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On nonsingular regular magic squares of odd order ☆

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ABSTRACT

Using centroskew matrices, we provide a necessary and sufficient condition for a regular magic square to be nonsingular. Using latin squares and circulant matrices we describe a method of construction of nonsingular regular magic squares of order n where n is an odd prime power.

1. Introduction

A *magic square* M is an n-by-n array of numbers in which the sum of entries along each row, each column, the main diagonal and the cross diagonal are the same constant μ called the *magic*

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sum. If the entries of M are integers from 1 through n^2 then M is said to be a *classical magic square* or *natural magic square* with magic sum $\frac{n(n^2+1)}{2}$. A magic square $M = [m_{ij}]$ is said to be *regular* if $m_{ij} + m_{n+1-i,n+1-j} = \frac{2\mu}{n}$. In other words, in a regular magic square the sum of any two entries that are symmetrically placed across the center of the square is equal to $\frac{2\mu}{n}$. In the case of classical magic square symmetrically placed entries would sum to $n^2 + 1$ and the entries are said to be *complements* of each other. Regular magic squares are also called *associated* or *symmetrical* magic squares.

The famous Dürer's magic square

$$\begin{bmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{bmatrix}$$

is an example of a regular magic square [7]. It was observed earlier (for example, see [6]) that regular magic squares of even order are singular i.e., the determinant of an even order regular magic square is zero. Mattingly [5] proved this result in 2000. For odd orders, the only classical magic square of order three is regular and nonsingular. In [4] the authors using a backtracking program found all the regular magic squares of order five and found a small number of them (656 out of 48,544) to be singular. One such 5-by-5 magic square exhibited below has zero as an eigenvalue with multiplicity two:

$$\begin{bmatrix} 15 & 12 & 21 & 10 & 7 \\ 2 & 6 & 17 & 18 & 22 \\ 25 & 23 & 13 & 3 & 1 \\ 4 & 8 & 9 & 20 & 24 \\ 19 & 16 & 5 & 14 & 11 \end{bmatrix}.$$

Therefore, in this paper we address the question of when an odd order regular magic square is nonsingular. We provide a necessary and sufficient condition for an odd order regular magic square to be nonsingular. Further we describe a method to construct nonsingular regular magic squares using circulant matrices and orthogonal latin squares when the order of the magic square is an odd prime power.

2. Eigenvalues of a magic square

In this section we collect some results on eigenvalues of magic squares that are needed in this paper. Since the field of complex numbers is algebraically closed, the characteristic polynomial $P_M(z)$ of an n-by-n magic square M can be written as a product of linear factors,

$$P_M(z) = det(zI - M) = \prod_{i=1}^{n} (z - m_i),$$

where I is the identity matrix and the complex numbers $m_1, \ldots, m_n$ are the eigenvalues of M. Since $Me = \mu e$, where e is a column vector of all 1's, we observe that the magic sum μ is an eigenvalue of M.

Let E denote the matrix of all 1's for its entries. The following theorem is found in [1]. We provide a simpler proof using Schur's unitary triangularization theorem [3].

Theorem 2.1. *If M is an n-by-n magic square and $p \in \mathbb{C}$, then $M + pE$ has the same eigenvalues of M except that μ is replaced with $\mu + pn$.*

Proof. Since $ME = \mu E = EM$ there is a unitary matrix U such that $U^*MU = T$ and $U^*EU = S$ where T and S are upper triangular matrices with the diagonal entries being the eigenvalues of M and E, respectively. We order the eigenvalues of M as $\mu, m_2, \ldots, m_n$. Since E has eigenvalues n with multiplicity one and 0 with multiplicity $n-1$ we may order them $s_1, s_2, \ldots, s_n$. Note that $M + pE$ is a magic square with magic sum $\mu + pn$. Hence its eigenvalues are $\mu + pn, k_2, \ldots, k_n$. Now $U^*(M + pE)U = T + pS$ has diagonal entries that are eigenvalues of $M + pE$. Thus, $\mu + pn = \mu + ps_1$, and $k_i = m_i + ps_i$ for $i = 2, \ldots, n$. Hence $s_1 = n$. Since $s_2, \ldots, s_n$ are all zero it follows that $k_i = m_i$ for $2 \leqslant i \leqslant n$. □

Corollary 2.2. *If M is a magic square, then* $M - \frac{\mu}{n}E$ *has the same eigenvalues* $m_2, \ldots, m_n$ *as M except that* μ *is replaced by 0.*

The above corollary is also proved in [5].

3. Regular magic squares

Let J denote the permutation matrix obtained by writing 1 in each of the cross diagonal entries and 0 elsewhere. Since multiplying a matrix on the left by J reverses the order of the rows and multiplying on the right by J reverses the order of the columns, we will call J the *reversal* matrix. Also, observe that $J^T = J$ and $J^2 = I$ (that is, J is self-adjoint and is its own inverse).

Observation 3.1. An n-by-n matrix M is a regular magic square if and only if

$$M + JMJ = \left(\frac{2\mu}{n}\right)E, \tag{1}$$

where μ is the magic sum.

Definition 3.2. If M is a regular magic square we define

$$Z = M - \frac{\mu}{n}E$$

to be the *corresponding zero regular magic square*.

Definition 3.3. An n-by-n matrix B with real entries is said to be *centrosymmetric* if $JBJ = B$ and is said to be *centroskew* if $JBJ = -B$.

The following lemma allows us to use properties of centroskew matrices to study regular magic squares.

Lemma 3.4. *If M is a regular magic square, the corresponding zero magic square Z is a centroskew matrix.*

Proof. Using (1) we see that $Z + JZJ = (M - \frac{\mu}{n}E) + (JMJ - \frac{\mu}{n}E) = 0$. □

In the following theorem, we give a necessary and sufficient condition for an odd order regular magic square to be nonsingular.

Theorem 3.5. *Let M be a regular magic square of order* $n = 2k + 1$ *with positive entries. We write the corresponding centroskew matrix Z in partitioned form as follows:*

$$Z = \begin{bmatrix} Z_{11} & a & Z_{13} \\ b^T & 0 & -b^TJ \\ -JZ_{13}J & -Ja & -JZ_{11}J \end{bmatrix},$$

where Z_{11}, Z_{13} are $k \times k$ matrices and a, b are $k \times 1$ vectors. Then M is nonsingular if and only if $Z_{11} + Z_{13}J$ and $Z_{11} - Z_{13}J$ are both nonsingular.

Proof. Using partitioned matrices

$$K = \begin{bmatrix} I & 0 & I \\ 0 & 1 & 0 \\ J & 0 & -J \end{bmatrix} \quad \text{and} \quad K^{-1} = \frac{1}{2}\begin{bmatrix} I & 0 & J \\ 0 & 2 & 0 \\ I & 0 & -J \end{bmatrix},$$

where I, J are $k \times k$ matrices we find Z' similar to Z, namely

$$Z' = K^{-1}ZK = \begin{bmatrix} 0 & 0 & Z_{11} - Z_{13}J \\ 0 & 0 & 2b^T \\ Z_{11} + Z_{13}J & a & 0 \end{bmatrix}.$$

Using elementary row operations we reduce $Z' - \lambda I$ to

$$Z' - \lambda I \sim \begin{bmatrix} -\lambda I & C_{12} \\ 0 & \frac{1}{\lambda}C_{21}C_{12} - \lambda I \end{bmatrix},$$

where $C_{12} = \begin{bmatrix} Z_{11} - Z_{13}J \\ 2b^T \end{bmatrix}$ and $C_{21} = \begin{bmatrix} Z_{11} + Z_{13}J & a \end{bmatrix}$. Therefore we can write the characteristic polynomial of Z' as

$$\begin{aligned} \det(Z' - \lambda I) &= (-1)^{k+1}\lambda^{k+1}\det\left(\tfrac{1}{\lambda}C_{21}C_{12} - \lambda I\right) \\ &= (-1)^{k+1}\lambda\det(C_{21}C_{12} - \lambda^2 I). \end{aligned}$$

Since Z is a zero regular magic square we observe that

$$a = -(Z_{11} + Z_{13})e \quad \text{and} \quad b^T = e^T(JZ_{13}J - Z_{11}).$$

Thus, using $JE = EJ = E$ we can write

$$\begin{aligned} C_{21}C_{12} &= \begin{bmatrix} Z_{11} + Z_{13}J & a \end{bmatrix}\begin{bmatrix} Z_{11} - Z_{13}J \\ 2b^T \end{bmatrix} \\ &= (Z_{11} + Z_{13}J)(Z_{11} - Z_{13}J) + (Z_{11} + Z_{13})2ee^T(Z_{11} - JZ_{13}J) \\ &= (Z_{11} + Z_{13}J)(I + 2E)(Z_{11} - Z_{13}J). \end{aligned}$$

Notice that $\det(I + 2E) = 2k + 1$ where I and E are $k \times k$ matrices. To see this, observe that adding columns 2 through k to the first column gives a constant first column consisting of $2k + 1$. Then a row reduction produces an upper triangular matrix with $(2k + 1)$ in $(1, 1)$ position and 1 for the remaining $k - 1$ diagonal entries.

Observe that λ^2 is a factor of the characteristic polynomial $\det(C_{21}C_{12} - \lambda^2 I)$ if and only if $\det(C_{21}C_{12}) = 0$. Using Corollary 2.2 the eigenvalues of Z are the same as M except 0 replaces the eigenvalue μ of M. Therefore zero is an eigenvalue of an odd order regular magic square M if and only if $\det(C_{21}C_{12}) = 0$. Hence M is nonsingular if and only if $Z_{11} + Z_{13}J$ and $Z_{11} - Z_{13}J$ are nonsingular. □

Remark. A result similar to Theorem 3.5 can be established for even order regular magic squares. As a consequence it is possible to give a different proof of Mattingly's result [5] mentioned earlier.

4. Latin squares and circulant matrices

In this section we describe properties of circulant matrices and latin squares that are needed in this paper. These properties are utilized in the next section to develop a method of construction that produces nonsingular regular magic squares.

Definition 4.1. A *latin square* is an n-by-n matrix containing n distinct numbers arranged in such a way that each number appears exactly once in every row and in every column.

Definition 4.2. Two n-by-n latin squares A and B are said to be *orthogonal* if the n^2 ordered pairs obtained using the corresponding entries of A and B are distinct. In this case, each matrix is said to be an *orthogonal mate* of the other.

For further information on latin squares the reader may consult [2].

Definition 4.3. An n-by-n matrix is said to be a *circulant* if each row other than the first row is obtained from the preceding row by shifting entries cyclically one column to the right.

Example 4.4. The following is a 5-by-5 circulant matrix

$$\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_5 & a_1 & a_2 & a_3 & a_4 \\ a_4 & a_5 & a_1 & a_2 & a_3 \\ a_3 & a_4 & a_5 & a_1 & a_2 \\ a_2 & a_3 & a_4 & a_5 & a_1 \end{bmatrix}.$$

Observation 4.5. Suppose A is a circulant matrix with n distinct entries in the first row. Then it is easy to verify each of the following conditions:

- A is a latin square.
- A is a Toeplitz matrix.
- When n is odd, the cross diagonal and all broken diagonals parallel to the cross diagonal of A have all the n distinct elements of the first row.

We will denote by P the right shifting permutation matrix with all entries equal to 1 on the super-diagonal and 1 in the lower left corner with remaining entries equal to zero. Note that if a matrix A is multiplied by P on the right then each column of A is shifted to its right and the last column of A is shifted to the first column of A. Moreover, $P^n = I$ and that the eigenvalues of P are the nth roots of unity [3].

For the rest of the paper let n denote an odd integer and S denote the set

$$S = \left\{-\frac{n-1}{2}, \ldots, -1, 0, 1, \ldots, \frac{n-1}{2}\right\}. \tag{2}$$

Definition 4.6. Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a list consisting of n distinct members from S in (2) and $a_1 = 0$. A circulant matrix A with its first row equal to $\vec{a}$ is called a S-circulant matrix.

For example

$$A_{5a} = \begin{bmatrix} 0 & -1 & -2 & 1 & 2 \\ 2 & 0 & -1 & -2 & 1 \\ 1 & 2 & 0 & -1 & -2 \\ -2 & 1 & 2 & 0 & -1 \\ -1 & -2 & 1 & 2 & 0 \end{bmatrix}$$

is a S-circulant matrix of order 5.

Lemma 4.7. *Suppose A is a S-circulant matrix. Then A is a zero magic square.*

Proof. Since the sum of the members of S in (2) is zero, the lemma follows from Observation 4.5. □

Lemma 4.8. *Suppose A is a S-circulant matrix. Then A is centroskew if and only if*

$$a_{j+1} + a_{n+1-j} = 0 \quad \text{for } j = 1, \ldots, n-1.$$

Proof. Let A be a S-circulant matrix. Then A can be written as $A = \sum_{j=0}^{n-1} a_{j+1}P^j$ where P is the right shifting permutation matrix. Since $JP^kJ = P^{n-k}$ and $a_1 = 0$ we get $A + JAJ = \sum_{j=0}^{n-1} a_{j+1}P^j + \sum_{k=0}^{n-1} a_{k+1}P^{n-k} = \sum_{j=1}^{n-1} a_{j+1}P^j + \sum_{j=1}^{n-1} a_{n+1-j}P^j = \sum_{j=1}^{n-1}(a_{j+1} + a_{n+1-j})P^j$. Hence it follows that A is centroskew if and only if $a_{j+1} + a_{n+1-j} = 0$ for $j = 1, \ldots, n-1$. □

Example 4.9. The following is a S-circulant matrix that is centroskew.

$$A_{5b} = \begin{bmatrix} 0 & 2 & -1 & 1 & -2 \\ -2 & 0 & 2 & -1 & 1 \\ 1 & -2 & 0 & 2 & -1 \\ -1 & 1 & -2 & 0 & 2 \\ 2 & -1 & 1 & -2 & 0 \end{bmatrix}.$$

Lemma 4.10. *Let A be a S-circulant matrix. Then JA and AJ are orthogonal mates. Moreover, if A is a centroskew matrix then JA and AJ are centroskew.*

Proof. Multiplication of A by J on the right reverses the order of columns of A. Therefore the broken diagonals parallel to the main diagonal interchange with broken diagonal parallel to the cross diagonal. If we pair up entries of A and AJ each constant diagonal entry in a diagonal of A parallel to the main diagonal pairs up with distinct entries from S in (2) in the corresponding position in AJ using Observation 4.5. Therefore A and AJ are orthogonal latin squares. Moreover, if A is centroskew then $J(AJ)J = (JAJ)J = -AJ$. Hence AJ is centroskew. Similar proof can be given to show A and JA are orthogonal latin squares and JA is centroskew when A is a centroskew matrix. □

5. Nonsingular regular magic squares

In this section we provide a method to construct nonsingular regular magic squares whose orders are odd primes and powers of odd primes. We utilize the results from Section 4 to develop this method.

Proposition 5.1. *Let A be a centroskew S-circulant matrix of order n. Define $Z = nA + AJ$. Then Z is a centroskew zero magic square with n^2 distinct entries from the set*

$$Q = \left\{ -(\frac{n^2-1}{2}), \ldots, -1, 0, 1, \ldots, (\frac{n^2-1}{2}) \right\}. \tag{3}$$

Proof. Since $JZJ = nJAJ + J(AJ)J = -nA - AJ = -Z$, using Lemma 4.10, we see that Z is centroskew. Since A has n distinct entries from the set S in (2) we see that nA has n distinct entries from the set nS. From Lemma 4.10 we know that A and AJ are orthogonal latin squares. Hence nA and AJ are orthogonal latin squares. Since entries of Z are sums of n^2 distinct ordered pairs of entries from nS and S, we see that Z has n^2 distinct entries from the set Q in (3). Since nA and AJ are zero magic squares Z is also a zero magic square. □

Theorem 5.2. *Let A be a centroskew S-circulant matrix of order n and $Z = nA + AJ$. If n is an odd prime then $\text{rank}(Z) = n - 1$.*

Proof. Let $Z = nA + AJ = A(nI + J)$. Suppose $B = nI + J$. It is known that $\text{rank}(B) = n$ and $\text{rank}(Z) = \text{rank}(AB) = \text{rank}(A)$. Since A is a S-circulant matrix its eigenvalues are determined by its first row and are given by

$$\left\{ \sum_{j=0}^{n-1} a_{j+1}\omega^{kj} : k = 0, 1, \ldots, n-1, \omega = e^{\frac{2\pi i}{n}} \right\}. \tag{4}$$

When $k = 0$ we find that the eigenvalue is zero since A is a zero magic square. For each value of $k = 1, \ldots, n-1$, note that kj will have distinct values mod n where n is prime.

Hence for $k \neq 0$, we have an eigenvalue $\sum_{l=0}^{n-1} a_{l+1}\omega^l$ where the a_l are distinct elements of the set S in (2). Since ω is a primitive nth root of unity where n is a prime number and the coefficients are distinct, the sum $\sum_{l=0}^{n-1} a_{l+1}\omega^l$ cannot be zero. Therefore zero is an eigenvalue of A of multiplicity one. Hence $\text{rank}(A) = n - 1$. This proves the theorem. □

Example 5.3. Using the matrix A_{5b} given in Example 4.9 we get the following matrix $Z_{5b} = 5A_{5b} + A_{5b}J$ namely,

$$Z_{5b} = \begin{bmatrix} -2 & 11 & -6 & 7 & -10 \\ -9 & -1 & 12 & -5 & 3 \\ 4 & -8 & 0 & 8 & -4 \\ -3 & 5 & -12 & 1 & 9 \\ 10 & -7 & 6 & -11 & 2 \end{bmatrix},$$

where $\text{rank}(Z_{5b}) = 4$.

Theorem 5.4. *Let A be a centroskew S-circulant matrix of order n and $Z = nA + AJ$. Let the first row of A be $\vec{a} = (a_1, a_2, \ldots, a_n)$ and set $a_j = j - 1$ for $j = 1, \ldots, \left(\frac{n+1}{2}\right)$. If $n = p^t$ where p is an odd prime, then $\text{rank}(Z) = n - 1$.*

Proof. Let $n = p^t$ where p is an odd prime and let $Z = nA + AJ = A(nI + J)$ where A is an S-circulant centroskew matrix. Since we have set $a_j = j - 1$ for $j = 1, \ldots, \left(\frac{n+1}{2}\right)$ and have made A an S-circulant centroskew matrix, we have defined all entries of the first row using Lemma 4.8. For if $j > \frac{n+1}{2}$, then $a_j = -a_{n-j+1}$. As in the proof of Theorem 5.2, $\text{rank}(Z) = \text{rank}(A)$ and the eigenvalues of A are given by (4).

We look at three different cases for k. They are $k = 0$, k relatively prime to p, and k a multiple of p^a where $a < t - 1$.

Case 1. As in the proof of Theorem 5.2, the eigenvalue is zero when $k = 0$.

Case 2. Assume now that k is relatively prime to p. This means that the kj are distinct values mod p^t. So the eigenvalues are $\sum_{j=0}^{n-1} a_l \omega^l$ where the a_l are distinct numbers in the set S in (2).

It is well known that for p prime, the pth cyclotomic polynomial is $\Phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$ and more generally, for any $b \in \mathbb{Z}$, the cyclotomic polynomial for p^b is $\Phi_{p^b}(x) = x^{(p-1)p^{b-1}} + x^{(p-2)p^{b-1}} + \cdots + x^{1p^{b-1}} + 1$. Notice that $\Phi_{p^b}(x) = \Phi_p(x^{p^{b-1}})$. The zeros of Φ_{p^b} are the primitive roots of unity of order p^b. With the above notation, this means $\Phi_{p^b}(\omega^{p^{t-b-1}}) = 0$.

The eigenvalue is a number which exists in the ring $\mathbb{Q}[\omega]$. For this reason, if the eigenvalue is zero, then it is a linear combination of $\Phi_{p^b}(\omega^{p^{t-b-1}})$. However, each $\Phi_{p^b}(\omega^{p^{t-b-1}})$ has the property that the coefficients on ω^g and $\omega^{g+p^{t-1}}$ are the same. Linear combinations of $\Phi_{p^b}(\omega^{p^{t-b-1}})$ must also have this property. The eigenvalue clearly does not have this property so the eigenvalue cannot be zero.

Case 3. Let $k = mp^s$ where $0 < s < t$ and m is relatively prime to p. The eigenvalue for this case becomes $\sum_{j=0}^{n-1} a_{j+1}(\omega^{mp^s})^j$. There are p^{t-s} distinct values for the $(\omega^{mp^s})^j$, so the eigenvalue will be $\sum_{l=0}^{p^{t-s}} C_l(\omega^{p^s})^l$ where each C_l is a sum of p^s a_j's.

Examine C_0. Notice that $j = up^{t-s}$ for some $u \in \mathbb{Z}$ if and only if $(\omega^{mp^s})^j = 1$. This means that $C_0 = \sum_{u=0}^{p^s-1} a_{up^{t-s}+1}$. We have defined the a_j's in such away as to have $a_1 = 0$ and $a_{j+1} = -a_{n-j+1}$. Since $a_1 = 0$ is in the sum of C_0 and also each $a_{n-up^{t-s}+1}$ is in the sum of C_0, we must have that $C_0 = 0$.

Examine $C_{mp^{t-s-1}}$. In this case, $j = p^{t-s-1} + up^{t-s}$ for some $u \in \mathbb{Z}$ if and only if $(\omega^{mp^s})^j = (\omega^{p^s})^{mp^{t-s-1}}$. This forces $C_{mp^{t-s-1}} = \sum_{u=0}^{p^s-1} a_{p^{t-s-1}+up^{t-s}+1}$. By the way we have defined our a_j, we have that $a_j + i = a_{j+i}$ as long as $j \neq \frac{n+1}{2}$ and $j + i \leqslant n$. This means that $C_{mp^{t-s-1}} = p^{t-s-1}(p^s) = p^{t-1}$ so long as $p^{t-s-1} + up^{t-s} \neq \frac{n+1}{2}$. One can see that $p^{t-s-1} + up^{t-s} \neq \frac{n+1}{2}$ because if it were, p would be a multiple of both n and $n + 1$ which cannot happen.

Recall that our eigenvalue is $\sum_{l=0}^{p^{t-s}} C_l(\omega^{p^s})^l$ which exists inside the group ring $\mathbb{Q}[\omega]$. To be zero in this ring, the eigenvalue would have to be a linear combination of $\Phi_{p^b}(\omega^{p^{t-b-1}})$ for different values of b. This would require our eigenvalue to have the property that ω^g and $\omega^{g+p^{t-1}}$ have the same coefficients, and by similar arguments, ω^g and $\omega^{g+jp^{t-1}}$ have the same coefficients for any j. Our eigenvalue does not have this property since $C_0 \neq C_{mp^{t-s-1}}$.

From cases 2 and 3 for $k \neq 0$ we have seen that the corresponding eigenvalue is not zero. Therefore zero is an eigenvalue of A of multiplicity one. Hence rank(A) =rank$(Z) = n - 1$. □

Theorem 5.5. *Let A be a centroskew S-circulant matrix of order n. If n is an odd prime and $Z = nA + AJ$, then $M = Z + \frac{n^2+1}{2}E$ is a classical regular magic square that is nonsingular. Similarly if n is the power of an odd prime and the first row of A is defined as $a_j = j - 1$ for $j = 1, \ldots, \left(\frac{n+1}{2}\right)$, then $M = Z + \frac{n^2+1}{2}E$ is a classical regular magic square that is nonsingular.*

Proof. Since $Z = nA + AJ$ is a zero regular magic square with n^2 distinct entries from the set Q in (3) we see that M is a classical regular magic square. Since Z is a centroskew zero magic square corresponding to M that has zero as an eigenvalue with multiplicity one (using Theorems 5.2 and 5.4), we conclude from Corollary 2.2 that M is nonsingular. □

6. Concluding remarks

The method of construction in the previous section gives at least one nonsingular classical regular magic square for every order n which is a prime power. We give one example below.

Example 6.1. Using Theorem 5.5 and Z_{5b} from Example 5.3 we obtain the classical regular magic square $M_{5b} = Z_{5b} + 13E$ namely,

$$M_{5b} = \begin{bmatrix} 11 & 24 & 7 & 20 & 3 \\ 4 & 12 & 25 & 8 & 16 \\ 17 & 5 & 13 & 21 & 9 \\ 10 & 18 & 1 & 14 & 22 \\ 23 & 6 & 19 & 2 & 15 \end{bmatrix}$$

which is nonsingular.

6.1. Case when $n = 9$

For odd primes, Theorem 5.2 requires that $a_j \in S, a_1 = 0$, and $a_{n-j+1} = -a_{j+1}$ for $j = 1, \ldots, n-1$. In Theorem 5.4 an assumption was made regarding the values of a_j, namely $a_j = j-1$ for $1 \leqslant j \leqslant \frac{n+1}{2}$. There are other choices for a_j that would also produce a Z so that $\text{rank}(Z) = n-1$. For instance, if one were to negate the a_j's in Theorem 5.4, the proof would still work. Some other assignments do not work. For example, let $n = 9$ and let the first row of a centroskew S-circulant matrix A_9 be $[0, 1, -2, 4, -3, 3, -4, 2, -1]$. This assignment has the properties $a_j \in S$, $a_1 = 0$, and $a_{n-j+1} = -a_{j+1}$ for $j = 1, \ldots, n-1$. The eigenvalues of the matrix are still given by (4), but when $k = 3$ the eigenvalue $(0+4-4)+(1-3+2)\omega^3+(-1+3-2)\omega^6 = 0$. Similarly the eigenvalue is zero when $k = 6$. Using this first row for A_9 yields a Z_9 such that $\text{rank}(Z_9) = 6$. This yields a classical regular magic square M_9 namely,

$$M_9 = \begin{bmatrix} 41 & 53 & 20 & 81 & 12 & 73 & 4 & 61 & 33 \\ 35 & 38 & 54 & 21 & 82 & 13 & 70 & 6 & 59 \\ 56 & 36 & 39 & 55 & 22 & 79 & 15 & 68 & 8 \\ 9 & 57 & 37 & 40 & 52 & 24 & 77 & 17 & 65 \\ 66 & 10 & 58 & 34 & 42 & 50 & 26 & 74 & 18 \\ 19 & 67 & 7 & 60 & 32 & 44 & 47 & 27 & 75 \\ 76 & 16 & 69 & 5 & 62 & 29 & 45 & 48 & 28 \\ 25 & 78 & 14 & 71 & 2 & 63 & 30 & 46 & 49 \\ 51 & 23 & 80 & 11 & 72 & 3 & 64 & 31 & 43 \end{bmatrix}$$

such that $\text{rank}(M_9) = 7$.

6.2. Case when $n = 15$

The methods in the previous section work only for odd primes and odd prime powers. One may ask if the method would work for products of distinct primes. When the variable n is the product of distinct primes the cyclotomic polynomials are more complex. Therefore, we do not know whether the theorem would hold when n is the product of distinct primes. However in the case that $n = 15$, we have found that the construction in Theorem 5.4 does work. If we let the first row of a centroskew S-circulant matrix A_{15} be $[0, 1, 2, 3, 4, 5, 6, 7, -7, -6, -5, -4, -3, -2, -1]$, we obtain the corresponding classical regular magic square M_{15} namely,

$$M_{15} = \begin{bmatrix} 112 & 126 & 140 & 154 & 168 & 182 & 196 & 225 & 14 & 28 & 42 & 56 & 70 & 84 & 98 \\ 96 & 110 & 124 & 138 & 152 & 166 & 195 & 209 & 223 & 12 & 26 & 40 & 54 & 68 & 82 \\ 80 & 94 & 108 & 122 & 136 & 165 & 179 & 193 & 207 & 221 & 10 & 24 & 38 & 52 & 66 \\ 64 & 78 & 92 & 106 & 135 & 149 & 163 & 177 & 191 & 205 & 219 & 8 & 22 & 36 & 50 \\ 48 & 62 & 76 & 105 & 119 & 133 & 147 & 161 & 175 & 189 & 203 & 217 & 6 & 20 & 34 \\ 32 & 46 & 75 & 89 & 103 & 117 & 131 & 145 & 159 & 173 & 187 & 201 & 215 & 4 & 18 \\ 16 & 45 & 59 & 73 & 87 & 101 & 115 & 129 & 143 & 157 & 171 & 185 & 199 & 213 & 2 \\ 15 & 29 & 43 & 57 & 71 & 85 & 99 & 113 & 127 & 141 & 155 & 169 & 183 & 197 & 211 \\ 224 & 13 & 27 & 41 & 55 & 69 & 83 & 97 & 111 & 125 & 139 & 153 & 167 & 181 & 210 \\ 208 & 222 & 11 & 25 & 39 & 53 & 67 & 81 & 95 & 109 & 123 & 137 & 151 & 180 & 194 \\ 192 & 206 & 220 & 9 & 23 & 37 & 51 & 65 & 79 & 93 & 107 & 121 & 150 & 164 & 178 \\ 176 & 190 & 204 & 218 & 7 & 21 & 35 & 49 & 63 & 77 & 91 & 120 & 134 & 148 & 162 \\ 160 & 174 & 188 & 202 & 216 & 5 & 19 & 31 & 47 & 61 & 90 & 104 & 118 & 132 & 146 \\ 144 & 158 & 172 & 186 & 200 & 214 & 3 & 17 & 31 & 60 & 74 & 88 & 102 & 116 & 130 \\ 128 & 142 & 156 & 170 & 184 & 198 & 212 & 1 & 30 & 44 & 58 & 72 & 86 & 100 & 114 \end{bmatrix}.$$

The magic square M_{15} has rank of 15. There are other assignments of the first row for a 15-by-15 centroskew S-circulant matrix A that do not force $\text{rank}(M) = 15$. For example, if the first row of a centroskew S-circulant matrix A is $[0, 1, 2, 5, 7, 4, 6, 3, -3, -6, -4, -7, -5, -2, -1]$ then the eigenvalue corresponding to $k = 3$ in (4) is zero. If we let the first row of a centroskew S-circulant matrix A be $[0, 3, 1, 2, 7, 4, 6, -5, 5, -6, -4, -7, -2, -1, -3]$ the eigenvalue corresponding to $k = 5$ in (4) is zero. Therefore, we leave as an open question whether the assignment of $a_j = j - 1$ for $1 \leqslant j \leqslant \binom{n+1}{2}$ always leads to getting nonsingular classical regular magic squares when n is a product involving two or more distinct primes.

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