tag{1.25}$$

Correspondingly, for $\alpha \in Q_{k,n}$, let T_α stand for the linear map on $\mathscr{H}_n$ such that

$$T_\alpha \vec{e}_i = \vec{e}_{\alpha_i}, \quad i = 1,2,\dots,k, \quad \text{and} \quad T_\alpha \vec{e}_{k+j} = \vec{e}_{\alpha'_j}, \quad j = 1,2,\dots,n-k. \tag{1.26}$$

Obviously T_α is unitary and

$$\det T_\alpha = \mathrm{sgn}(\alpha). \tag{1.27}$$

THEOREM 1.2. *Let A be an n-square matrix, $1 \leqslant k \leqslant n$, and $\alpha, \beta \in Q_{k,n}$. If $A[\alpha|\beta]$ is invertible, then*

$$\det A = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)\det A[\alpha|\beta]\det(A/[\alpha|\beta]). \tag{1.28}$$

If, in addition, A is invertible, so is $A/[\alpha|\beta]$ and

$$A^{-1}(\beta|\alpha) = (A/[\alpha|\beta])^{-1}. \tag{1.29}$$

Proof. Let us begin with the special case $\alpha = \beta = \{1,2,\dots,k\}$. Since A admits the factorization

$$A = \begin{bmatrix} I_n[\alpha] & 0 \\ A(\alpha|\alpha]A[\alpha]^{-1} & I_n(\alpha) \end{bmatrix} \begin{bmatrix} A[\alpha] & 0 \\ 0 & A/\alpha \end{bmatrix} \begin{bmatrix} I_n[\alpha] & A[\alpha]^{-1}A[\alpha|\alpha) \\ 0 & I_n(\alpha) \end{bmatrix}, \tag{1.30}$$

(1.28) is immediate, because the left and the right factors on the right hand side of (1.30) have determinant 1, while the middle factor has determinant $\det A[\alpha]\det(A/\alpha)$. Also, (1.29) follows from (1.30), on taking inverses of both sides.

Turning to the general case, consider the maps T_α, T_β in (1.26). Then it follows immediately from (1.2) and (1.26) that, with suitable identifications of indices,

$$\left(T_\alpha^{-1}AT_\beta\right)[1,2,\dots,k] = A[\alpha|\beta], \qquad \left(T_\alpha^{-1}AT_\beta\right)(1,2,\dots,k) = A(\alpha|\beta),$$

$$\left(T_\alpha^{-1}AT_\beta\right)(1,2,\dots,k|1,2,\dots,k] = A(\alpha|\beta],$$

$$\left(T_\alpha^{-1}AT_\beta\right)[1,2,\dots,k|1,2,\dots,k) = A[\alpha|\beta);$$

hence

$$\left(T_\alpha^{-1}AT_\beta\right)/\{1,2,\dots,k\} = A/[\alpha|\beta].$$

Now (1.28) follows from the special case proved above, by using (1.27). Finally (1.29) results from the following relation and the special case proved above:

$$A^{-1}(\beta|\alpha) = \left(T_\beta^{-1}A^{-1}T_\alpha\right)(1,2,\dots,k) = \left(T_\alpha^{-1}AT_\beta\right)^{-1}(1,2,\dots,k). \qquad \blacksquare$$

Since an n-square matrix is approximated arbitrarily closely by matrices all square submatrices of which are invertible, in deriving various determinantal identities from Theorem 1.2 we can assume that all square submatrices of A are invertible.

If an n-square matrix A is invertible, then

$$\det A^{-1}[\alpha|\beta] = \operatorname{sgn}(\alpha)\cdot\operatorname{sgn}(\beta)\,\frac{\det A(\beta|\alpha)}{\det A} \qquad \text{for} \quad \alpha,\beta \in Q_{k,n}. \tag{1.31}$$

This follows from (1.29) and (1.28), applied to α' and β' in place of α and β. Let $J_n := \operatorname{diag}(1,-1,1,-1,\dots,(-1)^{n-1})$. Since $\det J_n[\alpha|\omega] = \operatorname{sgn}(\alpha)\cdot(-1)^{k(k-1)/2}$ or $=0$ according as $\omega = \alpha$ or $\neq \alpha$, the following identity follows from (1.31), by using (1.23):

$$\det(J_n A^{-1} J_n)[\alpha|\beta] = \frac{\det A(\beta|\alpha)}{\det A} \qquad \text{for} \quad \alpha,\beta \in Q_{k,n}. \tag{1.32}$$

When $k=1$, (1.32) means that

$$[(i,j) \text{ entry of } A^{-1}] = (-1)^{i+j}\frac{\det A(j|i)}{\det A}, \qquad i,j = 1,2,\dots,n. \tag{1.33}$$

The following identity holds for a general n-square matrix A:

$$\sum_{\omega\in Q_{k,n}} \operatorname{sgn}(\omega)\det A[\alpha|\omega]\det A(\beta|\omega) = \operatorname{sgn}(\beta)\det A\cdot\delta_{\alpha,\beta} \qquad \text{for} \quad \alpha,\beta\in Q_{k,n}, \tag{1.34}$$

where $\delta_{\alpha,\beta} = 1$ or $=0$ according as $\alpha=\beta$ for $\neq\beta$. In fact, when A is invertible, by (1.31) the left hand side of (1.34) is equal to

$$\operatorname{sgn}(\beta)\det A \sum_{\omega\in Q_{k,n}} \det A[\alpha|\omega]\det A^{-1}[\omega|\beta],$$

which coincides with the right hand side by (1.23).

If $\alpha,\beta\in Q_{k,n}$ and $\omega,\tau\in Q_{l,n}$ are such that $\omega\subset\alpha'$ and $\tau\subset\beta'$, then

$$\begin{aligned}&\det A[\alpha|\beta]\det(A/[\alpha|\beta])[\omega|\tau]\\ &\qquad = \operatorname{sgn}(\alpha/\alpha\cup\omega)\operatorname{sgn}(\beta/\beta\cup\tau)\det A[\alpha\cup\omega|\beta\cup\tau],\end{aligned} \tag{1.35}$$

where $\operatorname{sgn}(\alpha/\alpha\cup\omega)$ and $\operatorname{sgn}(\beta/\beta\cup\tau)$ are defined as follows: let $\mu := \alpha\cup\omega$

$= (\mu_1, \mu_2, \dots, \mu_{k+l})$ and $\nu := \beta \cup \tau = (\nu_1, \nu_2, \dots, \nu_{k+l})$, and let $\alpha_i = \mu_{\gamma_i}$, $\beta_i = \nu_{\sigma_i}$, $i = 1,2,\dots,k$. Then

$$\operatorname{sgn}(\alpha/\alpha \cup \omega) = (-1)^{\Sigma_1^k \gamma_i - k(k+1)/2}, \qquad \operatorname{sgn}(\beta/\beta \cup \tau) = (-1)^{\Sigma_1^k \sigma_i - k(k+1)/2}. \tag{1.36}$$

To see (1.35), consider the $(k+l)$-square matrix $B = [b_{ij}]$, defined by $b_{ij} = a_{\mu_i \nu_j}$, $i, j = 1,2,\dots,k+l$. Then it is readily seen that (1.35) is just (1.28) with B, γ, σ in place of A, α, β respectively. An immediate consequence is

$$\{(\alpha_i', \beta_j') \text{ entry of } A/[\alpha|\beta]\}$$

$$= \operatorname{sgn}(\alpha/\alpha \cup \{\alpha_i'\}) \operatorname{sgn}(\beta/\beta \cup \{\beta_j'\}) \frac{\det A[\alpha \cup \{\alpha_i'\} | \cup \{\beta_j'\}]}{\det A[\alpha|\beta]}. \tag{1.37}$$

Further, for any n-square A and $\alpha, \beta \in Q_{k,n}$,

$$\det\left(\left[\det A[\alpha \cup \{\alpha_i'\} | \beta \cup \{\beta_j'\}]\right]_{i,j=1,2,\dots,n-k}\right) = \det A \det A[\alpha|\beta]^{n-k-1}. \tag{1.38}$$

In fact, with $\xi_{\alpha_i'} = \operatorname{sgn}(\alpha/\alpha \cup \{\alpha_i'\})$ and $\eta_{\beta_j'} = \operatorname{sgn}(\beta/\beta \cup \{\beta_j'\})$, $i, j = 1,2,\dots,n-k$, it follows from (1.37) that the left hand side of (1.38) is equal to

$$\det A[\alpha|\beta]^{n-k} \det\left[\operatorname{diag}(\xi_{\alpha_1'}, \xi_{\alpha_2'}, \dots, \xi_{\alpha_{n-k}'}) \cdot A/[\alpha|\beta] \cdot \operatorname{diag}(\eta_{\beta_1'}, \eta_{\beta_2'}, \dots, \eta_{\beta_{n-k}'})\right]$$

$$= \det A[\alpha|\beta]^{n-k} \cdot \prod_{i=1}^{n-k} \operatorname{sgn}(\alpha/\alpha \cup \{\alpha_i'\}) \det(A/[\alpha|\beta])$$

$$\times \prod_{j=1}^{n-k} \operatorname{sgn}(\beta/\beta \cup \{\beta_j'\}).$$

But is is readily seen from (1.25) and (1.36) that

$$\prod_{i=1}^{n-k} \operatorname{sgn}(\alpha/\alpha \cup \{\alpha_i'\}) = \operatorname{sgn}(\alpha) \quad \text{and} \quad \prod_{j=1}^{n-k} \operatorname{sgn}\big(\beta/\beta \cup \{\beta_j'\}\big) = \operatorname{sgn}(\beta),$$

and (1.38) follows from (1.28).

If A is an n-square matrix, $\alpha \in Q_{n-1,n}$, $\omega \in Q_{n-2,n}$, and $\omega \subset \alpha$, then for $1 < q < n$

$$\det A[\omega|1,n)\det A[\alpha|q) = \det A[\omega|1,q)\det A[\alpha|n) + \det A[\omega|q,n)\det A[\alpha|1). \quad (1.39)$$

To see (1.39), fix $p \in \omega$ and let $\mu := \omega \backslash \{p\}$ and $\nu := \{1,q,n\}'$. Further let $\{m\} = \alpha \setminus \omega$ and $B := A/[\mu|\nu]$. In view of (1.35) and (1.37), on dividing both sides of (1.39) by $\det A[\mu|\nu]^2$ and factoring out $\operatorname{sgn}(\mu/\mu \cup \{p\})\operatorname{sgn}(\mu/\mu \cup \{p,m\})$, it is readily seen that (1.39) is equivalent to the following relation:

$$\begin{aligned}
&\operatorname{sgn}(\nu/\nu \cup \{q\})\operatorname{sgn}(\nu/\nu \cup \{1,n\}) \cdot b_{pq} \det B[p,m|1,n] \\
&\quad = \operatorname{sgn}(\nu/\nu \cup \{n\})\operatorname{sgn}(\nu/\nu \cup \{1,q\}) \cdot b_{pn} \det B[p,m|1,q] \\
&\qquad + \operatorname{sgn}(\nu/\nu \cup \{1\})\operatorname{sgn}(\nu/\nu \cup \{q,n\}) \cdot b_{p1} \det B[p,m|q,n].
\end{aligned}$$

Next it follows from (1.36) that

$$\begin{aligned}
\operatorname{sgn}(\nu/\nu \cup \{q\})\operatorname{sgn}(\nu/\nu \cup \{1,n\}) &= \operatorname{sgn}(\nu/\nu \cup \{n\})\operatorname{sgn}(\nu/\nu \cup \{1,q\}) \\
&= \operatorname{sgn}(\nu/\nu \cup \{1\})\operatorname{sgn}(\nu/\nu \cup \{q,n\}) \\
&= (-1)^{q-1}.
\end{aligned}$$

Therefore (1.39) is finally equivalent to the following relation, which is easily checked:

$$b_{pq} \det B[p,m|1,n] = b_{pn} \det B[p,m|1,q] + b_{p1} \det B[p,m|q,n].$$

We close this preliminary section with an interesting chain rule for Schur complements, though it is not used explicitly in the subsequent part.

THEOREM 1.3. *Let A be an n-square matrix, and suppose that $A[\alpha|\beta]$ is invertible for some $\alpha, \beta \in Q_{k,n}$. If $\omega, \tau \in Q_{l,n}$, $\omega \subset \alpha'$, and $\tau \subset \beta'$, then the invertibility of $(A/[\alpha|\beta])[\omega|\tau]$ is equivalent to that of $A[\alpha \cup \omega|\beta \cup \tau]$. In this case the following relation holds;*

$$(A/[\alpha|\beta])/[\omega|\tau] = A/[\alpha \cup \omega|\beta \cup \tau]. \tag{1.40}$$

Proof. The first assertion is immediate from (1.35). And (1.40) is equivalent to the relation

$$\det((A/[\alpha|\beta])/[\omega|\tau])[\mu|\nu] = \det(A/[\alpha \cup \omega|\beta \cup \tau])[\mu|\nu] \tag{1.41}$$

for any $\mu, \nu \in Q_{p,n}$ such that $\mu \subset (\alpha \cup \omega)'$ and $\nu \subset (\beta \cup \tau)'$. But again, according to (1.35), the left hand side of (1.41) is equal to

$$\operatorname{sgn}(\omega/\omega \cup \mu)\operatorname{sgn}(\alpha/\alpha \cup \omega)\operatorname{sgn}(\alpha/\alpha \cup \omega \cup \mu)\operatorname{sgn}(\tau/\tau \cup \nu)$$
$$\times \operatorname{sgn}(\beta/\beta \cup \tau)\operatorname{sgn}(\beta/\beta \cup \tau \cup \nu) \times \frac{\det A[\alpha \cup \omega \cup \mu|\beta \cup \tau \cup \nu]}{\det A[\alpha \cup \omega|\beta \cup \tau]}$$

while the right hand side is equal to

$$\operatorname{sgn}(\alpha \cup \omega/\alpha \cup \omega \cup \mu)\operatorname{sgn}(\beta \cup \tau/\beta \cup \tau \cup \nu)\frac{\det A[\alpha \cup \omega \cup \mu|\beta \cup \tau \cup \nu]}{\det A[\alpha \cup \omega|\beta \cup \tau]}.$$

It is readily seen from (1.36) that

$$\operatorname{sgn}(\omega/\omega \cup \mu)\operatorname{sgn}(\alpha/\alpha \cup \omega)\operatorname{sgn}(\alpha/\alpha \cup \omega \cup \mu) = \operatorname{sgn}(\alpha \cup \omega/\alpha \cup \omega \cup \mu),$$

and

$$\operatorname{sgn}(\tau/\tau \cup \nu)\operatorname{sgn}(\beta/\beta \cup \tau)\operatorname{sgn}(\beta/\beta \cup \tau \cup \nu) = \operatorname{sgn}(\beta \cup \tau/\beta \cup \tau \cup \nu),$$

which proves (1.41), and hence (1.40). ∎

Notes and References to Section 1

We use mostly the notations of Marcus (1973). The $\binom{n}{k}$-square matrix $[\det A[\alpha|\beta]]_{\alpha,\beta \in Q_{k,n}}$ is called the kth compound of A. Theorem 1.1 is the Kronecker theorem, while (1.23) is the Binet-Cauchy theorem [see

Gantmacher (1953)]. More about tensor spaces and skew-symmetric spaces can be found in Marcus (1973).

Recently de Boor and Pinkus (1982) and Brualdi and Schneider (1983) also used the Schur complement as a unifying principle in deriving various classical determinantal identities. When $\alpha = \beta$ the identity (1.28) in Theorem 1.2 appeared first in Schur (1917), and the notion of Schur complement was explicitly introduced in Haynsworth (1968). The matrix $[(-1)^{i+j}\det A(j|i)]$ is called the adjugate of A and is denoted by $\operatorname{adj} A$; thus $\operatorname{adj} A = \det A \cdot J_n A^{-1} J_n$. Equations (1.31) and (1.38) are known as the Jacobi identity and the Sylvester identity respectively. The quotient formula (1.40) in Theorem 1.3 is due to Crabtree and Haynsworth (1969) and Ostrowski (1971). More about Schur complements can be found in Quellette (1981) and Carlson (1986).

2. CRITERIA FOR TOTAL POSITIVITY

In this section we introduce fundamental notions for our theme: sign regularity and total positivity.

By a *signature sequence* we mean an (infinite) real sequence $\vec{\varepsilon} = (\varepsilon_i)$ with $|\varepsilon_i| = 1$, $i = 1,2,\ldots$. The multiple of a signature sequence $\vec{\varepsilon}^{(1)} = (\varepsilon_i^{(1)})$ by a unimodular real ε and the product of $\vec{\varepsilon}^{(1)}$ and another signature sequence $\vec{\varepsilon}^{(2)} = (\varepsilon_i^{(2)})$ are those signature sequences defined by $(\varepsilon\varepsilon_i^{(1)})$ and $(\varepsilon_i^{(1)}\varepsilon_i^{(2)})$ respectively.

An $n \times m$ matrix A is called *sign-regular with signature* $\vec{\varepsilon}$ if

$$\varepsilon_k \cdot \bigwedge^k A \geqslant 0, \qquad k = 1,2,\ldots,\min(n,m). \tag{2.1}$$

The sign regularity of A is equivalent to the condition

$$\varepsilon_k \cdot \vec{a}_{\beta_1} \wedge \vec{a}_{\beta_2} \wedge \cdots \wedge \vec{a}_{\beta_k} \geqslant 0 \quad \text{for } \beta \in Q_{k,m}, \qquad k = 1,2,\ldots,\min(n,m), \tag{2.2}$$

or, by (1.22), in determinantal form,

$$\varepsilon_k \det A[\alpha|\beta] \geqslant 0 \qquad \text{for } \alpha \in Q_{k,n},\ \beta \in Q_{k,m},\ k = 1,2,\ldots,\min(n,m). \tag{2.3}$$

A is called *strictly sign-regular with signature* $\vec{\varepsilon}$ if $\geqslant$ in (2.1) is replaced by

$\gg$, or equivalently if $\geqslant$ in (2.2) [respectively (2.3)] is replaced by $\gg$ [$>$]. As a special case, A is called *totally positive* if

$$\bigwedge^{k} A \geqslant 0, \qquad k=1,2,\ldots,\min(n,m), \tag{2.4}$$

or equivalently if

$$\vec{a}_{\beta_1} \wedge \vec{a}_{\beta_2} \wedge \cdots \wedge \vec{a}_{\beta_k} \geqslant 0 \quad \text{for } \beta \in Q_{k,m}, \qquad k=1,2,\ldots,\min(n,m), \tag{2.5}$$

or equivalently if

$$\det A[\alpha|\beta] \geqslant 0 \quad \text{for } \alpha \in Q_{k,n},\ \beta \in Q_{k,m} \qquad k=1,2,\ldots,\min(n,m). \tag{2.6}$$

A is called *strictly totally positive* if $\geqslant$ in (2.4) is replaced by $\gg$, or equivalently if $\geqslant$ in (2.5) [(2.6)] is replaced by $\gg$ [$>$].

For the sign regularity of A it is required to check the signs of a very large number of determinants. But if the rank of A is known in advance—in particular, if A is invertible—the necessary number of determinants to check can be considerably reduced.

THEOREM 2.1. *Let A be an $n \times m$ matrix of rank r, and $\vec{\varepsilon}$ a signature sequence. If* (2.2), *or equivalently* (2.3), *is valid whenever $d(\beta) \leqslant m-r$, then A is sign-regular with signature $\vec{\varepsilon}$. In particular, if* (2.5), *or equivalently* (2.6), *is valid whenever $d(\beta) \leqslant m-r$, then A is totally positive.*

Proof of (2.3) *by induction on k.* When $k=1$, (2.3) is true because $d(\beta)=0$ for $\beta \in Q_{1,m}$. Suppose that (2.3) is true with $k-2$, $k-1$ in place of k, but not with k. Find $\beta \in Q_{k,m}$ for which there is $\alpha \in Q_{k,n}$ such that

$$\varepsilon_k \det A[\alpha|\beta] < 0, \tag{2.7}$$

and which has minimum $d(\beta)$ under the above requirement. Suppose first that $d(\alpha)=0$. Let $l:=d(\beta)$. Then (2.7) is possible only if

$$l>m-r. \tag{2.8}$$

We claim that for every p such that $\beta_1<p<\beta_k$ and $p\notin\beta$

$$\vec{a}_p\wedge\vec{a}_{\beta_2}\wedge\vec{a}_{\beta_3}\wedge\cdots\wedge\vec{a}_{\beta_{k-1}}=0. \tag{2.9}$$

For this, fix such p and let $\tau=\{\beta_2,\beta_3,\dots,\beta_{k-1}\}$. Then the claim means that $A[-|\tau\cup\{p\}]$ has rank $\leqslant k-2$. Let us use (1.39) in the form that for every $\omega\in Q_{k-1,n}$ with $\omega\subset\alpha$

$$\begin{aligned}\det A[\omega|\tau\cup\{p\}]&\det A[\alpha|\tau\cup\{\beta_1,\beta_k\}]\\ &=\det A[\omega|\tau\cup\{\beta_k\}]\det A[\alpha|\tau\cup\{\beta_1,p\}]\\ &\quad+\det A[\omega|\tau\cup\{\beta_1\}]\det A[\alpha|\tau\cup\{p,\beta_k\}].\end{aligned} \tag{2.10}$$

Since $\tau\cup\{\beta_1,\beta_k\}=\beta$, $d(\tau\cup\{\beta_1,p\})\leqslant l-1$, and $d(\tau\cup\{p,\beta_k\})\leqslant l-1$, it follows from (2.7), the induction assumption, and the minimal property of l that the above identity can be valid only when

$$\det A[\omega|\tau\cup\{p\}]=0 \qquad \text{for any} \quad \omega\in Q_{k-1,n}, \quad \omega\subset\alpha. \tag{2.11}$$

On the other hand, according to (1.34), by (2.7) there is $\gamma\in Q_{k-2,n}$ such that $\gamma\subset\alpha$ and $\det A[\gamma|\tau]\neq 0$. In order to prove the claim, that is, rank $A[-|\tau\cup\{p\}]\leqslant k-2$, it suffices to show that every row vector of $A[-|\tau\cup\{p\}]$ is a linear combination of the row vectors with indices in γ, or equivalently that

$$\det A[\gamma\cup\{q\}|\tau\cup\{p\}]=0 \qquad \text{for} \quad q\notin\gamma. \tag{2.12}$$

When $q\in\alpha$, (2.12) follows from (2.11). Therefore fix $q\notin\alpha$, and let $\mu=\{\mu_1,\mu_2,\mu_3\}:=(\alpha\backslash\gamma)\cup\{q\}$, and $\nu=\{\nu_1,\nu_2,\nu_3\}:=\{\beta_1,p,\beta_k\}$. Remark that $d(\alpha)=0$ implies $q=\mu_1$ or $=\mu_3$. Consider the 3-square matrix $B=[b_{ij}]$ defined by

$$b_{ij}=\det A[\gamma\cup\{\mu_i\}|\tau\cup\{\nu_j\}], \qquad i,j=1,2,3.$$

Then by induction assumption all b_{ij} have same sign ε_{k-1}, and in view of (1.38), all the determinants of 2×2 submatrices of $B[-|1)$ and $B[-|3)$ have the same sign $\varepsilon_{k-2}\varepsilon_k$. On the other hand, (2.11) implies that $b_{i2}=0$ whenever $\mu_i \neq q$. The claim asserts $\vec{b}_2 = 0$. If $\vec{b}_2 \neq 0$, all the above conditions can be consistent only when $b_{i1}=0$ whenever $\mu_i \neq q$ or $b_{i3}=0$ whenever $\mu_i \neq q$ according as $q=\mu_1$ or $=\mu_3$. Apply again (1.38) to see each case leads to the contradiction $\det A[\alpha|\beta]=0$. Therefore $\vec{b}_2=0$, which establishes (2.9). Since (2.9) is valid for l $\vec{a}_p$'s with $p \notin \{\beta_2, \beta_3, \dots, \beta_{k-1}\}$, we have $r = \operatorname{rank} A \leqslant m-l$, contradicting (2.8). This contradiction shows that (2.3) is valid for k whenever $d(\alpha)=0$. This restriction $d(\alpha)=0$ can be released again by appealing to (1.39). This completes the induction. ∎

Recall that a matrix $A=[a_{ij}]$ is called *lower* (*upper*) *triangular* if $a_{ij}=0$ whenever $i<j$ $(i>j)$. A is called a *Jacobi* (or *tridiagonal*) *matrix* if $a_{ij}=0$ whenever $|i-j|>1$.

COROLLARY 2.2. *An n-square invertible lower triangular matrix A is totally positive if* $\det A[\alpha|1,2,\dots,k] \geqslant 0$ *for every k and $\alpha \in Q_{k,n}$.*

Proof. Let A be lower triangular. Since rank $A=n$, according to Theorem 2.1 it suffices to show that $\det A[\alpha|\beta] \geqslant 0$ for $\alpha, \beta \in Q_{k,n}$ with $d(\beta)=0$. If $\alpha_1 < \beta_1$, then $\det A[\alpha|\beta]=0$ because of lower triangularity. If $\alpha_1 \geqslant \beta_1$, let $\tau = \{1,2,\dots,\beta_1-1\}$. Then by assumption and lower triangularity

$$\begin{aligned} 0 &\leqslant \det A[\alpha \cup \tau|1,2,\dots,\beta_k] \\ &= \det A[\alpha\cup\tau|\beta\cup\tau] \\ &= \det A[\tau]\det A[\alpha|\beta] \\ &= \prod_{i=1}^{\beta_1-1} a_{ii} \det A[\alpha|\beta]. \end{aligned}$$

Since $\det A = \prod_{i=1}^{n} a_{ii} \neq 0$ and each $a_{ii} \geqslant 0$, it follows $\det A[\alpha|\beta] \geqslant 0$. ∎

THEOREM 2.3. *Let A be an n-square Jacobi matrix. If A is positive, $A \geqslant 0$, and all principal minors are nonnegative, that is,* $\det A[\alpha] \geqslant 0$ *whenever $d(\alpha)=0$, then A is totally positive and for any $t_i>0$, $i=1,2,\dots,n$,*

$$\det(A + \operatorname{diag}(t_1,t_2,\dots,t_n)) \geqslant \det A + \prod_{i=1}^{n} t_i. \tag{2.13}$$

Proof by induction on n. The assertion is trivial for $n = 1$. Assume that the assertion is true with $n-1$ instead of n. We may obviously assume $a_{11} > 0$. Then by (1.35), $A/\{1\}$ is again an $(n-1)$-square positive Jacobi matrix with nonnegative principal minors, so that by the induction assumption $A/\{1\}$ is totally positive and

$$\det(A/\{1\} + \operatorname{diag}(t_2, t_3, \dots, t_n)) \geqslant \det(A/\{1\}) + \prod_{i=2}^{n} t_i.$$

Therefore by Theorem 1.2

$$\begin{aligned}
&\det(A + \operatorname{diag}(t_1, t_2, \dots, t_n)) \\
&\quad = (a_{11} + t_1)\det\left(A/\{1\} + \operatorname{diag}\left(t_2 + \frac{a_{12}a_{21}}{a_{11}} - \frac{a_{12}a_{21}}{a_{11}+t_1}, t_3, \dots, t_n\right)\right) \\
&\quad \geqslant a_{11} \det A/\{1\} + \prod_{i=1}^{n} t_i \\
&\quad = \det A + \prod_{i=1}^{n} t_i.
\end{aligned}$$

It remains to show that A is totally positive. But the above argument shows that, by adding small $t_i > 0$, we may assume $\det A > 0$. In view of Theorem 2.1 we have to check

$$\det A[\alpha|\beta] \geqslant 0 \qquad \text{for} \quad \alpha, \beta \in Q_{k,n} \quad \text{with } d(\beta) = 0.$$

For $k = n$, this is just the assumption. For $k \leqslant n-1$, this is derived from the total positivity assured in induction assumption. ■

COROLLARY 2.4. *If an n-square Jacobi matrix A is totally positive, so is* $A + \operatorname{diag}(t_1, t_2, \dots, t_n)$ *for any* $t_i \geqslant 0$, $i = 1, 2, \dots, n$.

Proof. It follows from Theorem 2.3, applied to principal submatrices, that $A + \operatorname{diag}(t_1, t_2, \dots, t_n)$ is a positive Jacobi matrix with nonnegative principal minors. ■

Now let us turn to criteria for strict sign regularity. The number of determinants to check is further reduced.

THEOREM 2.5. *An $n \times m$ matrix A is strictly sign-regular with signature $\vec{\varepsilon}$ if $\varepsilon_k \det A[\alpha|\beta] > 0$ whenever $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, and $d(\alpha) = d(\beta) = 0$, $k = 1,2,\ldots,\min(n,m)$. In particular, A is strictly totally positive if $\det A[\alpha|\beta] > 0$ whenever $\alpha \in Q_{k,n}$, $\beta \in Q_{k,m}$, and $d(\alpha) = d(\beta) = 0$, $k = 1,2,\ldots,\min(n,m)$.*

Proof. Let us prove the inequalities

$$\varepsilon_k \det A[\alpha|\beta] > 0 \quad \text{for } \alpha \in Q_{k,n},\ \beta \in Q_{k,m}, \qquad k = 1,2,\ldots,\min(n,m), \tag{2.14}$$

by induction on k. When $k = 1$, this is trivial because $d(\alpha) = d(\beta) = 0$ for $\alpha \in Q_{1,n}$, $\beta \in Q_{1,m}$. Assume that (2.14) is true with $k-1$ in place of k. First fix an $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$, and let us prove (2.14) with this α by induction on $l := d(\beta)$. When $l = 0$, this follows from the assumption of the theorem. Suppose that $\varepsilon_k \det A[\alpha|\gamma] > 0$ whenever $\gamma \in Q_{k,m}$ and $d(\gamma) \leqslant l - 1$. Take any $\beta \in Q_{k,m}$ with $d(\beta) = l$. Then there is p such that $\beta_1 < p < \beta_k$, $d(\tau \cup \{\beta_1, p\}) \leqslant l - 1$, and $d(\tau \cup \{p, \beta_k\}) \leqslant l - 1$, where $\tau = \{\beta_2, \beta_3, \ldots, \beta_{k-1}\}$. It follows from (1.39), as (2.10) in the proof of Theorem 2.1,

$$\begin{aligned}
&\det A[\omega|\tau \cup \{p\}] \det A[\alpha|\tau \cup \{\beta_1, \beta_k\}] \\
&\qquad = \det A[\omega|\tau \cup \{\beta_k\}] \det A[\alpha|\tau \cup \{\beta_1, p\}] \\
&\qquad\quad + \det A[\omega|\tau \cup \{\beta_1\}] \det A[\alpha|\tau \cup \{p, \beta_k\}]
\end{aligned}$$

for any $\omega \in Q_{k-1,n}$ with $\omega \subset \alpha$. Then it follows from the two induction assumptions that the right hand side of the above identity is nonzero with sign $\varepsilon_{k-1}\varepsilon_k$ while $\det A[\omega|\tau \cup \{p\}]$ on the left hand side is nonzero with sign ε_{k-1}. Therefore the identity is consistent only when $\varepsilon_k \det A[\alpha|\beta] > 0$. This proves (2.14) for $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$. Apply the same argument rowwise to conclude that (2.14) is generally true. ∎

The same argument, combined with Corollary 2.2, yields the following.

COROLLARY 2.6. *An n-square lower triangular matrix A is totally positive if $\det A[\alpha|1,2,\ldots,k] > 0$ for every k and $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$.*

We conclude this section with a theorem on approximation of a totally positive matrix by strictly totally positive ones.

THEOREM 2.7. *Every sign-regular matrix can be approximated arbitrarily closely by strictly sign-regular matrices with the same signature. In particular, every totally positive matrix can be approximated arbitrarily closely by strictly totally positive matrices.*

Proof. Let A be an $n \times m$ sign-regular matrix with signature $\vec{\varepsilon}$. We may assume $n = m$, by considering $[A,0]$ or $\begin{bmatrix} 0 \\ A \end{bmatrix}$ if necessary. As will be shown in Section 7, there is a sequence $\{G_p\}$ of n-square strictly totally positive matrices such that $G_p \to I_n$ as $p \to \infty$. Now let us proceed by backward induction on rank A. Remark that (1.17) implies

$$\varepsilon_i \cdot \bigwedge^{i}(G_p A G_p)\begin{cases} \gg 0 & \text{if} \quad i \leqslant \operatorname{rank} A, \\ = 0 & \text{if} \quad i > \operatorname{rank} A. \end{cases} \tag{2.15}$$

When rank $A = n$, the assertion follows immediately from (2.15). Assume that the assertion is true for all sign-regular matrices of rank $k+1$. Let rank $A = k$, and take p so large that $B := G_p A G_p$ is sufficiently close to A. According to (2.15) and (1.23), B has the property

$$\varepsilon_i \det B[\alpha|\beta] > 0 \quad \text{for } \alpha, \beta \in Q_{i,n}, \qquad i = 1,2,\ldots,k. \tag{2.16}$$

Let

$$\delta := \min_{1 \leqslant i \leqslant k} \frac{\min\limits_{\alpha,\beta \in Q_{i,n}} |\det B[\alpha|\beta]|}{\max\limits_{\omega,\tau \in Q_{i-1,n}} |\det B[\omega|\tau]|}.$$

Then, for any $0 < t < \delta$, the matrix $C := B + t\varepsilon_k \varepsilon_{k+1}[\vec{e}_1, 0, 0, \ldots, 0]$ is sign-regular with signature $\vec{\varepsilon}$ and is of rank $k+1$, because B is sign-regular with signature $\vec{\varepsilon}$ and

$$\det C[\alpha|\beta] = \begin{cases} \det B[\alpha|\beta] + t\varepsilon_k\varepsilon_{k+1} \det B[\alpha\backslash\{1\}|\beta\backslash\{1\}] & \text{if} \quad \alpha_1 = \beta_1 = 1, \\ \det B[\alpha|\beta] & \text{otherwise}. \end{cases}$$

For small t the matrix C is sufficiently close to B, and hence to A. Now by the induction assumption C can be approximated arbitrarily closely by strictly sign-regular matrices with signature $\vec{\varepsilon}$. This completes the induction. ∎

Notes and References to Section 2

The notions of total positivity and sign regularity were introduced by Gantmacher and Krein (1937, 1950) with special reference to vibration of mechanical systems. They established almost all the fundamental results that will be presented in this lecture. The theory was also developed by Schoenberg (1930) in connection with the variation-diminishing property. The monograph by Karlin (1968), which mainly concerns the theory of totally positive and sign-regular kernels, devotes some attention to the exposition of totally positive and sign-regular matrices.

The criterion for strict total positivity, Theorem 2.5, was proved by Fekete (1913); its improvement, Theorem 2.1, is due to Cryer (1976). As a generalization of Jacobi matrices, $A = [a_{ij}]$ is called m-banded, or an m-band matrix, if $a_{ij} = 0$ for $|i-j| > m$. A criterion for total positivity of a band matrix is found in Metelmann (1973). Lewin (1980) showed that a matrix of the form $A = I_n - B$ with positive B has totally positive inverse only when A is a Jacobi matrix. The approximation theorem 2.7 is in Whitney (1952).

3. PERMANENCE OF TOTAL POSITIVITY

This section is devoted to canonical methods of production of new totally positive matrices from given totally positive ones.

Obviously, if A is sign-regular with signature $\vec{\varepsilon}$, so are the adjoint A^* and the conversion $A^\#$.

THEOREM 3.1. *If A is an $n \times m$ sign-regular matrix with signature $\vec{\varepsilon}_A$, and B is an $m \times l$ sign-regular matrix with signature $\vec{\varepsilon}_B$, then the product AB is sign-regular with signature $\vec{\varepsilon}_A \cdot \vec{\varepsilon}_B$. In this case AB becomes strictly sign-regular if A is strictly sign-regular and B is of rank* $\min(m, l)$, *or if A is of rank* $\min(n, m)$ *and B is strictly sign-regular. In particular if A, B are (strictly) totally positive, so is AB.*

This is an immediate consequence of (1.17) or (1.23).

The sum of two totally positive matrices is not totally positive in general. Therefore a square matrix A can rarely generate a *totally positive one-parameter semigroup*, that is, $\exp(tA)$ rarely is totally positive for all $t > 0$.

THEOREM 3.2. *An n-square matrix A generates a totally positive one-parameter semigroup* $\exp(tA)$ *if and only if* $A = \xi I_n + B$ *for some real* ξ *and a totally positive Jacobi matrix B.*

Proof. Suppose first that A is of the form mentioned. Then since

$$\exp(tA) = e^{\xi t}\exp(tB)$$
$$= e^{\xi t}\lim_{p\to\infty}\left(I_n + \frac{t}{p}B\right)^p,$$

the total positivity of $\exp(tA)$ results from Theorem 3.1, because, for the totally positive Jacobi matrix B, $I_n + (t/p)B$ is again totally positive by Corollary 2.4.

Suppose conversely that $\exp(tA)$ is totally positive for all $t > 0$. In view of Theorem 2.3, it suffices to show that A is a real Jacobi matrix with nonnegative off-diagonal. Since

$$A = \lim_{t\downarrow 0}\frac{1}{t}\{\exp(tA) - I_n\},$$

all off-diagonal entries of A are nonnegative because $\exp(tA) \geqslant 0$. Finally $a_{ij} = 0$ whenever $|i-j| > 1$. In fact, if $i+1 < j$, say,

$$\det\exp(tA)[i, i+1|i+1, j] \geqslant 0 \qquad \text{for} \quad t > 0$$

implies

$$0 \leqslant \lim_{t\downarrow 0}\frac{1}{t}\det(I + tA)[i, i+1|i+1, j]$$
$$= \lim_{t\downarrow 0}\{ta_{i,i+1}a_{i+1,j} - (1 + ta_{i+1,i+1})a_{ij}\}$$
$$= -a_{ij}.$$ ■

THEOREM 3.3. *Let A be an* $n \times m$ *sign-regular matrix with signature* $\vec{\varepsilon}$.

(a) $A[\alpha|\beta]$ *is sign-regular with signature* $\vec{\varepsilon}$ *for every* $\alpha \in Q_{k,n}$ *and* $\beta \in Q_{l,m}$.

(b) A/α' *is sign-regular with signature* $\vec{\varepsilon}_\alpha = (\varepsilon_{n-k}\varepsilon_{n-k+i})_i$ *if* $n = m$, $\alpha \in Q_{k,n}$ *with consecutive components (i.e.* $d(\alpha) = 0$*), and* $A(\alpha)$ *is invertible.*

(c) $J_n A^{-1} J_n$ *is sign-regular with signature* $\vec{\varepsilon}_J = (\varepsilon_n \varepsilon_{n-i})_i$, *with convention* $\varepsilon_j = 1$ *for* $j \leqslant 0$, *if* $n = m$ *and* A *is invertible.*

In particular, if A *is totally positive, so are* $A[\alpha|\beta]$, A/α', *and* $J_n A^{-1} J_n$.

Proof. (a) is trivial. (b) follows from (1.35), and (c) from (1.32). ■

COROLLARY 3.4. *Let* $A = [a_{ij}] = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$ *be an n-square totally positive matrix. If* $a_{1k} \neq 0$, *then the matrix* $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n]$ *defined by*

$$\vec{b}_i = \vec{a}_i \quad i = 1,2,\ldots,k \text{ and } \vec{b}_i = \vec{a}_i - \frac{a_{1i}}{a_{1k}}\vec{a}_k, \qquad i = k+1,\ldots,n,$$

becomes totally positive.

Proof. By Theorem 2.7 we may assume $\det A > 0$. Since obviously $\det B = \det A$, according to Theorem 2.1 it suffices to show that

$$\vec{b}_i \wedge \vec{b}_{i+1} \wedge \cdots \wedge \vec{b}_j \geqslant 0 \qquad \text{for} \quad 1 \leqslant i \leqslant j \leqslant n. \tag{3.1}$$

If $j \leqslant k$ or $i \leqslant k \leqslant j$, then

$$\vec{b}_i \wedge \vec{b}_{i+1} \wedge \cdots \wedge \vec{b}_j = \vec{a}_i \wedge \vec{a}_{i+1} \wedge \cdots \wedge \vec{a}_j,$$

and (3.1) is valid because A is totally positive. If $k < i$, consider the n-square matrix $C = [\vec{a}_k, \vec{a}_{k+1}, \ldots, \vec{a}_n, 0, \ldots, 0]$. Then it is readily seen from the definition of $C/\{1\}$ that

$$[\vec{b}_{k+1}, \vec{b}_{k+2}, \ldots, \vec{b}_n, 0, \ldots, 0] = \begin{bmatrix} 0 \\ C/\{1\} \end{bmatrix}.$$

Now (3.1) follows from the total positivity of $C/\{1\}$ by Theorem 3.3. ■

A factorization $A = BC$ is called an *LU* (*UL*) *factorization* if B (C) is lower triangular and C (B) is upper triangular.

THEOREM 3.5. *Let* A *be an* $n \times m$ *totally positive matrix with* $n \geqslant m$. *Then* A *admits an LU factorization* $A = A_L A_U$ *and a UL factorization*

$A = \tilde{A}_U \tilde{A}_L$, where A_L, $\tilde{A}_U$ are n-square totally positive matrices, and A_U, $\tilde{A}_L$ are $n \times m$ totally positive matrices.

Proof. By considering the n-square matrix $[A, 0]$, our proof can be confined to the case $n = m$. Further, with the help of conversion, it suffices to treat only the LU-factorization.

When $n = 1$, everything is trivial. Assume that the assertion is true with $n - 1$ in place of n. Now let

$$S_j := \left[\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_{j-1}, 0, \vec{e}_j, \vec{e}_{j+1}, \ldots, \vec{e}_{n-1}\right], \qquad j = 1, 2, \ldots, n-1.$$

Clearly S_j is a positive, upper triangular Jacobi matrix, hence totally positive by Theorem 2.3. If $\vec{a}_1 = \vec{a}_2 = \cdots = \vec{a}_{k-1} = 0$ but $\vec{a}_k \neq 0$, then

$$A = \left[\vec{a}_k, \vec{a}_{k+1}, \ldots, \vec{a}_n, 0, \ldots, 0\right] S_1^{k-1},$$

and the matrix $[\vec{a}_k, \vec{a}_{k+1}, \ldots, \vec{a}_n, 0, \ldots, 0]$ is totally positive. Applying this procedure to $\{\vec{a}_{k+1}, \vec{a}_{k+2}, \ldots, \vec{a}_n\}$ and so on, we arrive at a factorization: for some $\omega \in Q_{l,n}$ and $k_i \geqslant 0$, $i = 1, 2, \ldots, l$,

$$A = B S_{\omega_l}^{k_l} S_{\omega_{l-1}}^{k_{l-1}} \cdots S_{\omega_1}^{k_1}.$$

where $B = [\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_n]$ is a totally positive matrix such that $\vec{b}_i = 0$ implies $\vec{b}_j = 0$, $j = i+1, i+2, \ldots, n$. If $B[1|-] \neq 0$, take the largest i for which $b_{1i} \neq 0$. We claim $b_{1,i-1} \neq 0$. Otherwise, $b_{1i} \neq 0$, $b_{1,i-1} = 0$, and det $B[1, j|i-1, i] \geqslant 0$ imply $b_{j,i-1} = 0$ for $j = 1, 2, \ldots, n$, that is, $\vec{b}_{i-1} = 0$, contradicting $\vec{b}_i \neq 0$. Now B admits a factorization $B = CU$ where

$$C := \left[\vec{b}_1, \ldots, \vec{b}_{i-1}, \vec{b}_i - \frac{b_{1,i}}{b_{1,i-1}} \vec{b}_{i-1}, \vec{b}_{i+1}, \ldots, \vec{b}_n\right]$$

and

$$U := \left[\vec{e}_1, \ldots, \vec{e}_{i-1}, \frac{b_{1,i}}{b_{1,i-1}} \vec{e}_{i-1} + \vec{e}_i, \vec{e}_{i+1}, \ldots, \vec{e}_n\right].$$

U is a positive, upper triangular Jacobi matrix, hence totally positive by Theorem 2.3. The total positivity of C follows from Corollary 3.4, because by the maximum property of i:

$$\vec{b}_j = \vec{b}_j - \frac{b_{1,j}}{b_{1,i-1}} \vec{b}_{i-1}, \qquad j = i+1, i+2, \ldots, n.$$

Repeating this procedure, we arrive at a factorization $B = D \cdot U_p U_{p-1} \cdots U_1$ where each U_j is upper triangular, tridiagonal, and totally positive while D is a totally positive matrix such that $D[1|1) = 0$:

$$A = D \cdot U_p U_{p-1} \cdots U_1 S_{\omega_l}^{k_l} S_{\omega_{l-1}}^{k_{l-1}} \cdots S_{\omega_1}^{k_1}. \tag{3.2}$$

Apply the corresponding procedure to the row vectors of D to get a factorization

$$A = S_{\tau_1}^{*j_1} S_{\tau_2}^{*j_2} \cdots S_{\tau_m}^{*j_m} L_1 L_2 \cdots L_q \cdot F \cdot U_p U_{p-1} \cdots U_1 S_{\omega_l}^{k_l} S_{\omega_{l-1}}^{k_{l-1}} \cdots S_{\omega_1}^{k_1}, \tag{3.3}$$

where each L_i is a lower triangular totally positive Jacobi matrix, and $\tau \in Q_{m,n}$ and $j_i \geqslant 0$, $i = 1, 2, \ldots, m$, while F is a totally positive matrix such that $F[1|1) = 0$ and $F(1|1] = 0$. Since $F(1)$ is an $(n-1)$-square totally positive matrix, according to the induction assumption it admits an LU factorization $F(1) = \hat{F}_L \hat{F}_U$ where $\hat{F}_L$ and $\hat{F}_U$ are $(n-1)$-square totally positive matrices. Now by (3.3) the n-square matrices A_L and A_U defined by

$$A_L = S_{\tau_1}^{*j_1} S_{\tau_2}^{*j_2} \cdots S_{\tau_m}^{*j_m} L_1 L_2 \ldots L_q \begin{bmatrix} \sqrt{f_{11}} & 0 \\ 0 & \hat{F}_L \end{bmatrix}$$

and

$$A_U = \begin{bmatrix} \sqrt{f_{11}} & 0 \\ 0 & \hat{F}_U \end{bmatrix} U_p U_{p-1} \cdots U_1 S_{\omega_l}^{k_l} S_{\omega_{l-1}}^{k_{l-1}} \cdots S_{\omega_1}^{k_1}$$

are totally positive, and produce an LU factorization $A = A_L A_U$. This completes the induction. ■

COROLLARY 3.6. *Every invertible n-square upper (lower) triangular, totally positive matrix is the product of a certain number of n-square upper (lower) triangular, totally positive Jacobi matrices.*

Proof by induction on n. The case $n = 1$ is trivial. Assume that the assertion is true with $n-1$ in place of n, and let A be an n-square invertible upper triangular totally positive matrix. Checking the proof of Theorem 3.5 will show that S's do not appear and the matrix D in the factorization (3.2) is also upper triangular. Since

$$D = \begin{bmatrix} d_{11} & 0 \\ 0 & D(1) \end{bmatrix}$$

and $D(1)$ is an $(n-1)$-square invertible upper triangular totally positive matrix, by the induction assumption we have $D(1) = \hat{W}_1 \hat{W}_2 \cdots \hat{W}_s$ for some $(n-1)$-square upper triangular totally positive Jacobi matrices $\hat{W}_i$, $i = 1, 2, \dots, s$. Let

$$W_i = \begin{bmatrix} d_{11}^{1/s} & 0 \\ 0 & \hat{W}_i \end{bmatrix}.$$

Then $A = W_1 W_2 \cdots W_s \cdot U_p U_{p-1} \cdots U_1$ is an expected factorization. ■

Beside the usual order relation $A \geqslant B$ between two n-square real matrices, let us introduce a stronger one: $A \overset{(t)}{\geqslant} B$ means by definition that $\bigwedge^k A \geqslant \bigwedge^k B$ for $k = 1, 2, \dots,$ in other words,

$$\det A[\alpha|\beta] \geqslant \det B[\alpha|\beta] \qquad \text{for any } k \text{ and } \alpha, \beta \in Q_{k,n}. \tag{3.4}$$

In this notation, $A \overset{(t)}{\geqslant} 0$ means that A is totally positive. The relation $A \overset{(t)}{\geqslant} B$ implies $A[\alpha|\beta] \overset{(t)}{\geqslant} B[\alpha|\beta]$ for any $\alpha, \beta \in Q_{k,n}$, but not $A - B \overset{(t)}{\geqslant} 0$. Also

$$A \overset{(t)}{\geqslant} B \overset{(t)}{\geqslant} 0$$

does not imply

$$A/\{1\} \overset{(t)}{\geqslant} B/\{1\} \quad \text{or} \quad J_n A^{-1} J_n \overset{(t)}{\leqslant} J_n B^{-1} J_n.$$

THEOREM 3.7. *If A is an n-square totally positive matrix, and $\alpha = \{1, 2, \dots, k\}$ or $= \{k, k+1, \dots, n\}$, then*

$$A[\alpha] \overset{(t)}{\geqslant} A/\alpha', \tag{3.5}$$

provided that $A(\alpha)$ is invertible.

Proof for the case $\alpha = \{1,2,\dots,k\}$. In view of (1.35), (3.5) is equivalent to the inequalities

$$\det A[\omega \cup \alpha' | \tau \cup \alpha'] \leqslant \det A[\omega|\tau] \det A(\alpha') \qquad \text{for} \quad \omega, \tau \in Q_{l,n} \quad \text{with } \omega, \tau \subset \alpha.$$

To prove these inequalities, by fixing ω, τ and considering the matrix $A[\omega \cup \alpha' | \tau \cup \alpha']$ in place of A, it suffices to establish the following general assertion: for any m-square totally positive matrix B

$$\det B \leqslant \det B[1,2,\dots,j] \det B[j+1, j+2,\dots,m] \qquad \text{for} \quad j = 1,2,\dots,m-1. \tag{3.6}$$

Let us prove (3.6) by induction on m. When $m = 2$, it is true because $b_{12} \geqslant 0$, $b_{21} \geqslant 0$ imply

$$\det B = b_{11}b_{22} - b_{12}b_{21} \leqslant b_{11}b_{22}.$$

Assume that the assertion is true for all the cases of order less than m. The m-square matrix B under consideration can be assumed to have $b_{11} > 0$. Then if $k > 1$, by (1.28)

$$\begin{aligned} &\det B[1,2,\dots,k] \det B[k+1, k+2,\dots,m] \\ &\quad = \det(B/\{1\})[2,3,\dots,k] \cdot b_{11} \cdot \det B[k+1, k+2,\dots,m]. \end{aligned}$$

Since the matrix $B[1, k+1, k+2,\dots,m]$ of order less than m is totally positive, the induction assumption yields

$$\begin{aligned} b_{11} \det B[k+1, k+2,\dots,m] &\geqslant \det B[1, k+1, k+2,\dots,m] \\ &= b_{11} \det(B/\{1\})[k+1, k+2,\dots,m]. \end{aligned}$$

Use again the induction assumption on the matrix $B/\{1\}$ of order $m-1$, which is totally positive by Theorem 3.3, to get

$$\begin{aligned} &\det B[1,2,\dots,k] \det B[k+1, k+2,\dots,m] \\ &\quad \geqslant b_{11} \det(B/\{1\})[2,3,\dots,k] \det(B/\{1\})[k+1, k+2,\dots,m] \\ &\quad \geqslant b_{11} \det(B/\{1\})[2,3,\dots,m] = \det B. \end{aligned}$$

When $k = 1$, proceed just as above with $B/\{m\}$ instead of $B/\{1\}$. ∎

COROLLARY 3.8. *If an n-square totally positive matrix A is invertible, then*

$$\det A[\alpha] > 0 \qquad \textit{for every } k \textit{ and } \alpha \in Q_{k,n}. \tag{3.7}$$

Proof by induction on n. The case $n = 1$ is trivial. Assume that the assertion is true with $n-1$ in place of n. If $\alpha_1 > 1$, then $\det A[\alpha] > 0$ follows from the induction assumption applied to $A(1)$, which is invertible by (3.6) and totally positive. If $\alpha_1 = 1$, then by (3.6) $a_{11} > 0$ and by (1.35)

$$\det A[\alpha] = a_{11} \det(A/\{1\})[\alpha\backslash\{1\}].$$

Now $\det A[\alpha] > 0$ follows from the induction assumption applied to $A/\{1\}$, which is invertible by (1.28) and totally positive by Theorem 3.3. ∎

Application of LU and UL factorization in Theorem 3.4 gives rise to other inequalities.

THEOREM 3.9. *If A is an n-square totally positive matrix, and* $\alpha = \{1,2,\dots,k\}$ *or* $= \{k+1, k+2, \dots, n\}$, *then*

$$A[\alpha] - A/\alpha' \overset{(t)}{\geqslant} 0 \tag{3.8}$$

provided that $A(\alpha)$ *is invertible.*

Proof for the case $\alpha = \{k+1, k+2, \dots, n\}$. Let $A = A_L A_U$ be an LU factorization with totally positive A_L, A_U, guaranteed by Theorem 3.5. Then by definition

$$\begin{aligned} A[\alpha] - A/\alpha' &= A[\alpha|\alpha)A(\alpha)^{-1}A(\alpha|\alpha] \\ &= A_L[\alpha|\alpha)A_U(\alpha)\cdot(A_L(\alpha)A_U(\alpha))^{-1}\cdot A_L(\alpha)A_U(\alpha|\alpha] \\ &= A_L[\alpha|\alpha)A_U(\alpha|\alpha]. \end{aligned}$$

Since $A_L[\alpha|\alpha)$ and $A_U(\alpha|\alpha]$ are totally positive, so is their product $A_L[\alpha|\alpha)A_U(\alpha|\alpha]$. Finally, a proof for the case $\alpha = \{1,2,\dots,k\}$ is accomplished by using UL factorization. ∎

Notes and References to Section 3

The characterization of totally positive semigroups, Theorem 3.2, is in Karlin (1968, p. 115) and related to Loewner (1955). A semigroup of totally positive Jacobi matrices with respect to the Hadamard (i.e. Schur) product was studied by Markham (1970). The LU factorization, Theorem 3.5, was proved by Cryer (1973, 1976); see also Rainey and Halbetler (1972). Incidentally the use of Schur complements in LU factorization is also seen in Neumann (1981). A check of the proof of Theorem 3.5 will show, on the basis of Theorem 1.3, that when A is invertible, the kth column of A_L is a positive scalar multiple of the kth column of $A/\{1,2,\dots,k-1\}$ augmented by 0 at the top $1,2,\dots,k-1$ positions. Representation of an (infinite) totally positive matrix as a product of totally positive Jacobi matrices was studied by de Boor and Pinkus (1982) and by Cavaretta, Dahmen, Miccelli, and Smith (1981). The fundamental determinantal inequality (3.6) is due to Gantmacher and Krein (1937). This inequality is valid under a slightly weaker condition; see in this respect Koteljanskiĭ (1963a).

4. OSCILLATORY MATRICES

An n-square matrix A is called *oscillatory* if it is totally positive and a certain power A^p becomes strictly totally positive. In this section we shall present a simple criterion for a totally positive matrix to be oscillatory.

Let us start from simple remarks. An oscillatory matrix is invertible, and its adjoint is also oscillatory. Therefore, by Corollary 3.8, if an n-square matrix $A = [a_{ij}]$ is oscillatory, then $\det A[\alpha] > 0$ for $\alpha \in Q_{k,n}$.

THEOREM 4.1. *Let A be an n-square oscillatory matrix. Then the following hold:*

(a) *$J_n A^{-1} J_n$ is oscillatory.*

(b) *$A[\alpha]$ and A/α' are oscillatory for every $\alpha \in Q_{k,n}$ with consecutive components, i.e. such that $d(\alpha) = 0$.*

Proof. Suppose that A is totally positive and A^p is strictly totally positive.

(a): $J_n A^{-1} J_n$ is totally positive, and $(J_n A^{-1} J_n)^p = J_n (A^p)^{-1} J_n$ is strictly totally positive by Theorem 3.3. Thus $J_n A^{-1} J_n$ is oscillatory.

(b): Let us prove first the oscillatoriness of $A[\alpha]$ for the case $\alpha = \{1,2,\dots,n-1\}$. Let $B = A[1,2,\dots,n-1]$. Take $\beta, \tau \in Q_{k,n-1}$, and let $\mu := \beta \cup \{n\}$ and $\nu := \tau \cup \{n\}$. By (1.23), $\det A^p[\mu|\nu] > 0$ implies that there is a sequence $\omega^{(i)} \in Q_{k+1,n}$, $i = 0,1,\dots,p$, such that $\omega^{(0)} = \mu$, $\omega^{(p)} = \nu$, and

$$\prod_{i=1}^{p} \det A\left[\omega^{(i-1)}|\omega^{(i)}\right] > 0.$$

Let $\tilde{\omega}^{(i)}$ be the element in $Q_{k,n-1}$ obtained by deleting the last component from $\omega^{(i)}$. Since $A[\omega^{(i-1)}|\omega^{(i)}]$ is totally positive with positive determinant, by (3.6)

$$\det B\left[\tilde{\omega}^{(i-1)}|\tilde{\omega}^{(i)}\right] = \det A\left[\tilde{\omega}^{(i-1)}|\tilde{\omega}^{(i)}\right] > 0, \qquad i = 1,2,\dots,p.$$

Then again, by the total positivity of B and (1.23),

$$\det B^p[\beta|\tau] \geqslant \prod_{i=1}^{p} \det B\left[\tilde{\omega}^{(i-1)}|\tilde{\omega}^{(i)}\right] > 0,$$

which proves the strict total positivity of B. The case $A[2,3,\dots,n]$ is treated similarly. The oscillatoriness of $A[\alpha]$ for $\alpha \in Q_{k,n}$ with $d(\alpha) = 0$ is shown by backward induction on k. Finally the oscillatoriness of A/α' follows from (1.29) by appealing to (a). ■

The following gives a surprisingly simple criterion for oscillatoriness.

THEOREM 4.2. *An n-square totally positive matrix $A = [a_{ij}]$ is oscillatory if and only if it is invertible and*

$$a_{i,i+1} > 0, \quad a_{i+1,i} > 0, \qquad i = 1,2,\dots,n-1. \tag{4.1}$$

The "only if" part is easy. In fact, by Theorem 4.1, $B := A[i, i+1]$ is oscillatory, and $B^p \gg 0$ for some p. But this is possible only when $a_{i,i+1} > 0$ and $a_{i+1,i} > 0$. The "if" part is more difficult, and is proved as a consequence of a more general result (Theorem 4.5).

COROLLARY 4.3. *Let A, B be n-square totally positive matrices. If A is oscillatory and B is invertible, then AB and BA are oscillatory.*

Proof. Since B is invertible, $b_{ii} > 0$ for $i = 1,2,\dots,n$. Then by (1.23) both AB and BA satisfy the condition (4.1) along with A. ■

The following theorem presents an extension of the condition (4.1) for oscillatory matrices.

THEOREM 4.4. *Let A be an n-square totally positive matrix. If A is invertible and satisfies* (4.1), *then* $\det A[\alpha|\beta] > 0$ *for every pair* $\alpha, \beta \in Q_{k,n}$ *such that*

$$|\alpha_i - \beta_i| \leqslant 1 \text{ and } \max(\alpha_i, \beta_i) < \min(\alpha_{i+1}, \beta_{i+1}), \qquad i = 1, 2, \ldots, k, \quad (4.2)$$

where $\alpha_{k+1} = \beta_{k+1} = \infty$.

Proof by induction on k. The case $k = 1$ follows from (3.7) and the assumption (4.1). Fix k, and assume that the assertion is true for every pair in $Q_{k-1,n}$ satisfying (4.2) with $k-1$ instead of k. Take any pair $\alpha, \beta \in Q_{k,n}$ satisfying (4.2). If $d(\alpha) = d(\beta) = 0$, then (4.2) is consistent only when $\alpha = \beta$. Thus in this case $\det A[\alpha|\beta] > 0$ results from (3.7), which is valid for every invertible totally positive matrix. Now assuming $d(\beta) > 0$, let $B = A[\alpha|\beta] = [\vec{b}_{\beta_1}, \vec{b}_{\beta_2}, \ldots, \vec{b}_{\beta_k}]$, each $\vec{b}_{\beta_i}$ being a k-vector. We have to show that $\det B = \det A[\alpha|\beta] = 0$ produces a contradiction. First, it follows from induction assumption that

$$\det B[\alpha_1, \alpha_2, \ldots, \alpha_{k-1}|\beta_1, \beta_2, \ldots, \beta_{k-1}] > 0$$

and

$$\det B[\alpha_2, \alpha_3, \ldots, \alpha_k|\beta_2, \beta_3, \ldots, \beta_k] > 0,$$

which implies, together with total positivity,

$$\vec{b}_{\beta_1} \wedge \vec{b}_{\beta_2} \wedge \cdots \wedge \vec{b}_{\beta_{k-1}} \gneqq 0 \quad \text{and} \quad \vec{b}_{\beta_2} \wedge \vec{b}_{\beta_3} \wedge \cdots \wedge \vec{b}_{\beta_k} \gneqq 0. \quad (4.3)$$

Then $\det B = 0$ guarantees that for some $\xi_i \in \mathbf{R}$

$$\vec{b}_{\beta_k} = \sum_{i=1}^{k-1} \xi_i \vec{b}_{\beta_i} \qquad \text{with} \quad \xi_1 \neq 0. \quad (4.4)$$

Now substitute the expression (4.4) for $\vec{b}_{\beta_k}$ in (4.3) to get

$$(-1)^{k-2} \xi_1 \cdot \vec{b}_{\beta_1} \wedge \vec{b}_{\beta_2} \wedge \cdots \wedge \vec{b}_{\beta_{k-1}} \geqslant 0. \quad (4.5)$$

Since $d(\beta) > 0$, the ordered set $\gamma := \{j \notin \beta : \beta_1 < j < \beta_k\}$ is nonempty. Let us show that for every $j \in \gamma$ the α-projection $\vec{b}_j$ of $\vec{a}_j$ is linearly dependent on $\vec{b}_{\beta_1}, \vec{b}_{\beta_2}, \dots, \vec{b}_{\beta_{k-1}}$, or equivalently

$$\vec{b}_j \wedge \vec{b}_{\beta_1} \wedge \vec{b}_{\beta_2} \wedge \cdots \wedge \vec{b}_{\beta_{k-1}} = 0. \tag{4.6}$$

To this end, take i such that $\beta_i < j < \beta_{i+1}$. Then since $A[\alpha|\beta \cup \{j\}]$ is totally positive,

$$\vec{b}_{\beta_1} \wedge \cdots \wedge \vec{b}_{\beta_i} \wedge \vec{b}_j \wedge \vec{b}_{\beta_{i+1}} \wedge \cdots \wedge \vec{b}_{\beta_{k-1}} \geqslant 0$$

and

$$\vec{b}_{\beta_2} \wedge \cdots \wedge \vec{b}_{\beta_i} \wedge \vec{b}_j \wedge \vec{b}_{\beta_{i+1}} \wedge \cdots \wedge \vec{b}_{\beta_k} \geqslant 0. \tag{4.7}$$

Now substitute the expression (4.4) for $\vec{b}_{\beta_k}$ in (4.7) to get

$$(-1)^{k-1}\xi_1 \cdot \vec{b}_{\beta_1} \wedge \cdots \wedge \vec{b}_{\beta_i} \wedge \vec{b}_j \wedge \vec{b}_{\beta_{i+1}} \wedge \cdots \wedge \vec{b}_{\beta_{k-1}} \geqslant 0. \tag{4.8}$$

It is clear that (4.3), (4.5), (4.7), (4.8), and $\xi_1 \neq 0$ are consistent only if the equality occurs in (4.8), or equivalently (4.6) is valid. This argument shows that the matrix $A[\alpha|\beta \cup \gamma]$ has rank $k-1$. Consider the ordered set $\tau := \{i \notin \alpha : \alpha_1 < i < \alpha_k\}$. Now the above argument applied to row vectors yields finally that $A[\alpha \cup \tau|\beta \cup \gamma]$ has rank $k-1$. Finally it follows from (4.2) and $d(\beta) > 0$ that there is $\omega \in Q_{k,n}$ such that $d(\omega) = 0$, $\omega \subset \alpha \cup \tau$, and $\omega \subset \beta \cup \gamma$. Then since $A[\alpha \cup \tau|\beta \cup \gamma]$ is of rank $k-1$, $\det A[\omega] = 0$, which is a contradiction as remarked earlier. This completes the proof. ∎

The "if" part of Theorem 4.2 will follow from the following more general result.

THEOREM 4.5. *Let A_i $i = 1,2,\dots,p$ be n-square, invertible totally positive matrices and $p \geqslant n-1$. If every A_i satisfies* (4.1), *then the product $A_1A_2 \cdots A_p$ is strictly totally positive.*

Proof. In view of Theorem 2.5 it suffices to show that

$$\det(A_1A_2 \cdots A_p)[\alpha|\beta] > 0$$

$$\text{whenever} \quad \alpha, \beta \in Q_{k,n}, \quad d(\alpha) = d(\beta) = 0. \tag{4.9}$$

Assuming $\beta_1 \geqslant \alpha_1$, let $\omega^{(0)} = \alpha$ and $\omega^{(p)} = \beta$. Define $\omega^{(l)} \in Q_{k,n}$ for $l = 1,2,\ldots,p-1$ by

$$\omega_i^{(l)} = \min\{\beta_i, \alpha_i + \max(l+i-k,0)\}, \qquad i = 1,2,\ldots,k.$$

Then it is readily seen that each pair $\omega^{(l-1)}, \omega^{(l)}$ satisfies (4.2); hence by Theorem 4.4, $\det A_l[\omega^{(l-1)}|\omega^{(l)}] > 0$, $l = 1,2,\ldots,p$. Therefore it follows from (1.23) and the total positivity that

$$\det(A_1 A_2 \cdots A_p)[\alpha|\beta] \geqslant \prod_{l=1}^{p} \det A_l\left[\omega^{(l-1)}|\omega^{(l)}\right] > 0,$$

proving (4.9). ∎

COROLLARY 4.6.

(a) *If A is an n-square oscillatory matrix, then A^{n-1} is strictly totally positive.*

(b) *If A is an n-square, invertible sign-regular matrix such that $a_{ii} \neq 0$, $i = 1,2,\ldots,n$, and $a_{i,i+1}a_{i+1,i} > 0$, $i = 1,2,\ldots,n-1$, then $A^{2(n-1)}$ is strictly totally positive.*

Proof. (a) follows immediately from Theorem 4.5. In (b), A^2 for A is totally positive and satisfies (4.1). Now appeal to (a). ∎

Notes and References to Section 4

The oscillatoriness of a Schur complement, Theorem 4.1(b), is in Markham (1970a). We closely followed Gantmacher and Krein (1937, 1950) in proving Theorem 4.5 and hence Theorem 4.2. Radke (1968) showed that an invertible totally positive matrix is oscillatory if it is irreducible.

5. VARIATION OF SIGNS

This section is devoted to characterizations of sign regularity of a matrix in terms of some variation-diminishing properties of the linear map it induces.

By a *sign sequence* of a real n-vector $\vec{x}$ we understand any signature sequence $\vec{\varepsilon}$ for which $\varepsilon_i x_i = |x_i|$, $i = 1,2,\ldots,n$. The number of sign changes of $\vec{x}$ associated to $\vec{\varepsilon}$, denoted by $\mathscr{C}(\vec{\varepsilon})$, is the number of indices i such that

$\varepsilon_i\varepsilon_{i+1} < 0$, $1 \leqslant i \leqslant n-1$, that is,

$$\mathscr{C}(\vec{\varepsilon}) = \frac{1}{2}\sum_{i=1}^{n-1}(1-\varepsilon_i\varepsilon_{i+1}).$$

Now the *maximum* [*minimum*] *variation of signs*, $\mathscr{V}_+(\vec{x})$ [$\mathscr{V}_-(\vec{x})$], is by definition the maximum [minimum] of $\mathscr{C}(\vec{\varepsilon})$ when $\vec{\varepsilon}$ runs over all sign sequences of $\vec{x}$. Obviously

$$0 \leqslant \mathscr{V}_-(\vec{x}) \leqslant \mathscr{V}_+(\vec{x}) \leqslant n-1 \qquad \text{for} \quad \vec{x} \in \mathbf{R}^n.$$

If any component of $\vec{x}$ does not vanish, a sign sequence of $\vec{x}$ is uniquely determined; hence $\mathscr{V}_-(\vec{x}) = \mathscr{V}_+(\vec{x})$. This common value is called the *exact variation* of signs and is denoted by $\mathscr{V}(\vec{x})$. The following hold:

$$\mathscr{V}_+(\vec{x}) + \mathscr{V}_-(J_n\vec{x}) = \mathscr{V}_-(\vec{x}) + \mathscr{V}_+(J_n\vec{x}) = n-1 \qquad \text{for} \quad \vec{x} \in \mathscr{R}^n. \tag{5.1}$$

In fact, when $\vec{\varepsilon}$ runs over all sign sequences of $\vec{x}$, $J_n\vec{\varepsilon}$ runs over all sign sequences of $J_n\vec{x}$, and

$$\mathscr{C}(\vec{\varepsilon}) + \mathscr{C}(J_n\vec{\varepsilon}) = n-1,$$

which immediately yields (5.1).

If a sequence $\vec{x}_p$, $p = 1,2,\ldots$, converges to $\vec{x}$, then

$$\mathscr{V}_-(\vec{x}) \leqslant \varliminf_{p\to\infty} \mathscr{V}_-(\vec{x}_p) \quad \text{and} \quad \varlimsup_{p\to\infty} \mathscr{V}_+(\vec{x}_p) \leqslant \mathscr{V}_+(\vec{x}). \tag{5.2}$$

This is also immediate from the definition.

LEMMA 5.1. *Let $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ be real n-vectors and $n > m$. In order that $\mathscr{V}_+(\sum_{i=1}^m \xi_i a_i) \leqslant m-1$ whenever $\xi_i \in \mathbf{R}$, $i = 1,2,\ldots,m$, and $\sum_{i=1}^m |\xi_i| \neq 0$, it is necessary and sufficient that $\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m$ be strictly definite, i.e. $\gg 0$ or $\ll 0$.*

Proof. To see sufficiency, suppose that $\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge a_m \gg 0$, say, and that $\mathscr{V}_+(\sum_{i=1}^m \xi_i \vec{a}_i) \geqslant m$ for some choice $\xi_i \in \mathbf{R}$, $i = 1,2,\ldots,m$, with $\sum_{i=1}^m |\xi_i| \neq 0$. Let $\vec{b} = \sum_{i=1}^m \xi_i \vec{a}_i$. Then $\vec{b}$ is nonzero, because $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ are linearly independent. It follows from $\mathscr{V}_+(\vec{b}) \geqslant m$ that for some $\alpha \in Q_{m+1,n}$

the α-projection of $\vec{b}$ has maximum variation m. Obviously the α-projections $\vec{a}_i'$ of $\vec{a}_i$, $i = 1,2,\ldots,m$, also satisfy $\vec{a}_1' \wedge \vec{a}_2' \wedge \cdots \wedge \vec{a}_m' \gg 0$. Therefore, by considering the α-projection if necessary, we may assume that $n = m+1$ and $(-1)^{i-1}\vec{b}_i \geqslant 0$, $i = 1,2,\ldots,n$. Further, $\vec{\mathbf{e}}_{(i)}^{\wedge} := \vec{e}_1 \wedge \cdots \wedge \vec{e}_{i-1} \wedge \vec{e}_{i+1} \wedge \cdots \wedge \vec{e}_n$, $i = 1,2,\ldots,n$, form a complete orthogonal basis of $\bigwedge^m \mathbf{R}^n$, so that

$$\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m = \sum_{i=1}^{n} \zeta_i \vec{\mathbf{e}}_{(i)}^{\wedge} \quad \text{and} \quad \zeta_i > 0, \quad i = 1,2,\ldots,n.$$

On the other hand, since $\vec{b}$ is a linear combination of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$, and $\vec{b} = \sum_{i=1}^{n} b_i \vec{e}_i$, by (1.11)

$$\begin{aligned} 0 &= \vec{b} \wedge \vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m \\ &= \left\{ \sum_{i=1}^{n} (-1)^{i-1} \zeta_i b_i \right\} \cdot \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n. \end{aligned}$$

Then $\zeta_i > 0$ and $(-1)^{i-1} b_i \geqslant 0$, $i = 1,2,\ldots,n$, imply $b_i = 0$, $i = 1,2,\ldots,n$, and hence $\vec{b} = 0$, a contradiction. This completes the proof of sufficiency.

Let us turn to the proof of necessity. Since $\mathscr{V}_+(0) = n-1$ and $m < n$, the assumption implies first that $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ are linearly independent, that is, $\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m \neq 0$. Let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m]$. Then by (1.11)

$$\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m = \sum_{\alpha \in Q_{m,n}} \det A[\alpha|\text{-}] \vec{\mathbf{e}}_{\alpha}^{\wedge},$$

and we have to show that $\det A[\alpha|\text{-}] > 0$ (or < 0) uniformly for all $\alpha \in Q_{m,n}$. Any two different $\alpha, \beta \in Q_{m,n}$ can be joined by a sequence $\omega^{(p)} \in Q_{m,n}$, $p = 0,1,\ldots,k$, such that $\alpha = \omega^{(0)}$, $\beta = \omega^{(k)}$ and for each $i = 1,2,\ldots,k$ there is $\tau^{(i)} \in Q_{m+1,n}$ such that $\omega^{(i-1)} \subset \tau^{(i)}$ and $\omega^{(i)} \subset \tau^{(i)}$. Since the inequality $\det A[\alpha|\text{-}] \det A[\beta|\text{-}] > 0$ follows from the inequalities

$$\det A\left[\omega^{(i-1)}|\text{-}\right] \det A\left[\omega^{(i)}|\text{-}\right] > 0, \qquad i = 1,2,\ldots,k,$$

considering, in the ith step, the $\tau^{(i)}$-projection, we may assume from the first that $n = m + 1$. Now as in the first part,

$$\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m = \sum_{i=1}^{n} \zeta_i \vec{e}^{\wedge}_{(i)} \qquad \text{with} \quad \zeta_i = \det A(i|\text{-}],$$

and we have to show that $\zeta_i\zeta_j > 0$, $i, j = 1,2,\ldots,n$. If $\zeta_i = 0$ for some i, by (1.11)

$$\vec{e}_i \wedge \vec{a}_1 \wedge \cdots \wedge \vec{a}_m = (-1)^{i-1}\zeta_i \cdot \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n = 0;$$

hence $\vec{e}_i$ becomes a linear combination of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$, but $\mathscr{V}_+(\vec{e}_i) = n - 1 = m$, a contradiction. Further, if not all ζ_j have the same sign, then $\zeta_l\zeta_{l+1} < 0$ for some l. Then, as above,

$$\begin{aligned}&(\zeta_{l+1}\vec{e}_l + \zeta_l\vec{e}_{l+1}) \wedge \vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m \\ &\quad = \left\{(-1)^{l-1}\zeta_{l+1}\zeta_l + (-1)^l\zeta_l\zeta_{l+1}\right\} \cdot \vec{e}_1 \wedge \vec{e}_2 \wedge \cdots \wedge \vec{e}_n = 0,\end{aligned}$$

and $\zeta_{l+1}\vec{e}_l + \zeta_l\vec{e}_{l+1}$ becomes a linear combination of $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$. But since $\zeta_l\zeta_{l+1} < 0$, we have $\mathscr{V}_+(\zeta_{l+1}\vec{e}_l + \zeta_l\vec{e}_{l+1}) = n - 1 = m$, a contradiction. This completes the proof of necessity. ■

THEOREM 5.2. *Let $\mathscr{M}$ be a (real linear) subspace of $\mathbf{R}^n$ and $0 < \dim_{\mathbf{R}}(\mathscr{M}) < n$. Then the following conditions are mutually equivalent:*

(a) $\mathscr{V}_+(\vec{x}) \leqslant \dim_{\mathbf{R}}(\mathscr{M}) - 1$ *for* $0 \neq \vec{x} \in \mathscr{M}$.
(b) $\mathscr{V}_-(\vec{y}) \geqslant \dim_{\mathbf{R}}(\mathscr{M})$ *for* $0 \neq \vec{y} \in \mathscr{M}^{\perp}$, *the orthocomplement in* $\mathbf{R}^n$.

Proof. Take complete orthonormal bases $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m$ for $\mathscr{M}$, and $\vec{a}_{m+1}, \vec{a}_{m+2}, \ldots, \vec{a}_n$ for $\mathscr{M}^{\perp}$, and let $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$. Then A is unitary and we may assume $\det A = 1$. According to Lemma 5.1, (a) implies that $\vec{a}_1 \wedge \vec{a}_2 \wedge \cdots \wedge \vec{a}_m \gg 0$ or $\ll 0$, which is equivalent, by (1.22), to the statement that $\det A[\alpha|1,2,\ldots,m]$ is nonzero and has one and the same sign for all $\alpha \in Q_{m,n}$. Since by (1.32) and $\det A = 1$

$$\begin{aligned}\det(J_nAJ_n)(\alpha|1,2,\ldots,m) &= \det\left(J_nA^{*-1}J_n\right)(\alpha|1,2,\ldots,m) \\ &= \det A^*\left[1,2,\ldots m|\alpha\right] = \det A\left[\alpha|1,2,\ldots,m\right],\end{aligned}$$

$\det(J_nAJ_n)[\tau|m+1,\ldots,n]$ is nonzero and has one and the same sign for all $\tau \in Q_{n-m,n}$, or equivalently, with $\vec{b}_i := J_nAJ_n\vec{e}_i$, $i = m+1, m+2, \ldots, n$, we have $\vec{b}_{m+1} \wedge \vec{b}_{m+2} \wedge \cdots \wedge \vec{b}_n \gg 0$ or $\ll 0$. But obviously $\vec{b}_i = (-1)^i J_n \vec{a}_i$; hence $J_n\vec{a}_{m+1} \wedge J_n\vec{a}_{m+2} \wedge \cdots \wedge J_n\vec{a}_n$ is strictly definite. Then again by Lemma 5.1, $\mathscr{V}_+(J_n\vec{y}) \leqslant n-m-1$ for $0 \neq \vec{y} \in \mathscr{M}^{\perp}$. Now apply (5.1) to get (b). Thus (a) implies (b). (b) $\Rightarrow$ (a) is proved similarly. ∎

A local version of Theorem 5.2 gives the following characterization of strict sign regularity in terms of a variation-diminishing property.

THEOREM 5.3. *Let A be an $n \times m$ real matrix with $n \geqslant m$. Then A is strictly sign-regular if and only if the real linear map A from $\mathbf{R}^m$ to $\mathbf{R}^n$ diminishes variation of signs in the sense that*

$$\mathscr{V}_+(A\vec{x}) \leqslant \mathscr{V}_-(\vec{x}) \qquad \textit{for} \quad 0 \neq \vec{x} \in \mathbf{R}^m. \tag{5.3}$$

Proof. Suppose that $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m]$ is strictly sign-regular with signature $\vec{\varepsilon}$. Take any $0 \neq \vec{x} \in \mathbf{R}^m$, and let $k := \mathscr{V}_-(\vec{x})$. Then there exist $\beta, \omega \in Q_{k+1,m}$ such that $\beta_i \leqslant \omega_i < \beta_{i+1}$, $i = 1,2,\ldots,k+1$ (with $\beta_{k+2} = \infty$) such that, for each $i = 1,2,\ldots,k+1$, the components x_j have constant sign for all j between β_i and ω_i, with sign alternating along i, and that $x_j = 0$ if $j < \beta_1$, $j > \omega_{k+1}$, or $\omega_i < j < \beta_{i+1}$ for some i. Let $\vec{b}_i := \sum_{\beta_i \leqslant j \leqslant \omega_i} x_j \vec{a}_j$, $i = 1,2,\ldots,k+1$. Then obviously $A\vec{x} = \sum_{i=1}^{k+1} \vec{b}_i$. Now the strict sign regularity of A implies that

$$\varepsilon_{k+1} \cdot \vec{a}_{j_1} \wedge \vec{a}_{j_2} \wedge \cdots \wedge \vec{a}_{j_{k+1}} \gg 0 \qquad \text{for} \quad \beta_i \leqslant j_i \leqslant \omega_i, \quad i = 1,2,\ldots,k+1,$$

so that

$$\vec{b}_1 \wedge \vec{b}_2 \wedge \cdots \wedge \vec{b}_{k+1} \gg 0 \text{ or } \ll 0.$$

Then Lemma 5.1 yields that

$$\mathscr{V}_+(A\vec{x}) = \mathscr{V}_+\left(\sum_{i=1}^{k+1} \vec{b}_i\right) \leqslant k = \mathscr{V}_-(\vec{x}),$$

proving (5.3).

Suppose conversely that $A = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_m]$ satisfies the condition (5.3). For each $\omega \in Q_{k,m}$ and $\xi_i \in \mathbf{R}$, $i = 1,2,\ldots,k$, with $\sum_{i=1}^{k} |\xi_i| \neq 0$, obviously $\mathscr{V}_-(\sum_{i=1}^{k} \xi_i \vec{e}_{\omega_i}) \leqslant k-1$; hence by assumption

$$\mathscr{V}_+\left(\sum_{i=1}^{k} \xi_i \vec{a}_{\omega_i}\right) = \mathscr{V}_+\left(A \sum_{i=1}^{k} \xi_i \vec{e}_{\omega_i}\right) \leqslant k-1.$$

Then it follows from Lemma 5.1 that $\vec{a}_{\omega_1} \wedge \vec{a}_{\omega_2} \wedge \cdots \wedge \vec{a}_{\omega_k}$ is strictly definite. A will be strictly sign-regular if the sign of $\vec{a}_{\omega_1} \wedge \vec{a}_{\omega_2} \wedge \cdots \wedge \vec{a}_{\omega_k}$ depends only on k. For $k = m$ this is trivial. Fix $1 \leqslant k \leqslant m-1$ and take $\alpha, \beta \in Q_{k,m}$. As remarked in the proof of Lemma 5.1, there is a sequence $\omega^{(p)} \in Q_{k,m}$, $p = 0, 1, \dots, l$, such that $\alpha = \omega^{(0)}$, $\beta = \omega^{(l)}$, and there is a sequence $\tau^{(p)} \in Q_{k+1,m}$ with $\omega^{(i)} \subset \tau^{(i)}$, $\omega^{(i-1)} \subset \tau^{(i)}$, $i = 1, 2, \dots, l$. Therefore, for our purpose it suffices to prove that, for each $\tau \in Q_{k+1,m}$ and $1 \leqslant i \leqslant k+1$, $\vec{a}_{\tau_1} \wedge \cdots \wedge \vec{a}_{\tau_{i-1}} \wedge \vec{a}_{\tau_{i+1}} \wedge \cdots \wedge \vec{a}_{\tau_{k+1}}$ and $\vec{a}_{\tau_1} \wedge \cdots \wedge \vec{a}_{\tau_i} \wedge \vec{a}_{\tau_{i+2}} \wedge \cdots \wedge \vec{a}_{\tau_{k+1}}$ have the same sign. By means of continuity argument this will be established if

$$\vec{a}_{\tau_1} \wedge \cdots \wedge \vec{a}_{\tau_{i-1}} \wedge \left\{(1-t)\vec{a}_{\tau_i} + t\vec{a}_{\tau_{i+1}}\right\} \wedge \vec{a}_{\tau_{i+2}} \wedge \cdots \wedge \vec{a}_{\tau_{k+1}}$$

is strictly definite for each $0 < t < 1$. But the strict definiteness follows from Lemma 5.1, via (5.3), because for any $\xi_i \in \mathbf{R}$, $i = 1, 2, \dots, k$, with $\sum_{i=1}^{k} |\xi_i| \neq 0$,

$$\mathscr{V}_-\left(\sum_{j=1}^{i-1} \xi_j \vec{e}_{\tau_j} + \xi_i\big((1-t)\vec{e}_{\tau_i} + t\vec{e}_{\tau_{i+1}}\big) + \sum_{j=i+2}^{k+1} \xi_j \vec{e}_{\tau_j} \right) \leqslant k-1. \qquad \blacksquare$$

Sign regularity is characterized by a weaker variation-diminishing property.

COROLLARY 5.4. *Let A be an $n \times m$ real matrix of rank m. Then A is sign-regular if and only if*

$$\mathscr{V}_-(A\vec{x}) \leqslant \mathscr{V}_-(\vec{x}) \qquad \textit{for} \quad 0 \neq \vec{x} \in \mathbf{R}^m. \tag{5.4}$$

Proof. As shown in Section 7, there is a sequence of n-square, strictly totally positive matrices G_p, $p = 1, 2, \dots$, such that $G_p \to I_n$ as $p \to \infty$. Suppose first that A is sign-regular. Since A is injective by assumption, $G_p A$ is strictly sign-regular, and $G_p A \to A$ as $p \to \infty$. Then Theorem 5.3 guarantees that

$$\mathscr{V}_+(G_p A\vec{x}) \leqslant \mathscr{V}_-(\vec{x}) \qquad \text{for} \quad 0 \neq \vec{x} \in \mathbf{R}^m,$$

which yields (5.4) via (5.2). Suppose conversely that (5.4) is valid. By

Theorem 5.3, applied to G_p,

$$\mathscr{V}_+(G_pA\vec{x}) \leqslant \mathscr{V}_-(A\vec{x}) \qquad \text{for} \quad 0 \neq \vec{x} \in \mathbf{R}^m,$$

because A is injective. Now (5.4) combined with Theorem 5.3 shows that G_pA is strictly sign-regular for $p = 1, 2, \ldots$. Then obviously A is sign-regular. ∎

By using the duality relation (5.1), we can speak about some variation-augmenting properties.

Corollary 5.5. *Let A be an $n \times m$ real matrix of rank m. Then J_nAJ_m is strictly sign-regular (respectively, sign-regular) if and only if*

$$n - m + \mathscr{V}_+(\vec{x}) \leqslant \mathscr{V}_-(A\vec{x})\ (\mathscr{V}_+(A\vec{x})) \qquad \textit{for} \quad 0 \neq \vec{x} \in \mathbf{R}^m.$$

When $n = m$, sign regularity admits several cousin characterizations.

Theorem 5.6. *Let A be an n-square invertible real matrix. Then the following conditions are mutually equivalent:*

(a) *A is sign-regular.*
(b) *$\mathscr{V}_+(A\vec{x}) \leqslant \mathscr{V}_+(\vec{x})$ for all $\vec{x} \in \mathbf{R}^n$.*
(c) *$\mathscr{V}_-(A\vec{x}) \leqslant \mathscr{V}_+(\vec{x})$ for all $\vec{x} \in \mathbf{R}^n$.*
(d) *$\mathscr{V}_-(A\vec{x}) \leqslant \mathscr{V}_-(\vec{x})$ for all $\vec{x} \in \mathbf{R}^n$.*

Proof. (a) $\Rightarrow$ (b): If A is sign-regular (and invertible), so is $J_nA^{-1}J_n$ by Theorem 3.2. Then (b) follows from Corollary 5.5, on replacing A and $\vec{x}$ by A^{-1} and $A\vec{x}$ respectively. (b) $\Rightarrow$ (c) is trivial. (c) $\Rightarrow$ (d) results on replacing $\vec{x}$ by $G_p\vec{x}$ and taking the limit as $p \to \infty$, where G_p is a strictly totally positive matrix in the proof of Corollary 5.4. Finally (d) $\Rightarrow$ (a) follows from Corollary 5.4. ∎

Notes and References to Section 5

The theory of variation-diminishing linear maps originated with Schoenberg (1930). Schoenberg and Whitney (1951) also studied cyclic variation-diminishing linear maps; the cyclic maximum variation of signs of a vector $\vec{x}$, for instance, is defined as the maximum of $\mathscr{V}_+(\vec{x}^{(k)})$, $k = 1, 2, \ldots, n$, where the ith component of $\vec{x}^{(k)}$ is given by x_{k+i-1} (mod n).

6. EIGENVALUES AND EIGENVECTORS

In this section we shall study spectral properties of sign-regular or totally positive matrices. The key tool for this is the classical results of Perron and Frobenius for positive matrices. Let us formulate the most elementary part of the Perron-Frobenius theorem, necessary for our purpose.

LEMMA 6.1. *If A is an n-square positive matrix, $A \geqslant 0$, then the first eigenvalue is real nonnegative, $\lambda_1(A) \geqslant 0$, and there is a positive eigenvector $\vec{u}_1 \geqslant 0$ corresponding to $\lambda_1(A)$. If A is strictly positive, $A \gg 0$, then $\lambda_1(A) > |\lambda_2(A)|$, and each eigenvector corresponding to $\lambda_1(A)$ is a scalar multiple of a strictly positive one $\vec{u}_1 \gg 0$.*

THEOREM 6.2. *If A is an n-square, strictly sign-regular matrix with signature $\vec{\varepsilon}$, then all eigenvalues of A are real and distinct, and*

$$\frac{\varepsilon_k}{\varepsilon_{k-1}}\lambda_k(A) > |\lambda_{k+1}(A)|, \qquad k = 1,2,\dots,n, \tag{6.1}$$

where $\varepsilon_0 = 1$ and $\lambda_{n+1}(A) = 0$, and the corresponding eigenvectors $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n$ can be so chosen that each $\vec{u}_k$ is a real vector and

$$\vec{u}_1 \wedge \vec{u}_2 \wedge \cdots \wedge \vec{u}_k \gg 0, \qquad k = 1,2,\dots,n. \tag{6.2}$$

Proof by induction. The case $k=1$ is immediate from Lemma 6.1, because $\varepsilon_1 A \gg 0$ by assumption. Suppose that $2 \leqslant m \leqslant n$ and (6.1) and (6.2) are true for all k with $1 \leqslant k \leqslant m-1$. Since $\varepsilon_m \cdot \bigwedge^m A \gg 0$ by assumption, and its first eigenvalue is $\varepsilon_m \prod_{i=1}^m \lambda_i(A)$ by Theorem 1.1, it follows again from Lemma 6.1 that

$$\prod_{i=1}^{m} \frac{\varepsilon_i}{\varepsilon_{i-1}}\lambda_i(A) = \varepsilon_m \prod_{i=1}^{m} \lambda_i(A) > \prod_{i=1}^{m-1} |\lambda_i(A)| \cdot |\lambda_{m+1}(A)|.$$

Then (6.1) for $k=m$ results from the induction assumption. Now since $\lambda_m(A)$ is real, $\vec{u}_m$ can be chosen a real vector, and $\vec{u}_1 \wedge \vec{u}_2 \wedge \cdots \wedge \vec{u}_m$ becomes a nonzero real eigenvector of $\varepsilon_m \cdot \bigwedge^m A$, corresponding to its first eigenvalue. Then by Lemma 6.1, with $\xi = 1$ or $= -1$, $\xi \cdot \vec{u}_1 \wedge \vec{u}_2 \wedge \cdots \wedge \vec{u}_m \gg 0$. Now replace $\vec{u}_m$ by $\xi \vec{u}_m$ if necessary, to get (6.2) for $k=m$. ∎

The set of real eigenvectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ possesses interesting oscillatory properties. For their formulations, we need some definitions. To each real n-vector $\vec{x}$, assign the piecewise linear function $x(t)$ for $1 \leqslant t \leqslant n$, defined by

$$x(t) = (k+1-t)x_k + (t-k)x_{k+1} \qquad \text{if} \quad k \leqslant t \leqslant k+1. \tag{6.3}$$

The *nodes* of $x(t)$ are the roots of the equation $x(t) = 0$, arranged in increasing order. Two ordered sequences $\xi_1 < \xi_2 < \cdots < \xi_k$ and $\eta_1 < \eta_2 < \cdots < \eta_{k+1}$ are said to be *interlacing* if $\eta_i < \xi_i < \eta_{i+1}$, $i = 1, 2, \dots, k$.

THEOREM 6.3. *Let A be an n-square, strictly sign-regular matrix. Then its real eigenvector $\vec{u}_k$, corresponding to the kth eigenvalue, has exact $k-1$ variations of sign:*

$$\mathscr{V}(\vec{u}_k) = k-1, \qquad k = 1, 2, \dots, n. \tag{6.4}$$

Furthermore the nodes of $u_k(t)$ and those of $u_{k+1}(t)$ are interlacing.

Proof. By Theorem 6.2, for each k we have $\vec{u}_1 \wedge \vec{u}_2 \wedge \cdots \wedge \vec{u}_k \gg 0$ or $\ll 0$; hence $\mathscr{V}_+(\vec{u}_k) \leqslant k-1$ by Lemma 5.1. To see $\mathscr{V}_-(\vec{u}_k) \geqslant k-1$, according to (5.1) it suffices to show $\mathscr{V}_+(J_n\vec{u}_k) \leqslant n-k$. Consider $J_nA^{-1}J_n$, which is again strictly sign-regular by Theorem 3.3. Since $J_n\vec{u}_k$ is an eigenvector of $J_nA^{-1}J_n$ corresponding to $1/\lambda_k(A) = \lambda_{n-k+1}(J_nA^{-1}J_n)$, the above argument yields $\mathscr{V}_+(J_n\vec{u}_k) \leqslant n-k$. This proves (6.4).

Next we claim that for $1 \leqslant k \leqslant n-1$

$$\mathscr{V}_+(\xi\vec{u}_k + \zeta\vec{u}_{k+1}) - 1 \leqslant \mathscr{V}_-(\xi\vec{u}_k + \zeta\vec{u}_{k+1})$$

$$\text{whenever} \quad \xi, \zeta \in \mathbf{R} \text{ and } |\xi| + |\zeta| \neq 0. \tag{6.5}$$

Since again $\vec{u}_1 \wedge \cdots \wedge \vec{u}_k \wedge \vec{u}_{k+1} \gg 0$ or $\ll 0$, Lemma 5.1 guarantees

$$\mathscr{V}_+(\xi\vec{u}_k + \zeta\vec{u}_{k+1}) \leqslant (k+1) - 1 = k.$$

Apply the same argument to $J_n\vec{u}_n, J_n\vec{u}_{n-1}, \dots, J_n\vec{u}_k$, which are the first $n-k+1$ eigenvectors of the strictly sign-regular matrix $J_nA^{-1}J_n$, to see

$$\mathscr{V}_+(\xi J_n\vec{u}_k + \zeta J_n\vec{u}_{k+1}) \leqslant n-k.$$

Hence (5.1) yields (6.5):

$$\mathscr{V}_-(\xi\vec{u}_k + \zeta\vec{u}_{k+1}) = n - 1 - \mathscr{V}_+(\xi J_n\vec{u}_k + \zeta J_n\vec{u}_{k+1})$$

$$\geqslant k - 1 \geqslant \mathscr{V}_+(\xi\vec{u}_k + \zeta\vec{u}_{k+1}) - 1.$$

Now let us turn to the proof of the second assertion. Let $x(t) = u_k(t)$ and $y(t) = u_{k+1}(t)$. In view of (6.4), $x(t)$ and $y(t)$ have $k-1$ and k nodes, respectively, and none of these nodes is integer. Let $t_1 < t_2 < \cdots < t_k$ be the nodes of $y(t)$. Then for the second assertion, it suffices to show that $x(t)$ has at least one node in each open interval (t_l, t_{l+1}), $l = 1,2,\dots,k-1$. For this purpose, (6.5) will be used in the following form: if $|\xi| + |\zeta| \neq 0$ and $1 < j < n$,

$$\{\xi x(j-1) + \zeta y(j-1)\}\{\xi x(j+1) + \zeta y(j+1)\} < 0$$

$$\text{whenever} \quad \xi x(j) + \zeta y(j) = 0. \tag{6.6}$$

Suppose that $x(t)$ has no node in the interval (t_l, t_{l+1}), that is, $x(t) > 0$, say, on this interval. We claim that $x(t) \geqslant \delta > 0$ uniformly on the closed interval $[t_l, t_{l+1}]$. Otherwise, $x(t_l) = 0$, say. Take i such that $i - 1 < t_l < i$. Since $x(t)$ is linear for $i - 1 \leqslant t \leqslant i$, we have $x(i-1)x(i) < 0$, and with the choice

$$\xi = -\frac{y(i) - y(i-1)}{x(i) - x(i-1)},$$

$\xi x(t) + y(t)$ vanishes for all $i - 1 \leqslant t \leqslant i$, contradicting (6.6). By the definition of nodes, $y(t)$ is definite, $\geqslant 0$ say, on the interval $[t_l, t_{l+1}]$. Now let η be the minimum of $\eta > 0$ for which $-\eta y(t) + x(t)$ has a node s say, $t_l \leqslant s \leqslant t_{l+1}$. Since by the minimum property $-\eta y(t) + x(t) \geqslant 0$ on the interval, and $-\eta y(t) + x(t)$ is piecewise-linear as (6.3), this is possible only when s is an integer or $-\eta y(t) + x(t)$ vanishes identically on the interval $j - 1 \leqslant t \leqslant j$ containing s. But each of these possibilities produces a contradiction to (6.6). ∎

If A is an n-square strictly sign-regular matrix, its adjoint A^* is again strictly sign-regular, and by Theorem 6.2 the real eigenvectors $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of A^* are so chosen that

$$\vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_k \gg 0, \qquad k = 1,2,\dots,n. \tag{6.7}$$

The properties (6.2) and (6.7) of the eigenvectors of A and A^* characterize strict sign-regularity in some sense.

THEOREM 6.4. *If an n-square invertible real matrix A has n real eigenvalues with distinct moduli and the real eigenvectors $\vec{u}_k$ of A and $\vec{v}_k$ of A^*, corresponding to $\lambda_k(A)=\lambda_k(A^*)$, are so chosen to satisfy* (6.2) *and* (6.7):

$$\vec{u}_1 \wedge \vec{u}_2 \wedge \cdots \wedge \vec{u}_k \gg 0 \ \text{ and } \ \vec{v}_1 \wedge \vec{v}_2 \wedge \cdots \wedge \vec{v}_k \gg 0, \qquad k=1,2,\ldots,n,$$

then some power of A is strictly sign-regular.

Proof. Let $\lambda_k=\lambda_k(A)$, $k=1,2,\ldots,n$, and let $U=[\vec{u}_1,\vec{u}_2,\ldots,\vec{u}_n]$ and $V=[\vec{v}_1,\vec{v}_2,\ldots,\vec{v}_n]$. Then U and V are invertible,

$$A=U\cdot\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\cdot U^{-1}, \qquad A^*=V\cdot\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\cdot V^{-1}, \tag{6.8}$$

and

$$|\lambda_1|>|\lambda_2|>\cdots>|\lambda_n|>0. \tag{6.9}$$

Since obviously $\langle \vec{u}_i,\vec{v}_j\rangle=0$ for $i\neq j$, (6.8) implies

$$U^{-1}=\operatorname{diag}(\rho_1,\rho_2,\ldots,\rho_n)V^* \tag{6.10}$$

for some nonzero ρ_i, $i=1,2,\ldots,n$. These ρ_i are all positive, because

$$0<\langle \vec{u}_1\wedge\vec{u}_2\wedge\cdots\wedge\vec{u}_k, \vec{v}_1\wedge\vec{v}_2\wedge\cdots\wedge\vec{v}_k\rangle=\prod_{i=1}^{k}\rho_i^{-1}, \qquad k=1,2,\ldots,n.$$

By (1.23), for any positive integer p and $\alpha,\beta\in Q_{k,n}$ it follows from (6.8) and (6.10) that

$$\begin{aligned}
\det A^p[\alpha|\beta] &= \sum_{\omega\in Q_{k,n}} \det U[\alpha|\omega]\cdot\left(\prod_{i=1}^{k}\lambda_{\omega_i}\right)^p\cdot\det U^{-1}[\omega|\beta] \\
&= \sum_{\omega\in Q_{k,n}} \det U[\alpha|\omega]\cdot\left(\prod_{i=1}^{k}\lambda_{\omega_i}\right)^p\cdot\left(\prod_{i=1}^{k}\rho_{\omega_i}\right)\cdot\det V[\beta|\omega] \\
&= \left(\prod_{i=1}^{k}\lambda_i\right)^p\left(\prod_{i=1}^{k}\rho_i\right)\det U[\alpha|1,2,\ldots,k]\det V[\beta|1,2,\ldots,k] \\
&\quad + \sum_{\substack{\omega\in Q_{k,n}\\ \omega\neq\{1,2,\ldots,k\}}} \det U[\alpha|\omega]\cdot\left(\prod_{i=1}^{k}\lambda_{\omega_i}\right)^p\left(\prod_{i=1}^{k}\rho_{\omega_i}\right)\cdot\det V[\beta|\omega].
\end{aligned}$$

(6.9) implies that

$$\prod_{i=1}^{k} |\lambda_i| > \prod_{i=1}^{k} |\lambda_{\omega_i}| \qquad \text{for} \quad \omega \in Q_{k,n}, \quad \omega \neq \{1,2,\dots,k\},$$

while (6.2) and (6.7) imply that

$$U[\alpha|1,2,\dots,k] > 0 \text{ and } V[\beta|1,2,\dots,k] > 0 \qquad \text{for} \quad \alpha,\beta \in Q_{k,n}.$$

Then for sufficiently large p, det $A^p[\alpha|\beta]$ is nonzero and has the same sign as $(\prod_{i=1}^{k}\lambda_i)^p$ for every α, $\beta \in Q_{k,n}$, that is, A^p is strictly sign-regular. ∎

Our next task is the comparison of the eigenvalues of A with those of $A[\alpha]$ for suitable α.

THEOREM 6.5. *If A is an n-square oscillatory matrix, then for every $\alpha \in Q_{k,n}$ $(1 \leqslant k \leqslant n-1)$ with consecutive components, i.e. $d(\alpha) = 0$,*

$$\lambda_j(A) > \lambda_j(A[\alpha]) > \lambda_{n+j-k}(A), \qquad j = 1,2,\dots,k, \tag{6.11}$$

and

$$\lambda_j(A) > \lambda_j(A/\alpha') > \lambda_{n+j-k}(A), \qquad j = 1,2,\dots,k. \tag{6.12}$$

Proof. Let us prove (6.11) by backward induction on k. When $k = n-1$, we have $\alpha = \{1,2,\dots,n-1\}$ or $= \{2,3,\dots,n\}$. Supposing $\alpha = \{1,2,\dots,n-1\}$, let $B = A[\alpha]$. Clearly $\lambda_j := \lambda_j(A)$, $j = 1,2,\dots,n$ are the only nodes of the polynomial $d_A(t) := \det A_t$ where $A_t := tI_n - A$, while $\lambda_j(B)$, $j = 1,2,\dots,n-1$, are the only nodes of the polynomial $d_B(t) := \det B_t$ where $B_t := tI_{n-1} - B$. To see (6.11) for this α, it suffices to show that

$$d_B(\lambda_i) d_B(\lambda_{i+1}) < 0, \qquad i = 1,2,\dots,n-1. \tag{6.13}$$

Consider the vectors $\vec{x}_t$ with real parameter t, defined by

$$\vec{x}_t := \left[(-1)^{n+i} \det A_t[\alpha|i)\right]_{1 \leqslant i \leqslant n}.$$

Then (1.31) yields that $A_t\vec{x}_t = d_A(t)\vec{e}_n$, so that

$$A\vec{x}_{\lambda_j} = \lambda_j \vec{x}_{\lambda_j}, \qquad j = 1,2,\dots,n. \tag{6.14}$$

The nth component $x_t(n)$ of $\vec{x}_t$ clearly coincides with $d_B(t)$, while the first component $x_t(1)$ admits the representation

$$x_t(1) = \sum_{j=2}^{n} t^{n-j} \sum_{\substack{\omega \in Q_{j,n} \\ \omega_1 = 1,\ \omega_j = n}} \det A[\omega \setminus \{n\} | \omega \setminus \{1\}]. \tag{6.15}$$

We claim that $x_t(1) > 0$ for all $t > 0$. In fact, since $d_B(t)$ has only $n-1$ nodes, for some j we have $x_{\lambda_j}(n) = d_B(\lambda_j) \neq 0$. Then by (6.14), $\vec{x}_{\lambda_j}$ is a nonzero real eigenvector of oscillatory matrix A, corresponding to $\lambda_j = \lambda_j(A)$, and its first component $x_{\lambda_j}(1)$ does not vanish, because $\vec{x}_{\lambda_j}$ has exactly $j-1$ variations of sign by Theorem 6.3. On the other hand, since A is totally positive, (6.15) shows that $x_t(1)$ is a polynomial of t with nonnegative coefficients. Then $x_t(1) > 0$ for all $t > 0$. Now by (6.14), for each i, $\vec{x}_{\lambda_i}$ is the ith eigenvector of A with positive first component. Then it follows from Theorem 6.3 that the nth component of $\vec{x}_{\lambda_i}$ has sign $(-1)^{i-1}$. This establishes (6.13), because $x_{\lambda_i}(n) = d_B(\lambda_i)$, $i = 1,2,\ldots,n$. The proof of (6.11) for $\alpha = \{2,3,\ldots,n\}$ is similar.

Suppose that (6.11) is true for $k > 1$, and take $\alpha \in Q_{k-1,n}$ with $d(\alpha) = 0$. We may assume that $\alpha = \{i, i+1,\ldots, i+k-2\}$ and $i+k-1 \leqslant n$. Now apply the above argument to the k-square oscillatory matrix $A[\alpha \cup \{i+k-1\}]$ to get

$$\lambda_j(A[\alpha \cup \{i+k-1\}]) > \lambda_j(A[\alpha]) > \lambda_{j+1}(A[\alpha \cup \{i+k-1\}]),$$
$$j = 1,2,\ldots,k-1.$$

On the other hand, the induction assumption implies

$$\lambda_j(A) > \lambda_j(A[\alpha \cup \{i+k-1\}]) > \lambda_{n+j-k}(A), \qquad j = 1,2,\ldots,k.$$

These together prove (6.11) for the case $k-1$, completing induction. The case $2 \leqslant i$ is treated similarly.

Finally (6.12) follows from (6.11). In fact, $J_n A^{-1} J_n$ is again oscillatory and $(J_n A^{-1} J_n)[\alpha] = J_\alpha (A/\alpha')^{-1} J_\alpha$ by Theorem 4.2. Now apply (6.11), remarking that

$$\frac{1}{\lambda_j(A)} = \lambda_{n-j+1}(J_n A^{-1} J_n) \quad \text{and} \quad \frac{1}{\lambda_j(A/\alpha')} = \lambda_{k-j+1}((J_n A^{-1} J_n)[\alpha]). \blacksquare$$

With the help of the approximation theorem 2.7, some of the above results can be generalized to the case A is sign-regular or totally positive. Let us present sample results.

COROLLARY 6.6. *If A is an n-square, sign-regular matrix with signature $\vec{\varepsilon}$, then all its eigenvalues are real, and*

$$\frac{\varepsilon_k}{\varepsilon_{k-1}}\lambda_k(A) > 0, \qquad k = 1,2,\dots,\operatorname{rank}(A).$$

If A is totally positive, then for any $\alpha \in Q_{k,n}$ $(1 \leqslant k \leqslant n-1)$ with consecutive components, i.e. $d(\alpha) = 0$,

$$\lambda_j(A) \geqslant \lambda_j(A[\alpha]) \geqslant \lambda_{n+j-k}(A), \qquad j = 1,2,\dots,k.$$

Given a real n-vector $\vec{x} = (x_i)$, let us denote by $\vec{x}^* = (x_i^*)$ its *decreasing rearrangement*:

$$x_1^* \geqslant x_2^* \geqslant \cdots \geqslant x_n^* \quad \text{and} \quad x_i^* = x_{\pi(i)} \quad \text{for some } \pi \in S_n. \tag{6.16}$$

A real n-vector $\vec{x}$ is said to be *majorized* by another, $\vec{y}$—in symbols $\vec{x} \prec \vec{y}$—if

$$\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i \quad \text{and} \quad \sum_{i=1}^{k} x_i^* \leqslant \sum_{i=1}^{k} y_i^*, \quad k = 1,2,\dots,n-1. \tag{6.17}$$

Obviously the inequality $\sum_{i=1}^{k} x_i^* \leqslant \sum_{i=1}^{k} y_i^*$ in (6.17) can be replaced by $\sum_{i=k+1}^{n} x_i^* \geqslant \sum_{i=k+1}^{n} y_i^*$.

The majorization relation is known to produce a lot of inequalities, based on the following fact: $\vec{x} \prec \vec{y}$ if and only if for any convex continuous function $\Phi(t)$ on $(-\infty,\infty)$

$$\sum_{i=1}^{n} \Phi(x_i) \leqslant \sum_{i=1}^{n} \Phi(y_i). \tag{6.18}$$

THEOREM 6.7. *Let $A = [a_{ij}]$ be an n-square matrix, and $\vec{\delta}(A) := (a_{ii})$, the diagonal of A. If A is totally positive, then $\vec{\delta}(A) \prec \vec{\lambda}(A)$.*

Proof by induction on n. When $n = 1$, everything is trivial. Assume that the assertion is true with $n - 1$ in place of n. Since $\sum_{i=1}^{n}\lambda_i(A) = \sum_{i=1}^{n}\delta_i(A)$ and $\lambda_i(A) = \lambda_i^*(A)$, $i = 1,2,\dots,n$, by definition and Corollary 6.6, it suffices to show that

$$\sum_{i=1}^{k}\lambda_i(A) \geqslant \sum_{i=1}^{k}\delta_i^*(A) \qquad \text{for} \quad k = 1,2,\dots,n-1. \tag{6.19}$$

Let $\delta_1(A) = \delta_p^*(A)$ and $\delta_n(A) = \delta_q^*(A)$. Considering the conversion if necessary, we may assume $p < q$. Let $B = A(n)$ and $C = A(1)$. Since B and C are $(n-1)$-square totally positive matrices, the induction assumption yields that

$$\sum_{i=1}^{k}\lambda_i(B) \geqslant \sum_{i=1}^{k}\delta_i^*(B) \quad \text{and} \quad \sum_{i=k}^{n-1}\lambda_i(C) \leqslant \sum_{i=k}^{n-1}\delta_i^*(C),$$
$$k = 1,2,\dots,n-1. \tag{6.20}$$

Since $\delta_i^*(B) = \delta_i^*(A)$, $i = 1,2,\dots,p$, by definition and since $\lambda_i(A) \geqslant \lambda_i(B)$, $i = 1,2,\dots,n-1$, by Corollary 6.6, (6.20) implies (6.19) for $k = 1,2,\dots,p$. Instead proving (6.19) for $k > p$, let us show the inequality

$$\sum_{i=k+1}^{n}\lambda_i(A) \leqslant \sum_{i=k+1}^{n}\delta_i^*(A). \tag{6.21}$$

Since $\delta_i^*(A) = \delta_{i-1}^*(C)$, $i = p+1, p+2,\dots,n$, and $\lambda_{i-1}(C) \geqslant \lambda_i(A)$, $i = 2,3,\dots,n$, by Corollary 6.6, (6.20) implies (6.21). ∎

Notes and References to Section 6

Lemma 6.1 is a small part of Perron (1907) and Frobenius (1908, 1909). More about positive matrices can be found in Gantmacher (1953). The reality and simplicity of eigenvalues (Theorem 6.2) and the interlacing property of nodes of eigenvectors (Theorem 6.3) were the starting points of the Gantmacher-Krein theory, with motivation from the vibration of mechanical systems. Theorem 6.4 is also due to Gantmacher and Krein (1960); see also Ševčuk (1978). For the reality of eigenvalues, Koteljanskiĭ (1963b) presented some generalization. See Karlin and Pinkus (1974) for some results related to Theorem 6.5. The majorization result, Theorem 6.7, was proved by Garloff (1982b). The majorization concept plays an important role in various places of analysis. More about majorization can be found in Ando (1986) and Marshall and Olkin (1979).

7. SOME EXAMPLES

In this last section we present some examples of totally positive matrices and characterizations of those matrices.

I. Totally Positive Kernels

Most of nontrivial totally positive matrices are obtained by restricting totally positive kernels to suitable finite subsets.

Let Γ, Λ be totally ordered sets (usually subintervals of $\mathbf{R}$ or $\mathbf{Z}$). A real-valued function $K(s,t)$ for $s \in \Gamma$, $t \in \Lambda$ is called a *totally positive kernel* if the matrix $[K(s_i, t_j)]_{i,j=1,2,\ldots,n}$ is totally positive for every choice $s_1 < s_2 < \cdots < s_n$ and $t_1 < t_2 < \cdots < t_n$. Strict total positivity of a kernel is defined correspondingly.

Here are some production formulas for totally positive kernels. If $K(s,t)$ is totally positive and $f(s), g(t)$ are positive functions on Γ and Λ respectively, then the kernel $f(s)K(s,t)g(t)$ is totally positive. If $K(s,t)$ is totally positive, and if $\phi(s)$ is a monotone increasing map from a totally ordered set Γ_1 to Γ, and $\psi(t)$ is a monotone increasing map from a totally ordered set Λ_1 to Λ, then $K(\phi(s), \psi(t))$ is a totally positive kernel on $\Gamma_1 \times \Lambda_1$. If both kernels $L(s,t)$ and $M(s,t)$ are totally positive and $d\sigma(\cdot)$ is a measure on Γ, then the kernel

$$K(u,v) := \int_T L(s,u)M(s,v)\,d\sigma(s), \qquad u,v \in \Lambda, \tag{7.1}$$

is totally positive on $\Lambda \times \Lambda$, provided that the integral exists. This is just a modification of Theorem 3.1.

Now let us turn to construction of concrete examples.

(a) For any real $\alpha_k \geqslant 0$, $k = 1,2,\ldots,n$, the kernel $K(s,t) := \sum_{k=0}^{n} \alpha_k s^k t^k$ is totally positive on $\mathbf{R}_+ \times \mathbf{R}_+$. Indeed, $K(s,t)$ is a composition of the type (7.1) with $L(k,t) = M(k,t) = t^k$ on $\mathbf{Z}_+ \times \mathbf{R}_+$. The total positivity of the kernel $L(k,t)$ is a consequence of the *Vandermonde determinant*:

$$\det\left[t_j^i\right]_{\substack{i=0,1,\ldots,n-1\\ j=1,2,\ldots,n}} = \prod_{1 \leqslant i < j \leqslant n} (t_j - t_i). \tag{7.2}$$

(b) For any $\sigma > 0$ the kernel $K(s,t) := \exp(\sigma st)$ is totally positive on $\mathbf{R}_+ \times \mathbf{R}_+$, as a limit of kernels of type (a). This kernel is strictly totally positive on $\mathbf{R} \times \mathbf{R}$ too. Consequently $\exp[-\sigma(s-t)^2]$ is strictly totally positive on $\mathbf{R}\times\mathbf{R}$ because

$$\exp\left[-\sigma(s-t)^2\right] = \exp(-\sigma s^2)\exp(2\sigma st)\exp(-\sigma t^2).$$

(c) For $p = 1,2,\dots$ the n-square matrices

$$G_p = \left[\exp\left(-\frac{(i-j)^2}{p}\right)\right]$$

are strictly totally positive by (b), and $G_p \to I_n$ as $p \to \infty$. This sequence has already been used several times in the previous sections.

(d) For each $0<\lambda<1$ and $0 \neq p \in \mathbf{R}$, consider the *weighted mean* on $\mathbf{R}_+ \times \mathbf{R}_+$

$$M_{\lambda,p}(s,t) := \{\lambda s^p + (1-\lambda)t^p\}^{1/p}. \tag{7.3}$$

Then $M_{\lambda,p}(s,t)$ or $1/M_{\lambda,p}(s,t)$ is totally positive according as $p<0$ or $p>0$. This follows from the observation that for any $\gamma>0$

$$\frac{1}{(s+t)^\gamma} = \frac{1}{\Gamma(\gamma)}\int_{-\infty}^0 e^{us}e^{ut}\frac{du}{|u|^{1-\gamma}}, \tag{7.4}$$

where $\Gamma(\cdot)$ is the gamma function, and that the kernel $\exp(us)$ is totally positive on $\mathbf{R}_- \times \mathbf{R}_+$.

(e) The kernel $K(s,t) := \min(s,t)$ is totally positive on $\mathbf{R}_+ \times \mathbf{R}_+$, because

$$K(s,t) = \lim_{p\to-\infty} M_{\lambda,p}(s,t). \tag{7.5}$$

(f) If $f(t)$, $g(t)$ are positive function on $\mathbf{R}_+$ such that $h(t) := f(t)/g(t)$ is nondecreasing, then the kernel

$$K(s,t) := f(\min(s,t))g(\max(s,t))$$

is totally positive on $\mathbf{R}_+ \times \mathbf{R}_+$, because

$$\begin{aligned} K(s,t) &= \min\{h(s),h(t)\}g(\min(s,t))g(\max(s,t)) \\ &= g(s)\cdot\min\{h(s),h(t)\}\cdot g(t). \end{aligned}$$

For $\sigma>0$, with $g(t) = \exp(-\sigma t)$ and $h(t) = \exp(2\sigma t)$, the kernel $\exp(-\sigma|s-t|)$ is totally positive on $\mathbf{R}_+ \times \mathbf{R}_+$.

(g) Let $\{b_i\}_{i=1,2,\ldots,n}$ and $\{c_i\}_{i=1,2,\ldots,n}$ be positive sequences. Then the n-square matrix $[b_{\min(i,j)}c_{\max(i,j)}]$ is totally positive if and only if $b_1/c_1 \leqslant b_2/c_2 \leqslant \cdots \leqslant b_n/c_n$. This follows immediately from (f). A matrix of this type is called a *Green matrix*.

II. Hurwitz Matrix

It is a celebrated theorem of A. Hurwitz that a polynomial $p(z) = d_0 z^n + d_1 z^{n-1} + \cdots + d_n$ of real coefficients ($d_0 > 0$) has all its zeros in the open left half plane $\operatorname{Re} z < 0$ if and only if the n-square matrix

$$H := \left[d_{2j-i}\right] = \begin{bmatrix} d_1 & d_3 & d_5 & d_7 & d_9 & \cdots & 0 \\ d_0 & d_2 & d_4 & d_6 & d_8 & \cdots & 0 \\ 0 & d_1 & d_3 & d_5 & d_7 & \cdots & 0 \\ 0 & d_0 & d_2 & d_4 & d_6 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & d_n \end{bmatrix}, \tag{7.6}$$

where $d_k = 0$ for $k < 0$ or $> n$, has positive leading principal minors:

$$\det H[1,2,\ldots,k] > 0, \qquad k = 1,2,\ldots,n. \tag{7.7}$$

Such a polynomial $p(z)$ is called a *Hurwitz polynomial*, and the matrix H is the *Hurwitz matrix* associated with it.

Let us show, by induction on n, that the Hurwitz matrix is totally positive. When $n = 1$, everything is trivial. Assume that the assertion is true with $n-1$ in place of n. Since $d_1 > 0$ for a Hurwitz matrix (7.6), it follows from (1.35) that the $(n-1)$-square matrix $G := H/\{1\}$, indexed by $2,3,\ldots,n$, has also positive leading principal minors:

$$\det G[2,3,\ldots,k] > 0, \qquad k = 2,3,\ldots,n. \tag{7.8}$$

Let $\overleftarrow{g}_j$, $j = 2,3,\ldots,n$, be the row $(n-1)$-vectors of G, and $c = d_0/d_1$. Then the $(n-1)$-square matrix F, indexed by $2,3,\ldots,n$, whose row vectors $\overleftarrow{f}_j$ are defined by

$$\overleftarrow{f}_2 := \overleftarrow{g}_2, \quad \text{and} \quad \overleftarrow{f}_{2j-1} := \overleftarrow{g}_{2j-1}, \quad \overleftarrow{f}_{2j} := \overleftarrow{g}_{2j} - c\overleftarrow{g}_{2j-1} \quad \text{for } j \geqslant 2, \tag{7.9}$$

has also positive leading principal minors. A glance will show that F is of the form (7.6) with $n-1$ instead of n, and d_j' instead of d_j, where

$$d_{2j}' = d_{2j+1} \text{ and } d_{2j-1}' = d_{2j} - c d_{2j+1}, \qquad j = 0,1,2,\dots. \tag{7.10}$$

Then according to the induction assumption, F is totally positive, and so is the n-square matrix

$$\tilde{F} := \begin{bmatrix} 0 & 0 \\ 0 & F \end{bmatrix}.$$

Now it is readily seen from (7.10) that

$$H(n-1,n) = \left(\left\{ S + \frac{c}{2}(I_n - J_n) \right\} S\tilde{F}S^* \right)(n-1,n), \tag{7.11}$$

where $S = [0, \vec{e}_1, \vec{e}_2, \dots, \vec{e}_{n-1}]$. The matrices S and S^* are totally positive, and so is the positive upper triangular matrix $S + (c/2)(I_n - J_n)$. Now the total positivity of H follows from (7.11) by Theorem 3.1 and Theorem 2.1.

III. Toeplitz Matrices

For a (bi-)infinite sequence $\{a_n : -\infty < n < \infty\}$, the matrix $[a_{i-j}]_{i,j=1,2,\dots}$ is called its *Toeplitz matrix*, and the function $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ its *generating function*. A complete characterization of the total positivity of (all finite sections of) a Toeplitz matrix has been established in a series of papers: Aissen, Schoenberg, and Whitney (1952), Whitney (1952), and Edrei (1952, 1953a,b).

A Toeplitz matrix $[a_{i-j}]$ is totally positive if and only if the generating function $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ is of the form

$$f(z) = Cz^k \exp\left(\gamma_1 z + \frac{\gamma_{-1}}{z}\right) \cdot \frac{\prod_1^\infty (1 + \alpha_n z) \prod_1^\infty \left(1 + \frac{\rho_n}{z}\right)}{\prod_1^\infty (1 - \beta_n z) \prod_1^\infty \left(1 - \frac{\delta_n}{z}\right)},$$

where k is an integer, $C \geqslant 0$, $\gamma_1, \gamma_{-1} \geqslant 0$, and $\alpha_n, \beta_n, \rho_n, \delta_n \geqslant 0$ are such that $\sum_1^\infty (\alpha_n + \beta_n + \rho_n + \delta_n) < \infty$.

When $a_n = 0$ for $n < 0$, the Toeplitz matrix is totally positive if and only if the generating function is of the form

$$f(z) = Ce^{\gamma z} \frac{\prod_1^\infty (1+\alpha_n z)}{\prod_1^\infty (1-\beta_n z)},$$

where $C \geqslant 0$, $\gamma \geqslant 0$, and $\alpha_n, \beta_n \geqslant 0$ are such that $\sum_1^\infty(\alpha_n + \beta_n) < \infty$.

The proofs of these results, based heavily on the theory of analytic functions, are beyond the scope of the present paper.

When applied to a polynomial, the above characterization implies that a polynomial $p(z) = d_0 z^n + d_1 z^{n-1} + \cdots + d_n$ $(d_0 > 0)$ has all its zeros on the negative real axis if and only if the infinite matrix $[d_{n+j-i}]_{i,j=1,2,\ldots}$ is totally positive, where $d_k = 0$ for $k < 0$ or $> n$. Remark that the Hurwitz matrix H, introduced in Section 7.II, is a submatrix of T, namely $H = T[n+1, n+2, \ldots, 2n|2, 4, \ldots, 2n]$.

IV Pólya Frequency Function

A function $f(t)$ on $(-\infty, \infty)$ is called a *Pólya frequency function* if the kernel $K(s,t) := f(s-t)$ is totally positive. The following remarkable characterization is due to Schoenberg (1953); $f(t)$ is a Pólya frequency function if and only if its bilateral Laplace transform exists in an open strip containing the imaginary axis and has the form

$$\int_{-\infty}^{\infty} e^{-st} f(s)\, ds = C \exp(\gamma t^2 + \delta t) \cdot \prod_1^\infty \frac{\exp(\alpha_n t)}{1 + \alpha_n t},$$

where $C > 0$, $\gamma \geqslant 0$, δ and α_n are real such that $0 < \gamma + \sum_1^\infty |\alpha_n|^2 < \infty$.

The proof of this result is beyond the scope of the present paper.

Notes and References to Section 7

The monograph by Karlin (1968) contains very many examples of totally positive kernels. Total positivity of various generalized means is seen in Carlson and Gustafson (1983). A kernel $K(s,t)$ on an inverval of the real line is called extended strictly totally positive if for every n

$$\det\left[\frac{\partial^{i+j-2}}{\partial s^{i-1} \partial t^{j-1}} K(s,t)\right]_{i,j=1,2,\ldots,n} > 0.$$

Extended strict total positivity implies strict total positivity. In this connection, Burbea (1974, 1976) defined the extended strict total positivity of a kernel $K(z, \bar{w})$ of complex variables by

$$\det\left[\frac{\partial^{i+j-2}}{\partial z^{i-1}\,\partial \bar{w}^{j-1}}K(z,\bar{w})\right]_{i,j=1,2,\dots,n} \neq 0$$

and established the extended strict total positivity of reproducing kernels of certain Hilbert spaces of analytic functions.

A proof of the Hurwitz theorem can be found in Gantmacher (1937). The total positivity of a Hurwitz matrix was proved by Asner (1970) and Kemperman (1982).

For the proof of characterizations of a totally positive Toeplitz matrix and a totally positive translation kernel, we refer to the original papers cited in the text and the monograph by Karlin (1968, Chapters 7–8). Lorenz and Mackens (1979) gave a characterization of total positivity of the inverse of a banded Toeplitz matrix.

I would like to express my deepest gratitude to Miss Yukiko Ito for her superb typing and unlimited patience.

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Received 14 September 1984; revised 17 April 1986