Bounds for the Greatest Characteristic Root of an Irreducible Nonnegative Matrix

ALFRED BRAUER AND IVEY C. GENTRY
Wake Forest University
Winston-Salem, North Carolina

Recommended by Hans Schneider

ABSTRACT

A new lower bound for the Perron root for irreducible, non-negative matrices is obtained which is, in particular, a better bound than the Frobenius bound $[\omega = \max(a_{\kappa\kappa})]$ if all the main diagonal elements are zero.

Let $A = (a_{\kappa\lambda})$ be an irreducible nonnegative matrix of order n and $r_1 \leqslant r_2 \leqslant \cdots \leqslant r_n$ its row-sums. We set $r = r_1$ and $R = r_n$. The following results of Frobenius [3] are well known. The absolute greatest characteristic root ω, sometimes called the Perron root, satisfies

$$\omega \geqslant \max(a_{\kappa\kappa}), \qquad (\kappa = 1, 2, \ldots, n), \tag{1}$$

$$r \leqslant \omega \leqslant R. \tag{2}$$

For positive matrices, (2) was improved by W. Ledermann [4], further by A. Ostrowski [5], and by the first of the authors [2]. It was shown in [2] that the bounds there cannot be improved in general as functions of R, r, and m, the smallest element of A. But they all may reduce to (2) in the case of nonnegative matrices.

For stochastic matrices, (2) cannot be improved since $R = r$. It was an unsolved problem for years to find better bounds for ω than (2) and which never reduce to (2) for all irreducible nonnegative matrices except stochastic matrices. A. Ostrowski and H. Schneider [6] succeeded in obtaining such bounds. They proved that for any such given matrix it is possible to find a number $\eta > 0$ such that

$$r + \eta < \omega < R - \eta. \tag{3}$$

In this paper, another improvement of (2) will be obtained which will often be better than (3) but which will reduce to (2) in some cases. We set

$$\sum_{\substack{\lambda=1 \\ \lambda \neq \kappa}}^{n} a_{\kappa\lambda} \doteq P_\kappa \tag{4}$$

and

$$M(\kappa, \lambda) = \tfrac{1}{2}\{a_{\kappa\kappa} + a_{\lambda\lambda} + [(a_{\kappa\kappa} - a_{\lambda\lambda})^2 + 4P_\kappa P_\lambda]^{1/2}\}. \tag{5}$$

Then

$$M_1 = \min_{\kappa \neq \lambda} M(\kappa, \lambda) \leqslant \omega \leqslant \max_{\kappa \neq \lambda} M(\kappa, \lambda) = M_2. \tag{6}$$

In the special case that all main diagonal elements are zero, we obtain from (5) and (6) that

$$(r_1 r_2)^{1/2} \leqslant \omega \leqslant (r_{n-1} r_n)^{1/2}.$$

If we replace $a_{\kappa\lambda}$ by $|a_{\kappa\lambda}|$ in (4), then the right-hand part of (6) holds for the absolute value of all characteristic roots of all matrices with real or complex elements, hence also for ω. This was shown by the first of the authors a number of years ago [1]. Therefore, we have only to prove the other part here.

Let us assume that $x_1, x_2, \ldots, x_n$ is a characteristic vector with positive coordinates belonging to ω, and that x_i is the smallest and x_j the second smallest coordinate. We consider the ith and jth equations of the corresponding system of linear equations

$$(\omega - a_{ii})x_i = \sum_{\substack{\nu=1 \\ \nu \neq i}}^{n} a_{i\nu} x_\nu \geqslant x_j P_i,$$

$$(\omega - a_{jj})x_j = \sum_{\substack{\nu=1 \\ \nu \neq j}}^{n} a_{j\nu} x_\nu \geqslant x_i P_j.$$

Multiplying these equations we get

$$(\omega - a_{ii})(\omega - a_{jj}) \geqslant P_i P_j,$$

since $x_i x_j \neq 0$. Hence ω lies in the exterior or on the boundary of this oval and on the real axis. This oval may be doubly connected, but since $\omega \geqslant \max(a_{ii}, a_{jj})$ by (1), it cannot lie between a_{ii} and a_{jj}. It follows that ω is greater than or equal to the greatest vertex of this oval,

$$\omega \geqslant \tfrac{1}{2}\{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4P_iP_j]^{1/2}\} = M(i, j).$$

This proves (6).

We want to prove that (6) is better than (2). It is no restriction to assume that $r_i = a_{ii} + P_i \leqslant r_j = a_{jj} + P_j$. Then

$$M(i, j) \geqslant \tfrac{1}{2}\{a_{ii} + a_{jj} + [(a_{ii} - a_{jj})^2 + 4P_i(a_{ii} - a_{jj} + P_i)]^{1/2}\}$$

$$\geqslant \tfrac{1}{2}\{a_{ii} + a_{jj} + [(a_{ii} - a_{jj} + 2P_i)^2]^{1/2}\}.$$

Hence

$$M(i, j) \geqslant a_{ii} + P_i = \min(r_i, r_j) \quad \text{or} \quad M_1 \geqslant r.$$

A sequel to this paper, containing an improvement of one result in a special case, has appeared in *Linear Algebra and its Applications* **5**(1972), 311–318.

REFERENCES

1 A. Brauer, Limits for the characteristic roots of a matrix, II, *Duke Math. J.* **14**(1947), 21–26.

2 A. Brauer, Limits for the characteristic roots of a matrix, IV: Applications to stochastic matrices, *Duke Math. J.* **19**(1952), 75–91.

3 F. G. Frobenius, *Gesammelte Abhandlungen*, Band III, Springer-Verlag, Heidelberg (1968), pp. 404–414, pp. 546–567.

4 W. Ledermann, Bounds for the greatest latent root of a positive matrix, *J. London Math. Soc.* **27**(1952), 265–268.

5 A. Ostrowski, Bounds for the greatest latent root of a positive matrix, *J. London Math. Soc.* **27**(1952), 253–256.

6 A. Ostrowski and H. Schneider, Bounds for the maximal characteristic root of a non-negative matrix, *Duke Math. J.* **27**(1960), 547–553.

Received April, 1970