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Minimum rank of skew-symmetric matrices described by a graph ☆

IMA-ISU research group on minimum rank *,[1]

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ABSTRACT

The minimum (symmetric) rank of a simple graph G over a field F is the smallest possible rank among all symmetric matrices over F whose ijth entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. The problem of determining minimum (symmetric) rank has been studied extensively. We define the minimum skew rank of a simple graph G to be the smallest possible rank among all skew-symmetric matrices over F whose ijth entry (for $i \neq j$) is nonzero whenever $\{i,j\}$ is an edge in G and is zero otherwise. We apply techniques from the minimum (symmetric) rank problem and from skew-symmetric matrices to obtain results about the minimum skew rank problem.

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1. Introduction

The classic minimum rank problem involves real symmetric matrices described by a graph. This problem has been studied extensively and generalized to symmetric matrices over other fields; see [9] for a survey of known results and a discussion of the motivation for the minimum rank problem. In this paper, we study the problem of determining the minimum rank of skew-symmetric matrices described by a graph.

If a field F is of characteristic 2, then the skew-symmetric matrices are the same as the symmetric matrices; and may have nonzero diagonal entries. Thus it is assumed throughout this paper that **the fields under consideration do not have characteristic** 2.

1.1. Notation and terminology

An $n \times n$ matrix A over a field F is *skew-symmetric* (respectively, *symmetric*) if $A^T = -A (A^T = A)$; for $A \in \mathbb{C}^{n\times n}$, A is *Hermitian* if $A^* = A$, where A^* denotes the conjugate transpose of A.

A *graph* is a pair $G = (V_G, E_G)$, where V_G is the (finite, nonempty) set of vertices of G (usually $\{1,\ldots,n\}$ or a subset thereof) and E_G is the set of edges (two-element subsets of vertices). These graphs are usually called simple undirected graphs. The *order* of a graph G, denoted $|G|$, is the number of vertices of G.

For a symmetric, skew-symmetric or Hermitian matrix, the *graph* of an $n \times n$ matrix A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1,\ldots,n\}$ and edges $\{\{i,j\} : a_{ij} \neq 0,\ 1 \leqslant i < j \leqslant n\}$. Note that the diagonal is ignored in determining $\mathcal{G}(A)$ for symmetric and Hermitian matrices (the diagonal must be 0 for a skew-symmetric matrix).

The set of *symmetric matrices over a field F described by G* is

$$\mathcal{S}(F,G) = \{A \in F^{n\times n},\ A^T = A,\ \mathcal{G}(A) = G\}.$$

The *minimum rank* of a graph G over F is $\mathrm{mr}(F,G) = \min\{\operatorname{rank} A : A \in \mathcal{S}(F,G)\}$, and the *maximum nullity* of G over F is $\mathrm{M}(F,G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}(F,G)\}$. Clearly $\mathrm{mr}\,(F,G) + \mathrm{M}(F,G) = |G|$. When the field is omitted it is assumed to be the real field, i.e. $\mathrm{mr}(G) = \mathrm{mr}(\mathbb{R},G)$.

The set of *skew-symmetric matrices over F described by G* is

$$\mathcal{S}^-(F,G) = \{A \in F^{n\times n} : A^T = -A,\ \mathcal{G}(A) = G\}.$$

The *minimum skew rank* of a graph G over F is defined to be

$$\mathrm{mr}^-(F,G) = \min\{\operatorname{rank} A : A \in \mathcal{S}^-(F,G)\},$$

and the *maximum skew nullity* of G over F is defined to be

$$\mathrm{M}^-(F,G) = \max\{\mathrm{null}(A) : A \in \mathcal{S}^-(F,G)\}.$$

Clearly $\mathrm{mr}^-(F,G) + \mathrm{M}^-(F,G) = |G|$. In this paper we say that the matrix $A \in F^{n\times n}$ is optimal for G (over F) if $A \in \mathcal{S}^-(F,G)$ and $\operatorname{rank} A = \mathrm{mr}^-(F,G)$.

Clearly the maximum rank among matrices in $\mathcal{S}(F,G)$ is $|G|$, but this need not be the case for skew rank. The *maximum skew rank* of a graph G is

$$\mathrm{MR}^-(F,G) = \max\{\operatorname{rank} A : A \in \mathcal{S}^-(F,G)\}.$$

The set of *Hermitian matrices described by G* is

$$\mathcal{H}(G) = \{A \in \mathbb{C}^{n\times n},\ A^* = A,\ \mathcal{G}(A) = G\}.$$

The *minimum Hermitian rank* of a graph G is $\mathrm{hmr}(G) = \min\{\operatorname{rank} A : A \in \mathcal{H}(G)\}$. Minimum Hermitian rank has been studied in [5], and is a lower bound on the skew rank over the real field (see Proposition 3.2).

The subgraph $G[R]$ of G *induced* by $R \subseteq V_G$ is the subgraph with vertex set R and edge set $\{\{i,j\} \in E_G | i,j \in R\}$. The subgraph induced by $V_G \setminus R$ is also denoted by $G - R$, or in the case $R = \{v\}$, by $G - v$. If A is an $n \times n$ matrix and $R \subseteq \{1,\ldots,n\}$, the *principal submatrix* $A[R]$ is the matrix consisting of the entries in the rows and columns indexed by R, and $A(R)$ is the complementary principal submatrix obtained from A by deleting the rows and columns indexed by R. In the special case when $R = \{k\}$, we use $A(k)$ to denote $A(R)$. If $A \in \mathcal{S}^-(F, G)$, then by a slight abuse of notation $\mathcal{G}(A[R])$ can be identified with $G[R]$.

The *adjacency matrix* of G, $A_G = [a_{ij}]$, is a 0, 1-matrix such that $a_{ij} = 1$ if and only if $\{i,j\} \in E_G$. The *formal skew adjacency matrix* of G is $X_G = A_G \circ X$ where X is a skew-symmetric matrix having ij-entry x_{ij} for $i < j$, x_{ij} are independent indeterminates, and $\circ$ denotes the Hadamard (entrywise) product.

A path, cycle, complete graph, and complete multipartite graph will be denoted by P_n, C_n, K_n, and $K_{n_1,n_2,\ldots n_t}$ $(t \geqslant 2, n_i \geqslant 1)$, respectively.

The *complement* of a graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$, where $\overline{E}$ consists of all two-element sets of V that are not in E. The *union* of $G_i = (V_i, E_i)$ is $\cup_{i=1}^h G_i = (\cup_{i=1}^h V_i, \cup_{i=1}^h E_i)$; a disjoint union is denoted $\dot{\cup}_{i=1}^h G_i$. The *intersection* of $G_i = (V_i, E_i)$ is $\cap_{i=1}^h G_i = (\cap_{i=1}^h V_i, \cap_{i=1}^h E_i)$. The *join* $G \vee G'$ of two disjoint graphs $G = (V, E)$ and $G' = (V', E')$ is the union of $G \cup G'$ and the complete bipartite graph with vertex set $V \cup V'$ and partition $\{V, V'\}$. A *cut-vertex* is a vertex whose deletion increases the number of connected components.

A *matching* in a graph G is a set of edges $\{i_1, j_1\}, \ldots, \{i_k, j_k\}$ such that all the vertices are distinct. A *perfect matching* in a graph G is a matching that includes all vertices of G. A *maximum matching* in G is a matching with the maximum number of edges among all matchings in G. The *matching number*, denoted $\mathrm{match}(G)$, is the number of edges in a maximum matching.

An important matrix function in the study of matchings is the pfaffian (see [12] for more details). Let $L = \{\{i_1, i_2\}, \ldots, \{i_{n-1}, i_n\}\}$ be a perfect matching in G, ordered so that $i_1 < i_2, i_3 < i_4, \ldots, i_{n-1} < i_n$ and $i_1 < i_3 < \cdots < i_{n-1}$. Let π_L be the permutation of $\{1, \ldots, n\}$ that maps k to i_k. For $A \in \mathcal{S}^-(F, G)$, the *weight* of L with respect to A is

$$\mathrm{wt}_A(L) = \mathrm{sgn}(\pi_L) a_{i_1,i_2} \cdots a_{i_{n-1},i_n},$$

where $\mathrm{sgn}(\pi)$ is the sign of the permutation π. Let $\mathcal{F}$ be the set of all perfect matchings of G. The *pfaffian* of A is

$$\mathrm{pf}(A) = \sum_{L \in \mathcal{F}} \mathrm{wt}_A(L),$$

where the sum over the empty set is 0.

1.2. Known results about matchings and skew-symmetric matrices

This subsection contains results that will be used in the next section; throughout F denotes a field (which, as we have already mandated, does not have characteristic 2). We note that Theorem 1.1 and Corollary 1.5 do extend to characteristic 2. However, Corollary 1.2, and Lemma 1.3 do not, as the identity matrix of odd order is a skew-symmetric matrix over the field of 2 elements has odd rank, determinant 1 and pfaffian 0.

The proof of the next result is similar to the proof for the symmetric case (cf. [10, Theorem 8.9.1]).

Theorem 1.1. *Let $A \in F^{n\times n}$ be skew-symmetric. Then* $\mathrm{rank}\, A = \max\{|S| : \det(A[S]) \neq 0\}$.

Corollary 1.2. *The rank of any skew-symmetric matrix over F is even.*

The proof of the next result is similar to the proof for the symmetric case (cf. [10, Lemma 8.9.3]).

Lemma 1.3. *For a nonzero skew-symmetric matrix $A \in F^{n\times n}$,* $\mathrm{rank}\, A \leqslant 2k$ *if and only if there exist* $\mathbf{x_1}, \ldots \mathbf{x_k}, \mathbf{y_1}, \ldots, \mathbf{y_k} \in F^n$ *such that* $A = \sum_{i=1}^k (\mathbf{x_i}\mathbf{y_i}^T - \mathbf{y_i}\mathbf{x_i}^T)$.

Theorem 1.4 [6, Theorem 9.5.2]. *If $A \in F^{n\times n}$ is skew-symmetric, then* $\det A = (\mathrm{pf}(A))^2$.

Corollary 1.5. *Let $A \in F^{n\times n}$ be skew-symmetric. If $\mathcal{G}(A)$ has a unique perfect matching then* $\mathrm{rank}\, A = n$.

Graphs with unique perfect matching have been characterized in [12, Corollary 5.3.12].

The statements in Observation 1.6 follow immediately from the preceding results or are established by applying the same methods used for the analogous results in the symmetric minimum rank problem.

Observation 1.6

1. $\mathrm{mr}^-(F,G)$ and $\mathrm{MR}^-(F,G)$ are always even.
2. If G has a unique perfect matching then $\mathrm{mr}^-(F,G) = |G|$.
3. If H is an induced subgraph of G, then $\mathrm{mr}^-(F,H) \leqslant \mathrm{mr}^-(F,G)$.
4. $\mathrm{mr}^-(F,G) = 0$ if and only if G has no edges.
5. If the connected components of G are $G_1,\ldots,G_t$, then

$$\mathrm{mr}^-(F,G) = \sum_{i=1}^{t} \mathrm{mr}^-(F,G_i).$$

Corollary 1.7. *Let G be a graph, and let F be a field. If G has a matching with k edges and this is the only perfect matching for the subgraph induced by the $2k$ vertices in the matching, then* $\mathrm{mr}^-(F,G) \geqslant 2$.

2. Results derived from the properties of skew-symmetric matrices

In this section we use properties specific to skew-symmetric matrices to obtain results about minimum skew rank. All of the results in this section are valid over any infinite field. Most are valid for finite fields, but some technical results about polynomials over finite fields are needed for the proofs; these are included in the Appendix (Section 5).

Theorem 2.1. *Let G be a connected graph with $|G| \geqslant 2$ and let F be an infinite field. Then the following are equivalent:*

1. $\mathrm{mr}^-(F,G) = 2$,
2. $G = K_{n_1,n_2,\ldots,n_t}$ *for some* $t \geqslant 2, n_i \geqslant 1, i = 1,\ldots,t$,
3. *G does not contain P_4 or the paw (see Fig. 1) as an induced subgraph.*

Without the assumption that G is connected, $\mathrm{mr}^-(F,G) = 2$ *if and only if G is a union of one $K_{n_1,n_2,\ldots,n_t}$ and possibly some isolated vertices.*

Proof. ($2 \implies 1$) Let $G = K_{n_1,n_2,\ldots,n_t} = (V_1 \dot{\cup} \cdots \dot{\cup} V_t, E)$ where the sets $V_k (k = 1,\ldots,t)$ are the partite sets, and let $n = \Sigma_{i=1}^{t} n_i$. Let $\alpha_1,\ldots,\alpha_t$ be distinct elements of F. Construct $\boldsymbol{x},\boldsymbol{y} \in F^n$ such that $x_i = 1$ for all i and $y_j = \alpha_k$ for each vertex j in V_k. Observe that by construction the matrix $A = \mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T$ is a skew-symmetric matrix with rank $A = 2$. If vertex i is in partite set V_k and vertex j is in partite set V_ℓ, then $a_{ij} = \alpha_\ell - \alpha_k$, and thus $a_{ij} = 0$ if and only if vertices i and j are in the same partite set. It follows that $\mathcal{G}(A) = K_{n_1,n_2,\ldots,n_t}$. Since $A \in \mathcal{S}^-(F,G)$ and rank $A = 2$, we conclude that

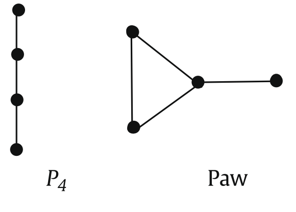

Fig. 1. Forbidden induced subgraphs for $\mathrm{mr}^-(F,G) \leqslant 2$.

Fig. 2. A path in the induced subgraph H that contains $K_2 \dot{\cup} K_1$.

$\mathrm{mr}^-(F, G) \leqslant 2$. Since $t \geqslant 2$, each matrix in $\mathcal{S}^-(F, G)$ has an invertible 2×2 principal submatrix, so $\mathrm{mr}^-(F, G) = 2$.

(1 $\Longrightarrow$ 3) This follows from Observation 1.6 since P_4 and the paw each have a unique perfect matching.

(3 $\Longrightarrow$ 2) Suppose that G is not a complete multipartite graph. Then $|G| \geqslant 4$ and G contains $K_2 \dot{\cup} K_1$ as an induced subgraph. Let H be the smallest connected induced subgraph of G that contains $K_2 \dot{\cup} K_1$ as an induced subgraph. Note that since H is connected, but has the induced subgraph $K_2 \dot{\cup} K_1$, we know that $|H| \geqslant 4$.

We show that if $|H| > 4$, then H is not the smallest such graph. Label the vertices of an induced $K_2 \dot{\cup} K_1$ by $x,\ y,\ z$ with x and y adjacent. Since H is connected, there is a path from one of x or y to z that does not include the other (say x). Label the additional vertices on this path $w_1, \ldots, w_k$. See Fig. 2 for the labeling, but note that this subgraph need not be an induced subgraph of G. Suppose $k > 1$. By the minimality of H, z is not adjacent to w_1. Then the subgraph induced by $y, w_1, \ldots, w_k, z$ is a smaller connected induced subgraph containing an induced $K_2 \dot{\cup} K_1$.

So $k = 1$, H contains the edges $\{x, y\},\ \{y, w_1\},\ \{w_1, z\}$ and H does not contain the edges $\{x, z\}$ or $\{y, z\}$. If $\{x, w_1\} \in E_H$, then H is the paw; if not $H = P_4$. Therefore if $G \neq K_{n_1,n_2,\ldots,n_t}$, then G must contain P_4 or the paw as an induced subgraph.

The result for disconnected graphs then follows from Observation 1.6.5. $\square$

Note that $K_n = K_{1,1,\ldots,1}$ and $G = K_{n_1,\ldots,n_t}$ if and only if $\overline{G} = K_{n_1} \dot{\cup} \cdots \dot{\cup} K_{n_t}$.

Remark 2.2. For a connected graph G, the equivalence that $G = K_{n_1,n_2,\ldots,n_t}$ for some $t \geqslant 2,\ n_i \geqslant 1,\ i = 1, \ldots, t$ if and only if G does not contain P_4 or the paw has been established.

The proof that (2) $\Rightarrow$ (1) is clearly valid for any field with at least t elements, and it can be modified to work in a field with $t - 1$ elements. The skew minimum rank $K_{n_1,\ldots,n_t}$ is larger than 2 for a finite field with fewer than $t - 1$ elements, as the next example shows computationally for a specific field and graph, and Corollary 2.4 below shows more generally.

Example 2.3. We claim that $\mathrm{mr}^-(\mathbb{Z}_3, K_5) = 4$. To see this, first note that the circulant matrix with first row $(0, 1, 1, -1, -1)$ is skew-symmetric and singular, and hence $\mathrm{mr}^-(\mathbb{Z}_3, K_5) \leqslant 4$. Second, note that if among any five vectors in $\mathbb{Z}_3^2$, there is a pair that are linearly dependent. Hence, each matrix of the form $xy^T - yx^T$ where $x, y \in \mathbb{Z}_3^5$ has an off-diagonal 0. We conclude that $\mathrm{mr}^-(\mathbb{Z}_3, K_5) > 2$. The result follows by noting that the rank of a skew-symmetric matrix is even.

Corollary 2.4. *In a finite field F of order q, the following are equivalent.*

1. *G is connected and* $\mathrm{mr}^-(F, G) = 2$.
2. *$G = K_{n_1,n_2,\ldots,n_t}$, where $2 \leqslant t \leqslant q + 1$.*

Proof. (2 $\Longrightarrow$ 1) Assume that $G = K_{n_1,n_2,\ldots,n_t}$ with $2 \leqslant t \leqslant q + 1$. In order to construct a matrix of rank 2 in $\mathcal{S}^-(F, G)$, we first notice that $(\mathbf{xy}^T - \mathbf{yx}^T)_{ij} = x_i y_j - y_i x_j$ is nonzero if and only if the nonzero vectors $[x_i, y_i]$ and $[x_j, y_j]$ are not parallel in F^2. In a field of order q, we know that there are $q + 1$ unique parallel classes of nonzero vectors in F^2. Let the elements of F be $0, 1, f_3, f_4, \ldots, f_q$. Take the vectors $[0, 1], [1, 0], [1, 1], [1, f_3], \ldots, [1, f_q]$ as representatives of these parallel classes. For $i = 1, \ldots, n$, define $[x_i, y_i]$ to be $[0, 1]$ if $i \in n_1$, $[1, 0]$ if $i \in n_2$, and $[1, f_j]$ if $i \in n_j$ and $j \geqslant 3$. The vectors $\mathbf{x} = [x_i]$ and $\mathbf{y} = [y_i]$ satisfy $\mathbf{xy}^T - \mathbf{yx}^T \in \mathcal{S}^-(F, G)$, so $\mathrm{mr}^-(F, G) = 2$.

($1 \Longrightarrow 2$) Assume that G is connected and $\mathrm{mr}^-(F,G) = 2$. Then we can find $\mathbf{x}, \mathbf{y} \in F^n$ so that $\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T \in \mathcal{S}^-(F,G)$. As above, $(\mathbf{x}\mathbf{y}^T - \mathbf{y}\mathbf{x}^T)_{ij} = x_iy_j - y_ix_j = 0$ if and only if vectors $[x_i, y_i]$ and $[x_j, y_j]$ are nonzero and parallel or one of them is the zero vector. Note that $[x_i, y_i] \neq [0,0]$ for all i because otherwise G would be disconnected. Partition the vertices into sets $V_1, V_2, \ldots, V_t$, where vertices i and j are in the same set if and only if the vectors $[x_i, y_i]$ and $[x_j, y_j]$ are parallel. Since there are only $q + 1$ parallel equivalence classes of nonzero vectors in F^2, we have $2 \leqslant t \leqslant q + 1$. Thus G will be a complete multipartite graph with partite sets $V_1, V_2, \ldots V_t$ of orders $n_1, n_2, \ldots, n_t$, respectively, with $2 \leqslant t \leqslant q + 1$. $\square$

Theorem 2.5. *For a graph G and a field F, $\mathrm{MR}^-(F,G) = 2\,\mathrm{match}(G)$, and every even rank between $\mathrm{mr}^-(F,G)$ and $\mathrm{MR}^-(F,G)$ is realized by a matrix in $\mathcal{S}^-(F,G)$.*

Proof. Let $A \in \mathcal{S}^-(F,G)$, $|G| = n$, and $\mathrm{match}(G) = m$. Then for any $\ell \times \ell$ principal submatrix B of A, $B \in \mathcal{S}^-(H)$ for an induced subgraph H of G. If $\ell > 2m$, then H has no perfect matching. Hence we have $\mathrm{pf}(B) = 0$, which implies that $\det B = 0$. This holds for all $\ell > 2m$, whence $\mathrm{rank}\, A \leqslant 2m$ by Theorem 1.1. Thus $\mathrm{MR}^-(F,G) \leqslant 2\,\mathrm{match}(G)$.

Renumber the vertices in the graph G (if necessary) such that the independent edges in a maximum matching are $\{\{1,2\},\{3,4\},\ldots,\{2m-1,2m\}\}$. If X_G is the formal skew adjacency matrix of G, then $\mathrm{pf}(X_G[\{1,\ldots,2m\}])$ is not the zero polynomial. Construct the matrix $B = [b_{ij}]$ over the field F by choosing values $b_{ij} \in F$ for the variables x_{ij} that make $\mathrm{pf}(B[\{1,\ldots,2m\}]) \neq 0$. Since F has at least 3 elements, Proposition 5.4 in the Appendix shows that we can make such a choice. Thus $\det(B[\{1,\ldots,2m\}]) \neq 0$, and we can complete $B \in \mathcal{S}^-(F,G)$ by choosing any nonzero values for the remaining nonzero entries. Since $B \in \mathcal{S}^-(F,G)$ and $\mathrm{rank} B \geqslant 2m$, $\mathrm{MR}^-(F,G) = 2m$.

We can go from any matrix $B \in \mathcal{S}^-(F,G)$ to any other matrix $A \in \mathcal{S}^-(F,G)$ by adding (one at a time) the matrix $S_{ij}, j > i$ such that $S_{ij}[\{i,j\}] = \begin{bmatrix} 0 & a_{ij} - b_{ij} \\ b_{ij} - a_{ij} & 0 \end{bmatrix}$ and all other entries are zero. Since $\mathrm{rank}\, S_{ij} = 2$, we must pass through every even rank in the transition from a maximum rank matrix B to a minimum rank matrix A. $\square$

Theorem 2.6. *For a graph G and a field F that has at least 5 elements, $\mathrm{mr}^-(F,G) = |G| = \mathrm{MR}^-(F,G)$ if and only if G has a unique perfect matching.*

Proof. If G has a unique perfect matching, then as noted in Observation 1.6, for any field F, $\mathrm{mr}^-(F,G) = |G|$.

Conversely, suppose $\mathrm{mr}^-(F,G) = |G|$. Clearly, this implies that $\mathrm{mr}^-(F,G) = \mathrm{MR}^-(F,G)$. Since every $A \in \mathcal{S}^-(F,G)$ has full rank, $\det A \neq 0$ for all $A \in \mathcal{S}^-(F,G)$. Applying Theorem 1.4 we determine that $\mathrm{pf}(A) \neq 0$ for $A \in \mathcal{S}^-(F,G)$. Since the nonzero terms of the pfaffian correspond to perfect matchings of G, G has at least one perfect matching.

It remains to show that the perfect matching is unique. Suppose that G contains at least two perfect matchings. If so, we show that there exists some $B = [b_{ij}] \in \mathcal{S}^-(F,G)$ with $\mathrm{pf}(B) = 0$. Let X_G be the formal skew adjacency matrix of G, and let the $\mathrm{pf}(X_G) = p(y_1,\ldots,y_k)$, where y_i are the entries of X_G that appear in the pfaffian. Since there are at least two nonzero terms, by Proposition 5.6 in the Appendix, we can choose nonzero values $b_1,\ldots,b_k$ for $y_1,\ldots,y_k$ so that $p(b_1,\ldots b_k) = 0$. By setting the entry corresponding to y_j equal to b_j, $j = 1,\ldots,k$, and all other nonzero entries to any nonzero value, we can find a $B \in \mathcal{S}^-(F,G)$ having $\mathrm{pf}(B) = 0$, which is a contradiction. $\square$

Theorem 2.7. *Let T be a tree and let F be a field. Then $\mathrm{mr}^-(F,T) = 2\,\mathrm{match}(T) = \mathrm{MR}^-(F,T)$.*

Proof. By Theorem 2.5, $\mathrm{mr}^-(F,T) \leqslant 2\,\mathrm{match}(T)$. Let $\{v_1,\ldots,v_k\}$ be the vertices in a maximum matching of a graph G. The induced subgraph $H = G[\{v_1, v_2, \ldots v_k\}]$, is a forest that has a perfect matching. This perfect matching is unique, because if we choose any leaf of H, it is incident to only one edge, so it must be matched with its only neighbor. Excluding these two vertices, we are left with a forest which

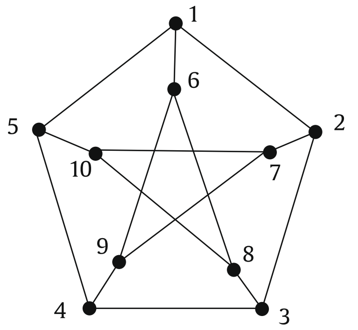

Fig. 3. The Petersen graph P.

still has a perfect matching and still has a leaf. We continue this procedure until each vertex in H is matched. Thus $\mathrm{mr}^-(F,T) \geqslant 2\,\mathrm{match}(T)$. $\square$

It is straightforward to find a maximum matching of a tree. Start with an empty edge set M, an empty vertex set W, and the tree (note that as vertices are deleted, the tree may become a forest). At the kth step, choose a vertex v_k of degree 1, denote its unique neighbor by w_k, remove w_k (and its incident edges) from the forest, add edge $\{v_k, w_k\}$ to the matching M and add w_k to W. Continue with this procedure until all edges are gone. Since every edge has been removed by being incident to a w_k, W is a vertex cover, i.e. a subset of vertices that contains at least one endpoint of every edge. Since $\deg v_k = 1$, when w_k is removed, v_k has no more edges, so M is a matching. For any graph G and vertex cover U, $\mathrm{match}(G) \leqslant |U|$ [13, p. 112]. Since $|M| = |W|$, M is a maximum matching.

Observation 2.8. For a tree T, $\mathrm{match}(T)$ can be determined by starting with a vertex of degree 1, matching it, removing both matched vertices from the graph, and continuing in this manner.

In the proof of Theorem 2.7 it was shown that a tree T has an induced subgraph H such that $\mathrm{mr}^-(F,T) = |H| = \mathrm{mr}^-(F,H)$ (and H has a unique perfect matching). This need not be true in general, as the next example shows.

Example 2.9. Let P be the Petersen graph (shown in Fig. 3).
Any matrix $A \in \mathcal{S}^-(F,P)$ can be put in the form

$$A = \begin{bmatrix} 0 & a & 0 & 0 & b & 1 & 0 & 0 & 0 & 0 \\ -a & 0 & c & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -c & 0 & d & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -d & 0 & e & 0 & 0 & 0 & 1 & 0 \\ -b & 0 & 0 & -e & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & g & h & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & s & q \\ 0 & 0 & -1 & 0 & 0 & -g & 0 & 0 & 0 & r \\ 0 & 0 & 0 & -1 & 0 & -h & -s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -q & -r & 0 & 0 \end{bmatrix}$$

by use of a diagonal congruence. It is straightforward to verify that every induced subgraph of order 8 has two perfect matchings. However, $\mathrm{mr}^-(F,P) = 8$, because any choice of values of the variables makes at least one order 8 principal submatrix nonsingular. Specifically,

$$\det(A[\{1,2,3,4,5,6,7,8\}]) = (e - bdg)^2, \tag{1}$$

$$\det(A[\{1,2,3,4,5,6,9,10\}]) = (c - adh)^2, \tag{2}$$

$$\det(A[\{1,2,3,4,5,6,8,9\}]) = (bcg + aeh)^2. \tag{3}$$

Substituting $e = bdg$ and $c = adh$ into Eq. (3) results in

$$\det(A[\{1,2,3,4,5,6,8,9\}]) = 4a^2b^2d^2g^2h^2 \neq 0.$$

3. Results derived using minimum rank techniques

In this section, we examine connections between the classical minimum rank (using symmetric matrices) and minimum skew rank. Minimum rank and minimum skew rank are noncomparable, but minimum Hermitian rank is a lower bound on minimum skew rank (over the real numbers).

Example 3.1. The minimum skew rank of a graph can be greater than the minimum rank of the graph: $\operatorname{mr}(F, K_2) = 1 < 2 = \operatorname{mr}^-(F, K_2)$. The minimum skew rank can also be less than the minimum rank: $\operatorname{mr}^-(F, K_{3,3,3}) = 2 < 3 = \operatorname{mr}(F, K_{3,3,3})$ [5] (as always, char $F \neq 2$).

Proposition 3.2. $\operatorname{hmr}(G) \leqslant \operatorname{mr}^-(\mathbb{R}, G)$.

Proof. If $A \in \mathcal{S}^-(\mathbb{R}, G)$ then $iA \in \mathcal{H}(G)$ and $\operatorname{rank}(iA) = \operatorname{rank} A$, so $\operatorname{hmr}(G) \leqslant \operatorname{mr}^-(\mathbb{R}, G)$. $\square$

Proposition 3.3. *Let $G = \cup_{i=1}^h G_i$. If F is an infinite field or if G_i and G_j have no edges in common for all $i \neq j$, then* $\operatorname{mr}^-(F, G) \leqslant \sum_{i=1}^h \operatorname{mr}^-(F, G_i)$.

Proof. A skew-symmetric matrix $A \in F^{n \times n}$ of rank at most $\sum_{i=1}^h \operatorname{mr}^-(F, G_i)$ having $\mathcal{G}(A) = G$ can be constructed by choosing (for each $i = 1, \ldots, h$) a matrix A_i that realizes $\operatorname{mr}^-(F, G_i)$, embedding A_i in a matrix $\widetilde{A_i}$ of size $|G|$, choosing $a_i \in F$ such that no cancellation of nonzero entries occurs, and letting $A = \sum_{i=1}^h a_i \widetilde{A_i}$. $\square$

3.1. Zero forcing number

An upper bound for $\operatorname{M}(F, G)$, which yields an associated lower bound for $\operatorname{mr}(F, G)$, is the zero forcing number $\operatorname{Z}(G)$ introduced in [1]. The zero forcing number is a useful tool for determining the minimum rank of structured families of graphs and small graphs, and is motivated by simple observations about null vectors of matrices. In this subsection we extend these ideas to minimum skew rank by revising the color change rule to better exploit properties of skew-symmetric matrices, thereby creating a new zero forcing parameter.

Definition 3.4. Let $G = (V, E)$ be a graph.

- A subset $Z \subset V$ defines an *initial coloring* by coloring all vertices in Z black and all the vertices not in Z white.
- The *skew color change rule* says: If a vertex $v \in V$ has exactly one white neighbor, w, change the color of w to black. In this case we say that v forces w.
- The *skew derived set* of an initial coloring Z is the result of applying the skew color change rule until no more changes are possible.
- A *skew zero forcing set* is a subset $Z \subseteq V$ such that the skew derived set of Z is V.
- The *skew zero forcing number*, $\operatorname{Z}^-(G)$, is the minimum size of a skew zero forcing set.

We note that the skew color change rule differs from the conventional color change rule in that it does not require the vertex $v \in V$ with exactly one white neighbor to be black.

If $\mathbf{x} = [x_k]$ is a nonzero null vector of the skew-symmetric matrix A whose graph is G, and i is a vertex of G, then either $x_j = 0$ for each neighbor j of i or x_j is nonzero for at least two neighbors j of i. If A is a skew-symmetric matrix of nullity k, then for every set Z of cardinality $k-1$, there is a nonzero null vector $\mathbf{x}$ of A with $x_j = 0$ for all $j \in Z$. Thus if Z is a skew zero forcing set of G, then for each matrix in $\mathcal{S}^-(F, G)$ the only null vector with 0's in positions indexed by Z is the zero vector. These ideas provide the proof of the next proposition, just as analogous statements about symmetric matrices provide the proof of Proposition 2.4 in [1].

Proposition 3.5. *For any graph G and any field F, $\mathrm{M}^-(F, G) \leqslant \mathrm{Z}^-(G)$ and $\mathrm{mr}^-(F, G) \geqslant |G| - \mathrm{Z}^-(G)$.*

The next example illustrates a skew zero forcing set and computation of the skew zero forcing number.

Example 3.6. Let H be the paw (see Fig. 1) with the vertices numbered as follows: the degree one vertex is number 1, the degree three vertex is number 2, and the two degree two vertices are numbers 3 and 4. With this numbering, 1 can force 2, then 3 can force 4 and 4 can force 3, and finally 2 can force 1. Thus the empty set is a zero forcing set, so $\mathrm{Z}^-(H) = 0$.

Proposition 3.7. *Let G be a graph and let F be a field. Then $\mathrm{Z}^-(G) \leqslant Z(G)$. If $\mathrm{mr}(F, G) = |G| - Z(G)$, then $\mathrm{mr}^-(F, G) \geqslant \mathrm{mr}(F, G)$.*

Proof. Let Z be an optimal zero forcing set for the graph G, i.e, $|Z| = \mathrm{Z}(G)$. The set Z is also a skew zero forcing set for G, although Z may not be an optimal skew zero forcing set. Thus $\mathrm{Z}^-(G) \leqslant |Z| = \mathrm{Z}(G)$.

Therefore, if $\mathrm{mr}(F, G) = |G| - \mathrm{Z}(G)$, it follows by Proposition 3.5 that $\mathrm{mr}^-(F, G) \geqslant |G| - \mathrm{Z}^-(G) \geqslant |G| - \mathrm{Z}(G) = \mathrm{mr}(F, G)$. $\square$

See [1] for a list of graphs G for which it is known that $\mathrm{mr}(\mathbb{R}, G) = |G| - Z(G)$. The zero forcing number $\mathrm{Z}(G)$ of a graph G is never zero, because the color change rule requires a vertex to be black to force another vertex, whereas (as we saw in Example 3.6), it is possible to have $\mathrm{Z}^-(G) = 0$.

The *Cartesian product* of two graphs G and H, denoted $G \,\square\, H$, is the graph with vertex set $V_G \times V_H$ such that (u, v) is adjacent to (u', v') if and only if (1) $u = u'$ and $\{v, v'\} \in E_H$, or (2) $v = v'$ and $\{u, u'\} \in E_G$.

Corollary 3.8. *For any field F and any graph G, $\mathrm{mr}^-(F, G \,\square\, P_t) \geqslant (t-1)|G|$. If t is even and $|G|$ is odd, then $\mathrm{mr}^-(F, G \,\square\, P_t) \geqslant (t-1)|G| + 1$.*

Proof. The set of vertices in a pendant copy of G is a zero forcing set, and minimum skew rank must be even. $\square$

3.2. Cut-vertex reduction

The rank-spread of a graph G was defined in [4] and used to establish cut-vertex reduction, whereby the computation of the minimum rank of a graph with a cut-vertex could be reduced to computing the minimum rank of certain proper subgraphs. In this subsection we extend these ideas to minimum skew rank.

The *skew-rank-spread* of G at vertex v over a field F is defined to be

$$\mathrm{r}_v^-(F, G) = \mathrm{mr}^-(F, G) - \mathrm{mr}^-(F, G - v).$$

Clearly for any vertex v of G, $\mathrm{r}_v^-(F, G)$ is either 0 or 2.

Lemma 3.9. *Let $G=(V=\{v_1,\dots,v_n,v\},E)$ be a graph and F a field. Then $\mathrm{r}_v^-(F,G)=0$ if and only if there exist an optimal matrix $A'\in F^{n\times n}$ for $G-v$ and a vector $\mathbf{b}=[b_i]\in \operatorname{range} A'$ such that $b_i\neq 0$ if and only if v is adjacent to v_i, and $\mathrm{r}_v^-(F,G)=2$ otherwise.*

Proof. Suppose there exists an optimal matrix $A'\in F^{n\times n}$ for $G-v$ and a vector $\mathbf{b}=[b_i]\in \operatorname{range} A'$ such that $b_i\neq 0$ if and only if v is adjacent to v_i. Then

$$A=\begin{bmatrix} A' & \mathbf{b}\\ -\mathbf{b}^T & 0\end{bmatrix}\in \mathcal{S}^-(F,G). \tag{4}$$

Since $\mathbf{b}\in\operatorname{range} A'$, there exists $\mathbf{x}\in F^n$ such that $\mathbf{b}=A'\mathbf{x}$. Since $\mathbf{x}^TA'\mathbf{x}=(\mathbf{x}^TA'\mathbf{x})^T=-\mathbf{x}^TA'\mathbf{x}$, $\mathbf{x}^TA'\mathbf{x}=0$ and $\operatorname{rank} A=\operatorname{rank} A'$. Thus $\mathrm{r}_v^-(F,G)=0$. Conversely, if $r_v^-(G)=0$, any optimal matrix A will have the form (4) with $\operatorname{rank} A'=\mathrm{mr}^-(F,G-v)$ and $\mathbf{b}\in\operatorname{range} A'$. Since $0\leqslant r_v^-(F,G)\leqslant 2$ and the rank of a skew matrix is even, $\mathrm{r}_v^-(F,G)=2$ if and only if $\mathrm{r}_v^-(F,G)\neq 0$. $\square$

Theorem 3.10 [8]. *Let v be a cut-vertex of G. For $i=1,\dots,h$, let $W_i\subseteq V(G)$ be the vertices of the ith component of $G-v$ and let G_i be the subgraph induced by $\{v\}\cup W_i$. Then over a field F,*

$$r_v^-(F,G)=\max_{i=1,\dots,h}\mathrm{r}_v^-(F,G_i), \text{ and}$$

$$\mathrm{mr}^-(F,G)=\begin{cases}\sum_1^h \mathrm{mr}^-(F,G_i-v) & \text{if } \mathrm{r}_v^-(F,G_i)=0 \text{ for all } i=1,\dots,h\\ \sum_1^h \mathrm{mr}^-(F,G_i-v)+2 & \text{if } \mathrm{r}_v^-(F,G_i)=2 \text{ for some } i,\ 1\leqslant i\leqslant h\end{cases}$$

Proof. In both cases, $\sum_1^h \mathrm{mr}^-(F,G_i-v)=\mathrm{mr}^-(F,G-v)\leqslant \mathrm{mr}^-(F,G)$. First assume that $\mathrm{r}_v^-(F,G_i)=0$ for all $i=1,\dots,h$. Then $\sum_1^h \mathrm{mr}^-(F,G_i-v)=\sum_1^h \mathrm{mr}^-(F,G_i)$. Since v is a cut-vertex, there are no overlapping edges, and by Proposition 3.3, $\mathrm{mr}^-(F,G)\leqslant\sum_1^h \mathrm{mr}^-(F,G_i)$. Thus $\mathrm{mr}^-(F,G)=\sum_1^h \mathrm{mr}^-(F,G_i-v)$.

Now assume $\mathrm{r}_v^-(F,G_k)=2$for some k. Then by Lemma 3.9, for every matrix $A^{(k)}$ that is optimal for G_k-v and vector $\mathbf{b}^{(k)}$ having a nonzero pattern reflecting the adjacencies of v within G_k, $\mathbf{b}^{(k)}\notin \operatorname{range} A^{(k)}$. Thus for every matrix A' that is optimal for $G-v$ and vector $\mathbf{b}$ having a nonzero pattern reflecting the adjacencies of v within G, $\mathbf{b}\notin\operatorname{range} A'$ because A' is block-diagonal. Thus by Lemma 3.9, $\mathrm{r}_v^-(F,G)=2$. $\square$

Proposition 3.11. *If F is an infinite field, G' is connected, $|G|\geqslant 2$, and $G=G'\vee K_1$, then $\mathrm{mr}^-(F,G)=\mathrm{mr}^-(F,G')$.*

Proof. Let A' be an optimal matrix for G', and let $V(K_1)=\{v\}$. Since every row of A' has a nonzero entry, there exists $\mathbf{b}\in\operatorname{range} A'$ such that every entry of $\mathbf{b}$ is nonzero. Then by Lemma 3.9, $\mathrm{r}_v^-(G)=0$. $\square$

4. Computation of minimum skew rank of selected graphs

In this section we apply the results in the preceding sections to determine the minimum skew rank of some additional families of graphs. The minimum (symmetric) rank of these graphs is known and listed in the AIM minimum rank graph catalog [2]. We begin by defining several families of graphs.

The wheel on n vertices, denoted by W_n, is constructed by adding a new vertex adjacent to all vertices of the cycle C_{n-1}. The sth *hypercube*, Q_s, is defined inductively by $Q_1=K_2$ and $Q_{s+1}=Q_s\,\square\,K_2$. Clearly $|Q_s|=2^s$. The m,k-*pineapple* (with $m\geqslant 3, k\geqslant 1$) is $P_{m,k}=K_m\cup K_{1,k}$ such that $K_m\cap K_{1,k}$ is the vertex of $K_{1,k}$ of degree k; $P_{5,3}$ is shown in Fig. 4.

The sth *half-graph*, denoted H_s, is constructed from (disjoint) graphs K_s and $\overline{K_s}$, having vertices $u_1,\dots,u_s,v_{s+1},\dots,v_{2s}$, respectively, by adding all edges $\{u_i,v_j\}$ such that $i+j\leqslant 2s+1$. Fig. 5 shows

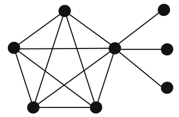

Fig. 4. The pineapple $P_{5,3}$.

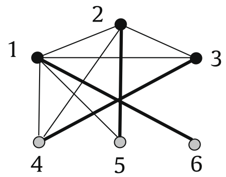

Fig. 5. The 3rd half-graph H_3.

H_3, with the vertices of the K_3 being colored black and the vertices of the $\overline{K_3}$ colored grey. Note that half graph H_s is *the* graph on $2s$ vertices with the largest number of edges among graphs G such that G has a unique perfect matching (in Fig. 5, the three heavy lines are the unique perfect matching of H_3) [12, Corollary 5.3.14].

The *necklace* with s diamonds, denoted N_s, is a 3-regular graph on $4s$ vertices that can be constructed from a $3s$-cycle by appending s extra vertices, with each "extra" vertex adjacent to three sequential cycle vertices; N_3 is shown in Fig. 6 (the coloring of the vertices is explained in the proof of Proposition 4.4).

The *corona* of G with H, denoted $G \circ H$, is the graph of order $|G||H| + |G|$ obtained by taking one copy of G and $|G|$ copies of H, and joining all the vertices in the ith copy of H to the ith vertex of G.

For many of the graphs we discuss, the minimum skew rank is the same over all fields (of characteristic not 2), but as we saw in Example 2.3, the minimum skew rank can differ for finite fields, and it seems plausible that like minimum (symmetric) rank, minimum skew rank can differ even over fields of characteristic zero, although we do not have an example of such a graph.

Proposition 4.1. *Let F be a field.*

1. $\mathrm{mr}^-(F, P_n) = \begin{cases} n & \textit{if } n \textit{ is even,} \\ n-1 & \textit{if } n \textit{ is odd.} \end{cases}$
2. $\mathrm{mr}^-(F, P_{m,k}) \geqslant 4$ $(m \geqslant 3,\ k \geqslant 1)$.
3. $\mathrm{mr}^-(F, H_s) = 2s = |H_s|$.
4. $\mathrm{mr}^-(F, G \circ K_1) = 2|G| = |G \circ K_1|$.

Proof

1. This is an immediate consequence of Theorem 2.7.
2. $P_{m,k} = K_m \cup K_{1,k}$, so by Proposition 3.3, $\mathrm{mr}^-(F, P_{m,k}) \leqslant \mathrm{mr}^-(F, K_m) + \mathrm{mr}^-(F, K_{1,k}) = 4$. Since $P_{m,k}$ contains the paw as an induced subgraph, $\mathrm{mr}^-(P_{m,k}) \geqslant 4$.
3. H_s has a unique perfect matching, so Observation 1.6 applies.
4. $G \circ K_1$ has a unique perfect matching, so again Observation 1.6 applies. □

Proposition 4.2. *Over any field* F, $\mathrm{mr}^-(F, C_n) = \begin{cases} n-1 & \textit{if } n \textit{ is odd,} \\ n-2 & \textit{if } n \textit{ is even.} \end{cases}$

Proof. Note that C_n has an induced P_{n-1}, so $\mathrm{mr}^-(F, C_n)$ is at least the stated rank. Define $A_n = [a_{ij}] \in \mathcal{S}^-(F, C_n)$ by $a_{i,i+1} = 1,\ a_{i+1,i} = -1,\ i = 1, \ldots, n-1,\ a_{n,1} = 1,\ a_{1,n} = -1$ and all other entries are

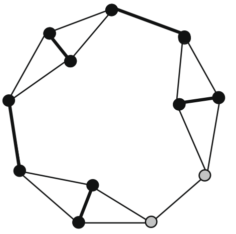

Fig. 6. The necklace N_3.

zero. Since $[1, 1, \ldots, 1, 1]^T \in \ker A$, and if n is even, $[1, -1, \ldots, 1, -1]^T \in \ker A$, rank A realizes the stated minimum rank. $\square$

Since $W_n = C_{n-1} \vee K_1$, by Proposition 3.11 we have the following corollary.

Corollary 4.3. *Over an infinite field* F, $\mathrm{mr}^-(F, W_n) = \begin{cases} n-2 & \text{if } n \text{ is even,} \\ n-3 & \text{if } n \text{ is odd.} \end{cases}$

Proposition 4.4. *Over any field F with at least five elements*, $\mathrm{mr}^-(F, N_s) = 4s - 2$.

Proof. Since N_s has $4s$ vertices and more than one perfect matching (because it contains a $4s$-cycle), by Theorem 2.6, $\mathrm{mr}^-(N_s) \leqslant 4s - 2$. The deletion of two vertices from the $3s$-cycle that are the ends of consecutive diamonds leaves an induced subgraph with a unique perfect matching (in Fig. 6, if the two grey vertices are deleted, then the heavy edges are the unique perfect matching), so $\mathrm{mr}^-(N_s) \geqslant 4s - 2$. $\square$

Proposition 4.5. *Over any field F, for* $s \geqslant 2$, $\mathrm{mr}^-(F, C_t \circ K_s) = \begin{cases} 3t-1 & \textit{if } t \textit{ is odd,} \\ 3t-2 & \textit{if } t \textit{ it even.} \end{cases}$

Proof. Since $C_t \circ K_s$ can be covered by t copies of K_{s+1} and one C_t, intersecting only at cycle vertices, by Proposition 3.3, $\mathrm{mr}^-(F, C_t \circ K_s) \leqslant 2t + (t - 1$ if t is odd, or $t - 2$ if t is even$) = 3t - 1$if t is odd, or $3t - 2$ if t is even.

Let Z be the set of vertices consisting of all but 2 of the vertices in each K_s and two consecutive vertices on the cycle. Note that $|Z| = t(s-2) + 2$. Then Z is a zero forcing set for $C_t \circ K_s$, so $ts + t - (t(s-2) + 2) = 3t - 2 \leqslant \mathrm{mr}^-(C_t \circ K_s)$. So if t is even, $\mathrm{mr}^-(C_t \circ K_s) = 3t - 2$. If t is odd, $3t - 2$ is odd, so $\mathrm{mr}^-(C_t \circ K_s) = 3t - 1$. $\square$

Proposition 4.6. *Over a field F such that the characteristic of F is* 0, *or* $|F| \geqslant 6$, $\mathrm{mr}^-(F, Q_s) = 2^{s-1}$ *for* $s \geqslant 2$.

Proof. Over any field, $\mathrm{mr}^-(F, Q_s) \geqslant 2^{s-1}$ by Corollary 3.8.

Let F be as prescribed. As noted in [7, Theorem 3.14], there are nonzero scalars α, β in F such that $\alpha^2 + \beta^2 = 1$. We define the matrices L_s as follows:

$$L_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad L_s = \begin{bmatrix} \alpha L_{s-1} & \beta I \\ -\beta I & -\alpha L_{s-1} \end{bmatrix}.$$

Each $L_s \in F^{2^s \times 2^s}$ is a skew-symmetric matrix. We show by induction that $L_s^2 = -I_{2^s}$. This is clearly true for $s = 1$. Next, we assume $L_{s-1}^2 = -I_{2^{s-1}}$, so

$$L_s^2 = \begin{bmatrix} \alpha L_{s-1} & \beta I \\ -\beta I & -\alpha L_{s-1} \end{bmatrix}^2 = \begin{bmatrix} \alpha^2 L_{s-1}^2 - \beta^2 I & 0 \\ 0 & -\beta^2 I + \alpha^2 L_{s-1}^2 \end{bmatrix} = \begin{bmatrix} -I & 0 \\ 0 & -I \end{bmatrix}.$$

Define

$$H_s = \begin{bmatrix} L_{s-1} & I \\ -I & L_{s-1} \end{bmatrix}.$$

Each $H_s \in F^{2^s \times 2^s}$ is a skew-symmetric matrix such that $H_s \in \mathcal{S}^-(Q_s)$. Since

$$\begin{bmatrix} I & 0 \\ -L_{s-1} & I \end{bmatrix} \begin{bmatrix} L_{s-1} & I \\ -I & L_{s-1} \end{bmatrix} = \begin{bmatrix} L_{s-1} & I \\ 0 & 0 \end{bmatrix},$$

rank $H_s = 2^{s-1}$. Therefore, $\mathrm{mr}^-(F, Q_s) \leqslant 2^{s-1}$ for $s \geqslant 2$. $\square$

4.1. Minimum skew rank over the real numbers

In this subsection we apply techniques that are specific to the real numbers.

A standard technique for establishing the minimum (symmetric) rank of a Cartesian product $G \,\square\, H$ is to use a Kronecker product construction to produce a matrix in $\mathcal{S}(G \,\square\, H)$ (cf. [1]) (and use the zero forcing number to bound the minimum rank from below). We adapt this method to minimum skew rank.

If A is an $s \times s$ real matrix and B is a $t \times t$ real matrix, then $A \otimes B$ is the $s \times s$ block matrix whose ijth block is the $t \times t$ matrix $a_{ij}B$. Note that $(A \otimes B)^T = A^T \otimes B^T$, so if one of A, B is symmetric and the other is skew-symmetric, $A \otimes B$ is skew-symmetric. Let G be a graph on s vertices, let H be a graph on t vertices, let $A \in \mathcal{S}^-(G)$ and $B \in \mathcal{S}^-(H)$. Then $A \otimes I_t + I_s \otimes B \in \mathcal{S}^-(G \,\square\, H)$ (cf. [10, 9.7]). If $\mathbf{x}$ is an eigenvector of A for eigenvalue λ and $\mathbf{y}$ is an eigenvector of B for eigenvalue μ, then $\mathbf{x} \otimes \mathbf{y}$ is an eigenvector of $A \otimes I_t + I_s \otimes B$ for eigenvalue $\lambda + \mu$.

Lemma 4.7. *Let $A \in \mathbb{R}^{n \times n}$ be skew-symmetric and let the distinct eigenvalues of A be $\lambda_1, \ldots, \lambda_k$ with multiplicities $m_1, \ldots, m_k$. Then* $\mathrm{rank}(A \otimes I_n - I_n \otimes A) \leqslant n^2 - \sum_{i=1}^k m_i^2$.

Proof. Since A is skew-symmetric, over $\mathbb{C}$ there exist independent eigenvectors $\mathbf{x}_j^{(i)}$, $j = 1, \ldots, m_i$ for λ_i, and thus independent null vectors $\mathbf{x}_j^{(i)} \otimes \mathbf{x}_\ell^{(i)}$, $1 \leqslant j, \ell \leqslant m_i$, $1 \leqslant i \leqslant k$. Thus viewing $A \in \mathbb{C}^{n \times n}$, rank $A \leqslant n^2 - \sum_{i=1}^k m_i^2$, and viewing A as a real matrix does not increase its rank. $\square$

Proposition 4.8. $\mathrm{mr}^-(\mathbb{R}, P_s \,\square\, P_s) = s^2 - s = \mathrm{mr}(\mathbb{R}, P_s \,\square\, P_s)$.

Proof. Since $\mathrm{Z}(P_s \,\square\, P_s) = M(\mathbb{R}, P_s \,\square\, P_s) = s$ [1], by Proposition 3.7, $s^2 - s = \mathrm{mr}(\mathbb{R}, P_s \,\square\, P_s) \leqslant \mathrm{mr}^-(\mathbb{R}, P_s \,\square\, P_s)$. But by Lemma 4.7, for any $A \in \mathcal{S}^-(\mathbb{R}, P_s)$, $\mathrm{rank}(A \otimes I_n - I_n \otimes A) \leqslant s^2 - s$ and $A \otimes I_n - I_n \otimes A \in \mathcal{S}^-(\mathbb{R}, P_s \,\square\, P_s)$, so $\mathrm{mr}^-(\mathbb{R}, P_s \,\square\, P_s) \leqslant s^2 - s$. $\square$

Lemma 4.9. *There exists $A \in \mathcal{S}^-(K_n)$ such that* $\mathrm{mult}_A(i) = \mathrm{mult}_A(-i) = \lfloor \frac{n}{2} \rfloor$ *(and zero is an eigenvalue of multiplicity one if n is odd).*

Proof. Let $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ if n is even and $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus [0]$ if n is odd. Choose a real orthogonal matrix U such that UBU^* has all off-diagonal entries nonzero.[2]
$\square$

Proposition 4.10

$$\mathrm{mr}^-(K_s \,\square\, P_t) = \begin{cases} st - s + 1 & \textit{if } s \textit{ is odd and } t \textit{ is even}; \\ st - s & \textit{otherwise}. \end{cases}$$

[2] The existence of such a U can be argued by noting that for any i, j and $k \notin \{i, j\}$, and any skew-symmetric matrix $C = [c_{r,s}]$ with $c_{ik} \neq 0$, there is a suitable Given's rotation Q such that the (i, k) and (j, k) entries of $Q^T CQ$ are both nonzero and the (r, s)-entry of QCQ^T is nonzero whenever the (r, s)-entry of C is nonzero. Thus, if $C \neq 0$, then C is orthogonally similar to a matrix with no off-diagonal zeros.

Proof. $s = \mathrm{Z}(K_s \,\square\, P_t) \leqslant \mathrm{Z}^-(K_s \,\square\, P_t)$ (the equality was established in [1]), so $st - s \leqslant \mathrm{mr}^-(K_s \,\square\, P_t)$. In the case s is odd and t is even, $st - s$ is odd, so $st - s + 1 \leqslant \mathrm{mr}^-(K_s \,\square\, P_t)$.

Construct $A_s \in \mathcal{S}^-(K_s)$ such that $\mathrm{mult}_A(i) = \mathrm{mult}_A(-i) = \lfloor \frac{s}{2} \rfloor$ (and 0 as an eigenvalue of multiplicity one if s is odd). By scalar multiplication we can construct $B_t \in \mathcal{S}^-(P_t)$ having eigenvalues $\pm i$, and also 0 if t is odd. Then $\mathrm{mult}_{A_s \otimes I_t + I_s \otimes B_t}(0) = s$, except if s is odd and t is even, $\mathrm{mult}_{A_s \otimes I_t + I_s \otimes B_t}(0) = s - 1$. Thus $st - s \geqslant \mathrm{mr}^-(K_s \,\square\, P_t)$, except if s is odd and t is even, $st - s + 1 \geqslant \mathrm{mr}^-(K_s \,\square\, P_t)$. $\square$

5. Open questions

In this section we list some open questions about minimum skew rank. We assume throughout this section that the field F is infinite, because the answers differ for finite fields.

Note that for n even, [12] completely characterizes those G for which there is a unique perfect matching, hence by Theorem 2.6, the graphs for which $\mathrm{mr}^-(F, G)$ is as large as possible. It is natural to ask the same question for n odd, namely:

Question 5.1. Characterize G such that $\mathrm{mr}^-(F, G) = |G| - 1$.

Examples of graphs with this property include any graph G with a vertex v such that $G - v$ has a unique perfect matching. To date these are the only known examples (over an infinite field). Example 2.3 shows $\mathrm{mr}^-(\mathbb{Z}_3, K_5) = |K_5| - 1$, despite the fact that $K_5 - v = K_4$ does not have a unique perfect matching for any vertex v.

Question 5.2. Characterize the graphs G such that $\mathrm{mr}^-(F, G) = 4$.

Since 4 is the second smallest possible minimum skew rank of a graph that has an edge, Question 5.2 is related to the interesting and important results characterizing $\mathrm{mr}(G) = 2$ (for symmetric matrices) in [5]. Again, Example 2.3 shows that the answer can be different over a finite field.

Question 5.3. Characterize G such that $\mathrm{mr}^-(F, G) = \mathrm{MR}^-(F, G)$.

Again, Example 2.3 shows that the answer can be different over a finite field. A graph G satisfying $\mathrm{mr}^-(F, G) = \mathrm{MR}^-(F, G)$ is said to have *fixed rank* (over F), since rank$A$ is constant for $A \in \mathcal{S}^-(F, G)$.

Appendix. Polynomials over finite fields

In this appendix we establish some results about polynomials over finite fields that are needed for the proofs given in Section 2. These results may be known, but we do not have a reference.

Proposition 5.4. *Let F be a field with $q \geqslant 3$ elements, and let $p(x_1, x_2, \ldots, x_m)$ be a nonzero homogeneous polynomial in $F[x_1, \ldots, x_m]$ of degree d such that each monomial $x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$ satisfies $e_k \leqslant 1$ for $k = 1, 2, \ldots, m$. Then there exist $a_1, a_2, \ldots, a_m \in F \setminus \{0\}$ such that $p(a_1, a_2, \ldots, a_m) \neq 0$.*

Proof. The proof is by induction on m. If $m = 1$, p has the form cx_1 or c for some nonzero c, and we may simply take $x_1 = 1$.

Assume $m \geqslant 2$ and proceed by induction. Write

$$p(x_1, x_2, \ldots, x_m) = x_m r(x_1, \ldots, x_{m-1}) + s(x_1, \ldots, x_{m-1})$$

for some homogeneous polynomials r and s in $F[x_1, \ldots, x_{m-1}]$. If s is not the zero polynomial, then s is homogeneous of degree d and by the inductive assumption, there exist nonzero $a_1, \ldots, a_{m-1}$ such that $s(a_1, \ldots, a_{m-1}) \neq 0$. If $r(a_1, a_2, \ldots, a_{m-1}) = 0$, then $p(a_1, \ldots, a_{m-1}, 1) \neq 0$. Otherwise,

$$p(a_1, \ldots, a_{m-1}, a_m) \neq 0$$

for each a_m other than $-\frac{s(a_1,\ldots,a_{m-1})}{r(a_1,\ldots,a_{m-1})}$. Since F has at least two nonzero elements, there is such a nonzero a_m.

Next consider the case that s is the zero polynomial. Since p is not the zero polynomial, r is not the zero polynomial, and hence is a nonzero homogeneous polynomial in $m-1$ variables. By induction there exist $a_1, \ldots, a_{m-1} \in F \setminus \{0\}$ with $r(a_1, a_2, \ldots, a_{m-1}) \neq 0$, and hence $p(a_1, a_2, \ldots, a_{m-1}, 1) \neq 0$. □

Lemma 5.5. *Let F be a field with $q \geqslant 4$ elements, and let $t(x_1, x_2, \ldots, x_m)$ be a nonzero homogeneous polynomial in $F[x_1, \ldots, x_m]$ of degree d such that each monomial $x_1^{e_1} x_2^{e_2} \cdots x_m^{e_m}$ satisfies $e_k \leqslant 2$ for $k = 1, 2, \ldots, m$. Then there exist $a_1, \ldots, a_m \in F \setminus \{0\}$ such that $t(a_1, \ldots, a_m) \neq 0$.*

Proof. By induction on m. If $m = 1$, then $t(x_m)$ is cx_m^2, cx_m or c for some nonzero c, and we may take $x_m = 1$.

Assume $m \geqslant 2$ and proceed by induction. Write

$$t(x_1, \ldots, x_m) = x_m^2 j(x_1, x_2, \ldots, x_{m-1}) + x_m k(x_1, \ldots, x_{m-1}) + \ell(x_1, \ldots, x_{m-1}).$$

For $a_1, \ldots, a_{m-1} \in F \setminus \{0\}$,

$$t(a_1, \ldots, a_{m-1}, x_m) = x_m^2 j(a_1, a_2, \ldots, a_{m-1}) + x_m k(a_1, \ldots, a_{m-1}) + \ell(a_1, a_2, \ldots, a_{m-1})$$

is a polynomial in $F[x_m]$. If there is an $a_m \in F \setminus \{0\}$ such that $t(a_1, a_2, \ldots, a_{m-1}, x_m)$ evaluated at $x_m = a_m$ is nonzero, then we are done.

Otherwise, for each choice of $a_1, \ldots, a_{m-1} \in F \setminus \{0\}$, each nonzero element of F is a root of $t(a_1, a_2, \ldots, a_{m-1}, x_m)$. We claim that this cannot occur. As F has at least four elements, $t(a_1, a_2, \ldots, a_{m-1}, x_m)$ has at least three roots and degree at most two. Thus, $t(a_1, a_2, \ldots, a_{m-1}, x_m)$ is the zero polynomial for each choice of $a_1, \ldots, a_{m-1} \in F \setminus \{0\}$. In particular, each of the homogeneous polynomials, j, k, ℓ vanishes at each choice of $(a_1, a_2, \ldots, a_{m-1})$ with $a_1, \ldots, a_{m-1} \in F \setminus \{0\}$. Hence by induction, each of j, k and ℓ is the zero polynomial, which cannot happen since t is nonzero. □

Note that if F is the field with three elements, and $p(x, y) = x^2 - y^2$, then $p(a, b) = 0$ for each choice of $a, b \in F \setminus \{0\}$. So Lemma 5.5 needs $q \geqslant 4$.

Proposition 5.6. *Let F be a field with more than three elements, and let $p(x_1, x_2, \ldots, x_m)$ be a nonzero homogeneous polynomial in $F[x_1, \ldots, x_m]$ of degree d such that each monomial $x_1^{e_1} x_2^{e_2} \cdots x_m^{e^m}$ satisfies $e_k \leqslant 1$ for $k = 1, 2, \ldots, m$. Then either $p(x_1, x_2, \ldots, x_m)$ has exactly one nonzero term or there exist $a_i \in F \setminus \{0\}$ such that $p(a_1, a_2, \ldots, a_m) = 0$.*

Proof. Assume that $p(x_1, x_2, \ldots, x_m)$ has at least two nonzero terms. Since p is homogeneous and has at least two nonzero terms, there is an i such that p has one term involving x_i and another term that does not involve x_i. Without loss of generality, we may take $i = m$. Write $p(x_1, \ldots, x_m) = x_m r(x_1, \ldots, x_{m-1}) + s(x_1, \ldots, x_{m-1})$. Since x_m is in some term of $p(x_1, \ldots, x_m)$, r is not the zero polynomial. Since x_m is not in some term of $p(x_1, \ldots, x_m)$, s is not the zero polynomial.

Consider the polynomial $t(x_1, x_2, \ldots, x_{m-1}) = r(x_1, \ldots, x_{m-1}) s(x_1, \ldots x_{m-1})$. Note that t is homogeneous, nonzero, and the exponent of each x_j in each monomial is at most 2. Thus, by Lemma 5.5, there exist nonzero $a_1, \ldots, a_{m-1}$ such that $t(a_1, \ldots, a_{m-1}) \neq 0$. Now observe that

$$p\left(a_1, a_2, \ldots, a_{m-1}, \frac{-s(a_1, \ldots, a_{m-1})}{r(a_1, \ldots, a_{m-1})}\right) = 0$$

and each of $a_1, a_2, \ldots, a_{m-1}$, and $\frac{-s(a_1, \ldots, a_{m-1})}{r(a_1, \ldots, a_{m-1})}$ is nonzero. □

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