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Black hole spectra in holography: Consequences for equilibration of dual gauge theories

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Abstract

For a closed system to equilibrate from a given initial condition there must exist an equilibrium state with the energy equal to the initial one. Equilibrium states of a strongly coupled gauge theory with a gravitational holographic dual are represented by black holes. We study the spectrum of black holes in Pilch–Warner geometry. These black holes are holographically dual to equilibrium states of strongly coupled $SU(N)$ $\mathcal{N}=2^*$ gauge theory plasma on S^3 in the planar limit. We find that there is no energy gap in the black hole spectrum. Thus, there is a priori no obstruction for equilibration of arbitrary low-energy states in the theory via a small black hole gravitational collapse. The latter is contrasted with phenomenological examples of holography with dual four-dimensional CFTs having non-equal central charges in the stress–energy tensor trace anomaly.

1. Introduction and summary

Consider an interacting system in a finite volume. Suppose that the theory is gapless — there are arbitrary low-energy excitations. If a generic state in a theory equilibrates, there cannot be a gap in the spectrum of equilibrium states in the theory. This obvious statement has a profound im-

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plication for strongly coupled gauge theories with an asymptotically AdS gravitational dual [1]. In a holographic dual the equilibrium states are realized by black holes [2]. Thus, if it is possible to prepare an arbitrary low-energy initial configurations in a holographic dual with a gapped spectrum of black holes, such states of the boundary gauge theory will never equilibrate. Correspondingly, the asymptotically AdS dual is guaranteed to be stable against gravitational collapse for sufficiently small amplitude of the perturbations. Examples of this type would violate ergodicity from the field theory perspective.

In this paper we show that while it is possible to realize above scenario in a phenomenological (bottom-up) holographic example — the Einstein–Gauss–Bonnet (EGB) gravity with a negative cosmological constant, it does not occur in a specific model of gauge theory/supergravity correspondence we consider — the holographic duality between $\mathcal{N}=2^*$ $SU(N)$ gauge theory and the gravitational Pilch–Warner (PW) flow [3–5].

From the gauge theory perspective, $SU(N)$ $\mathcal{N}=2^*$ gauge theory is obtained from the parent $\mathcal{N}=4$ SYM by giving a mass to $\mathcal{N}=2$ hypermultiplet in the adjoint representation. In $R^{3,1}$ space–time, the low-energy effective action of the theory can be computed exactly [6]. The theory has quantum Coulomb branch vacua $\mathcal{M}_{\mathcal{C}}$, parameterized by the expectation values of the complex scalar Φ in the $\mathcal{N}=2$ vector multiplet, taking values in the Cartan subalgebra of the gauge group,

$$\Phi=\text{diag}(a_1,a_2,\cdots,a_N)\,,\qquad \sum_i a_i=0\,, \tag{1.1}$$

resulting in complex dimension of the moduli space

$$\dim_{\mathbb{C}}\mathcal{M}_{\mathcal{C}}\;=\;N-1\,. \tag{1.2}$$

In the large-N limit, and for strong ’t Hooft coupling, the holographic duality reduces to the correspondence between the gauge theory and type IIb supergravity. Since supergravities have finite number of light modes, one should not expect to see the full moduli space of vacua in $\mathcal{N}=2$ examples of gauge/gravity correspondence. This is indeed what is happening: the PW flow localizes on a semi-circle distribution of (1.1) with a linear number density [4],

$$\begin{aligned}
&\text{Im}(a_i)=0\,,\qquad a_i\in[-a_0,a_0]\,,\qquad a_0^2=\frac{m^2g_{YM}^2N}{4\pi^2}\,,\\
&\rho(a)=\frac{8\pi}{m^2g_{YM}^2}\sqrt{a_0^2-a^2}\,,\qquad \int_{-a_0}^{a_0}da\;\rho(a)=N\,,
\end{aligned} \tag{1.3}$$

where m is the hypermultiplet mass. This holographic localization can be deduced entirely from the field theory perspective [7], using the S^4-supersymmetric localization techniques [8]. To summarize, $\mathcal{N}=2^*$ holography is a well-understood nontrivial example of gauge/gravity correspondence that passes a number of highly nontrivial tests [4,7,9].

We would like to compactify the background space of the $\mathcal{N}=2^*$ strongly coupled gauge theory on S^3 of radius ℓ — in a dual gravitational picture we prescribe the boundary condition for the non-normalizable component of the metric in PW effective action to be that of $R\times S^3$. This is in addition to specifying non-normalizable components (corresponding to m in (1.3)) for the two PW scalars, dual to the mass deformation operators of dimensions $\Delta=2$ and $\Delta=3$ of the gauge theory hypermultiplet mass term. Thus, we produced a holographic example of a strongly interacting system in a finite volume. The single dimensionless parameter,[1] so far, is $m\ell$. We proceed to construct regular solutions of the PW effective gravitational action with the prescribes boundary condition, interpreting them as vacua of S^3-compactified strongly coupled $\mathcal{N}=2^*$ gauge theory. Using the standard holographic renormalization technique[2] we compute the vacuum energy of the theory as a function of $m\ell$, $E_{vacuum}=E_{vacuum}(m\ell)$. We do not verify in this work whether described S^3-compactifications preserve any supersymmetry; thus, it is important to check the stability of the vacuum solutions. Previously, careful analysis of the S^4-compactified PW holographic flows of [11] pointed to the discrepancy in the free energy of the solutions, compared with the localization prediction in [7]. This discrepancy was resolved by identifying a larger truncation [9] (BEFP),[3] where it was pointed out that preservation of the S^4-supersymmetry necessitates turning on additional bulk scalar fields. Stability of the PW embedding inside BEFP was discussed in [12]. We verify here that S^3-compactified PW vacua are stable within BEFP truncation. Having constructed vacuum solutions, we move to the discussion of the black hole spectrum. We construct regular Schwarzschild black hole solutions in PW effective action, and compute $\delta E\equiv\delta E(m\ell,\ell_{BH}/L)\equiv E-E_{vacuum}(m\ell)$. We argue that there is no obstruction of initializing arbitrary low-energy excitations over the vacuum. Thus, one would expect no gap in the energy spectrum of PW black hole solutions, realizing equilibrium configurations of the strongly coupled $\mathcal{N}=2^*$ gauge theory in the planar limit. Indeed, we find strong numerical evidence that

$$\lim_{\ell_{BH}/L\to 0}\ \frac{\delta E(m\ell,\ell_{BH}/L)}{E_{vacuum}(m\ell=0)}=0\,. \tag{1.4}$$

The rest of the paper is organized as follows. In the next section we discuss the spectrum of black holes in five-dimensional EGB gravity with a negative cosmological constant. These gravitational backgrounds can be interpreted as holographic duals to equilibrium states of strongly coupled conformal gauge theories with non-equal central charges in the stress–energy tensor trace anomaly. We show that there is a gap in the spectrum of black holes. However, as one imposes constraints on EGB gravity coming from interpreting it as an effective description of gauge theory/string theory correspondence, the claim about the gap becomes unreliable — higher derivative corrections, which are not under control, make order-one corrections to the gap. We follow up with the discussion in the $\mathcal{N}=2^*$ holographic example. In Section 3 we review the PW effective action and its embedding within a larger BEFP truncation. In Section 4 we construct gravitational dual to vacuum states of $\mathcal{N}=2^*$ gauge theory on S^3. Stability of the latter states within BEFP truncation is discussed in Section 5. In Section 6 we study the spectrum of black holes in PW effective action.

2. Black hole spectrum in Einstein–Gauss–Bonnet gravity

Effective action of a five-dimensional Einstein–Gauss–Bonnet gravity with a negative cosmological constant takes form:

[1] $\mathcal{N}=2^*$ theory in Minkowski space–time has a scale associated with the Coulomb branch moduli distribution (1.3). Once the theory is compactified on the S^3 the moduli space is lifted.

[2] For the model in hand this was developed in [10].

[3] Of course, BEFP can itself be consistently truncated to PW.

$$S = \frac{1}{2\ell_p^3} \int_{\mathcal{M}_5} d^5 z \sqrt{-g} \left(\frac{12}{L^2} + R + \frac{\lambda_{GB}}{2} L^2 \left(R^2 - 4 R_{\mu\nu} R^{\mu\nu} + R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right) \right) . \tag{2.1}$$

When interpreted in a framework of gauge theory/gravity correspondence,[4] EGB action (2.1) represents a holographic dual to a putative strongly coupled conformal theory with non-equal central charges, $c \neq a$, of the boundary stress–energy tensor,

$$\begin{aligned} \langle T^\mu{}_\mu \rangle_{\rm CFT} &= \frac{c}{16\pi^2} I_4 - \frac{a}{16\pi^2} E_4 \,, \\ E_4 &= r_{\mu\nu\rho\lambda} r^{\mu\nu\rho\lambda} - 4 r_{\mu\nu} r^{\mu\nu} + r^2 \,, \\ I_4 &= r_{\mu\nu\rho\lambda} r^{\mu\nu\rho\lambda} - 2 r_{\mu\nu} r^{\mu\nu} + \frac{1}{3} r^2 \,, \end{aligned} \tag{2.2}$$

where E_4 and I_4 are the four-dimensional Euler density and the square of the Weyl curvature of the CFT background space–time. The precise identification of the central charges is as follows:

$$\begin{aligned} c &= \frac{\pi^2 \tilde{L}^3}{\ell_p^3} \left(1 - 2 \frac{\lambda_{GB}}{\beta^2} \right) , \qquad a = \frac{\pi^2 \tilde{L}^3}{\ell_p^3} \left(1 - 6 \frac{\lambda_{GB}}{\beta^2} \right) , \\ \tilde{L} &\equiv \beta L \,, \qquad \beta^2 \equiv \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\lambda_{GB}} \,. \end{aligned} \tag{2.3}$$

The gravitational dual to the vacuum state of a CFT on a three-sphere S^3 is a global AdS_5,

$$ds^2 = \frac{L^2 \beta^2}{\cos^2 x} \left(-dt^2 + dx^2 + \sin^2 x \, d\Omega_3^2 \right) , \qquad x \in [0, \pi/2] \,, \tag{2.4}$$

where $d\Omega_3^2$ is the metric of S^3. Notice that λ_{GB} is restricted to be

$$\lambda_{GB} \le \frac{1}{4} \,; \tag{2.5}$$

otherwise, there is simply no asymptotic AdS solution. Following holographic renormalization of EGB gravity developed in [14,13], we find that the vacuum energy (the mass) of (2.4), or the Casimir energy from the boundary CFT perspective, is

$$E_{vacuum} = \frac{3a}{4\tilde{L}} \,. \tag{2.6}$$

Black holes (equilibrium configurations of EGB CFT) are found as a regular horizon solutions within the metric ansatz,

$$ds^2 = \frac{L^2 \beta^2}{\cos^2 x} \left(-A(x) dt^2 + \frac{dx^2}{A(x)} + \sin^2 x \, d\Omega_3^2 \right) . \tag{2.7}$$

The most general solution of equations of motion obtained from (2.1) determine $A(x)$ is terms of a single parameter $M > 0$,

$$\begin{aligned} A = 1 - \frac{1}{2\lambda_{GB}} \Bigg(&(2\lambda_{GB} - \beta^2) \sin^2 x + \Big(4\lambda_{GB} (\beta^2 - 2\lambda_{GB}) M \cos^4 x \\ &+ (2\lambda_{GB} - \beta^2)^2 \cos^4 x - \beta^4 (1 - 4\lambda_{GB}) \cos(2x) \Big)^{1/2} \Bigg) . \end{aligned} \tag{2.8}$$

4 See [13] for a recent review.

Furthermore, using the machinery of the holographic renormalization, the energy of the boundary CFT is

$$E=\frac{3c}{4L\beta}\left(\frac{\beta^2-6\lambda_{GB}}{\beta^2-2\lambda_{GB}}+4M\right)=\frac{3c}{4\tilde{L}}\left(\frac{a}{c}+4M\right). \tag{2.9}$$

It is remarkable that the regular Schwarzschild horizon in the geometry (2.7), (2.8) exists only provided [15,16]

$$M\ge\begin{cases}\frac{1-\beta^2}{2\beta^2-1}\,, & \text{if } \lambda_{GB}>0\,,\\ (\beta^2-1)(2\beta^2-1)\,, & \text{if } \lambda_{GB}<0\,.\end{cases} \tag{2.10}$$

For positive λ_{GB}, the bound comes requiring that S^3 remains finite at the location of the horizon (otherwise the curvature at the horizon diverges). For negative λ_{GB}, violating the bound would render geometry complex (expression inside the square root in (2.8) would turn negative for some $x\in(0,\pi/2)$).

Constraints (2.10) imply the gap in $\delta E\equiv E-E_{vacuum}$ in the spectrum of EGB black holes,

$$\frac{\delta E}{|E_{vacuum}|}\ \ge\ \epsilon_{gap}=\frac{4(1-\beta^2)}{|6\beta^2-5|}\times\begin{cases}1\,, & \lambda_{GB}>0\,,\\ -(2\beta^2-1)^2\,, & \lambda_{GB}<0\,,\end{cases} \tag{2.11}$$

with the only restriction (2.5) on λ_{GB}, ϵ_{gap} is unbounded as $\lambda_{GB}\to-\infty$ and $\lambda_{GB}\to 5/36$.

We argue now that attempts to interpret EGB holography as an effective description of some gauge theory/string theory correspondence make the gap claim (2.11) unreliable. First, causality of the holographic GB hydrodynamics requires that [17]

$$-\frac{7}{36}\le\lambda_{GB}\le\frac{9}{100}\qquad\Longrightarrow\qquad\epsilon_{gap}\le\begin{cases}1\,,\ \lambda_{GB}>0\,,\\ \frac{16}{27}\,,\ \lambda_{GB}<0\,.\end{cases} \tag{2.12}$$

Additionally, it was pointed out [18] that pure EGB gravity with a negative cosmological constant cannot arise as a low-energy limit of a gauge theory/string theory correspondence — the difference of central charges $(c-a)/c$ is bounded by Δ_{gap}^{-2}, where Δ_{gap} is the dimension of the lightest single particle operators with spin $J>2$ in the holographically dual conformal gauge theory. Integrating out massive $J>2$ spin states generically produces new higher-curvature contributions, in addition to the Gauss–Bonnet term. These higher curvature corrections are as important as the Einstein–Hilbert term and the GB term in (2.1) when the size of a black hole becomes of order $\lambda_{GB}L$. The latter is true even as $\lambda_{GB}\ll 1$, as the Ricci scalar evaluated on the horizon of $\sim\lambda_{GB}L$ size black hole (2.8) diverges as $\frac{1}{\lambda_{GB}}$.

3. PW/BEFP effective actions

We begin with description of the PW effective action [3]. The action of the effective five-dimensional supergravity including the scalars α and χ (dual to mass terms for the bosonic and fermionic components of the hypermultiplet respectively) is given by

$$\begin{aligned}S&=\int_{\mathcal{M}_5} d\xi^5\sqrt{-g}\,\mathcal{L}_{PW}\\ &=\frac{1}{4\pi G_5}\int_{\mathcal{M}_5} d\xi^5\sqrt{-g}\left[\tfrac{1}{4}R-3(\partial\alpha)^2-(\partial\chi)^2-\mathcal{P}\right],\end{aligned} \tag{3.1}$$

where the potential[5]

$$\mathcal{P}=\frac{1}{16}\left[\frac{1}{3}\left(\frac{\partial W}{\partial\alpha}\right)^2+\left(\frac{\partial W}{\partial\chi}\right)^2\right]-\frac{1}{3}W^2\,, \tag{3.2}$$

is a function of α and χ, and is determined by the superpotential

$$W=-e^{-2\alpha}-\frac{1}{2}e^{4\alpha}\cosh(2\chi)\,. \tag{3.3}$$

In our conventions, the five-dimensional Newton's constant is

$$G_5\equiv\frac{G_{10}}{2^5\,\mathrm{vol}_{S^5}}=\frac{4\pi}{N^2}\,. \tag{3.4}$$

Supersymmetric vacuum of $\mathcal{N}=2^*$ gauge theory in Minkowski space–time is given by

$$ds_5^2=e^{2A}\left(-dt^2+d\vec{x}^2\right)+dr^2\,,\qquad \rho=\rho(r)\equiv e^{\alpha(\rho)}\,,\qquad \chi=\chi(r)\,, \tag{3.5}$$

with

$$e^A=\frac{k\rho^2}{\sinh(2\chi)}\,,\qquad \rho^6=\cosh(2\chi)+\sinh^2(2\chi)\,\ln\frac{\sinh(\chi)}{\cosh(\chi)}\,,\qquad \frac{dA}{dr}=-\frac{1}{3}W\,, \tag{3.6}$$

where the single integration constant k is related to the hypermultiplet mass m according to [4]

$$k=mL=2m\,. \tag{3.7}$$

The BEFP effective action [9] is given by

$$\begin{aligned}S_{BEFP}&=\int_{\mathcal{M}_5} d\xi^5\sqrt{-g}\,\mathcal{L}_{BEFP}\\&=\frac{1}{4\pi G_5}\int_{\mathcal{M}_5} d\xi^5\sqrt{-g}\left[R-12\frac{(\partial\eta)^2}{\eta^2}-4\frac{(\partial\vec{X})^2}{(1-\vec{X}^2)^2}-\mathcal{V}\right],\end{aligned} \tag{3.8}$$

with the potential

$$\mathcal{V}=-\left[\frac{1}{\eta^4}+2\eta^2\,\frac{1+\vec{X}^2}{1-\vec{X}^2}-\eta^8\,\frac{(X_1)^2+(X_2)^2}{(1-\vec{X}^2)^2}\right], \tag{3.9}$$

where $\vec{X}=(X_1,X_2,X_3,X_4,X_5)$ are five of the scalars and η is the sixth. The symmetry of the action reflects the symmetries of the dual gauge theory: the two scalars (X_1,X_2) form a doublet under the $U(1)_R$ part of the gauge group, while (X_3,X_4,X_5) form a triplet under $SU(2)_V$ and η is neutral. The PW effective action is recovered as a consistent truncation of (3.8) with

$$X_2=X_3=X_4=X_5=0\,, \tag{3.10}$$

[5] We set the five-dimensional supergravity coupling to one. This corresponds to setting the radius L of the five-dimensional sphere in the undeformed metric to 2.

provided we identify the remaining BEFP scalars (η, X_1) with the PW scalars (α, χ) as follows

$$e^\alpha \equiv \eta\,, \qquad \cosh 2\chi = \frac{1+(X_1)^2}{1-(X_1)^2}\,. \tag{3.11}$$

Note that once $m \neq 0$ (correspondingly $X_1 \neq 0$), the $U(1)_R$ symmetry is explicitly broken; on the contrary, $SU(2)_V$ remains unbroken in truncation to PW.

4. Holographic duals to $\mathcal{N} = 2^*$ vacuum states on S^3

We derive bulk equations of motion and specify boundary conditions representing gravitational dual to vacuum states of strongly coupled $\mathcal{N} = 2^*$ gauge theory on S^3. We assume that the vacua are $SO(4)$-invariant. We argue that there is no obstruction of exciting these vacua by arbitrarily small perturbations of the bulk scalar fields α and χ. We review holographic renormalization of the theory and compute the vacuum energy. Next, we solve static gravitational equations perturbatively in the mass deformation parameter $m\ell \ll 1$ — this would serve as an independent check for the general $\mathcal{O}(m\ell)$ numerical solutions. We conclude with the plot representing $\epsilon \equiv E_{vacuum}(m\ell)/E^{\mathcal{N}=4}_{vacuum}$,

$$E^{\mathcal{N}=4}_{vacuum} \equiv E_{vacuum}(m\ell = 0) = \frac{3N^2}{16\ell}\,, \tag{4.1}$$

as a function of $m\ell$. Interestingly, while the vacuum energy of the $\mathcal{N} = 4$ SYM is positive, it is negative[6] for $\mathcal{N} = 2^*$ gauge theory once $m\ell \gtrsim 0.87$.

4.1. Equations of motion and the boundary conditions

We consider the general time-dependent $SO(4)$-invariant ansatz for the metric and the scalar fields:

$$ds_5^2 = \frac{4}{\cos^2 x}\left(-Ae^{-2\delta}(dt)^2 + \frac{(dx)^2}{A} + \sin^2 x (d\Omega_3)^2\right), \tag{4.2}$$

where $(d\Omega_3)^2$ is a metric on a unit[7] round S^3, and $\{A, \delta, \alpha, \chi\}$ being functions of a radial coordinate x and time t. Introducing

$$\Phi_\alpha \equiv \partial_x \alpha\,, \quad \Phi_\chi \equiv \partial_x \chi\,, \quad \Pi_\alpha \equiv \frac{e^\delta}{A}\partial_t \alpha\,, \quad \Pi_\chi \equiv \frac{e^\delta}{A}\partial_t \chi\,, \tag{4.3}$$

we obtain from (3.1) the following equations of motion:

- the evolution equations, $\dot{} = \partial_t$,

$$\begin{aligned}
&\dot\alpha = Ae^{-\delta}\Pi_\alpha\,, \qquad \dot\chi = Ae^{-\delta}\Pi_\chi\,,\\
&\dot\Phi_\alpha = \left(Ae^{-\delta}\Pi_\alpha\right)_{,x}\,, \qquad \dot\Phi_\chi = \left(Ae^{-\delta}\Pi_\chi\right)_{,x}\,,\\
&\dot\Pi_\alpha = \frac{1}{\tan^3 x}\left(\tan^3 x\, Ae^{-\delta}\Phi_\alpha\right)_{,x} - \frac{2}{3\cos^2 x}e^{-\delta}\frac{\partial \mathcal{P}}{\partial\alpha}\,,\\
&\dot\Pi_\chi = \frac{1}{\tan^3 x}\left(\tan^3 x\, Ae^{-\delta}\Phi_\chi\right)_{,x} - \frac{2}{\cos^2 x}e^{-\delta}\frac{\partial \mathcal{P}}{\partial\chi}\,,
\end{aligned} \tag{4.4}$$

[6] Prior to imposing causality constraints in EGB gravity, its vacuum energy becomes negative once $\lambda_{GB} > 5/36$. Vacuum energy of a different nonconformal gauge theory on S^3 was also observed to be negative in [19].

[7] We set $\ell = 1$; the ℓ dependence can be easily recovered from dimensional analysis.

- the spatial constraint equations,

$$
\begin{aligned}
A_{,x} = & \frac{2+2\sin^2 x}{\sin x\cos x}(1-A) - 2\sin(2x)A\left(\Phi_\alpha^2+\Pi_\alpha^2+\frac13\Phi_\chi^2+\frac13\Pi_\chi^2\right) \\
& -4\tan x\left(1+\frac43\mathcal{P}\right), \\
\delta_{,x} = & -2\sin(2x)\left(\Phi_\alpha^2+\Pi_\alpha^2+\frac13\Phi_\chi^2+\frac13\Pi_\chi^2\right),
\end{aligned} \tag{4.5}
$$

- and the moment constraint equation,

$$
A_{,t}+4\sin(2x)A^2e^{-\delta}\left(\Phi_\alpha\Pi_\alpha+\frac13\Phi_\chi\Pi_\chi\right)=0\,. \tag{4.6}
$$

It is straightforward to verify that the spatial derivative of (4.6) is implied by (4.4) and (4.5); thus is it sufficient to impose this equation at a single point. As $x\to 0_+$, the momentum constraint implies that $A(0,t)$ is a constant,[8] and as $x\to\frac{\pi}{2}{}_-$ the latter constraint is equivalent to the conservation of the boundary stress–energy tensor (see 4.2 for details).

The general non-singular solution of (4.4), (4.5) at the origin takes form

$$
\begin{aligned}
& A(t,x)=1+\mathcal{O}(x^2)\,, \qquad \delta(t,x)=d_0^h(t)+\mathcal{O}(x^2)\,, \\
& \alpha(t,x)=\alpha_0^h(t)+\mathcal{O}(x^2)\,, \qquad \chi(t,x)=\chi_0^h(t)+\mathcal{O}(x^2)\,.
\end{aligned} \tag{4.7}
$$

It is completely characterized by three time-dependent functions:

$$
\{d_0^h\,,\alpha_0^h\,,\chi_0^h\}\,. \tag{4.8}
$$

At the outer boundary $x=\frac{\pi}{2}$ we introduce $y\equiv\cos^2 x$ so that we have

$$
\begin{aligned}
A = & 1+y\,\frac23 c_{1,0}\;+y^2\left(a_{2,0}(t)+\left(\frac23 c_{1,0}(c_{1,0}+1)+8\rho_{1,1}^2+16\rho_{1,1}\rho_{1,0}(t)\right)\ln y\right. \\
& \left. +8\rho_{1,1}^2\ln^2 y\right)+\mathcal{O}(y^3\ln^3 y)\,, \\
\delta = & y\,\frac13 c_{1,0}+y^2\left(\frac12 c_{2,0}(t)-\frac1{36}c_{1,0}^2+4\rho_{1,0}^2(t)-\frac18 c_{1,0}+2\rho_{1,1}^2+4\rho_{1,0}(t)\rho_{1,1}\right. \\
& \left.+\left(\frac14 c_{1,0}+\frac13 c_{1,0}^2+4\rho_{1,1}^2+8\rho_{1,0}(t)\rho_{1,1}\right)\ln y+4\rho_{1,1}^2\ln^2 y\right)+\mathcal{O}(y^3\ln^3 y)\,, \\
e^\alpha = & 1+y\,\left(\rho_{1,0}(t)+\rho_{1,1}\ln y\right)+y^2\left(\frac1{12}c_{1,0}^2+\rho_{1,0}(t)-3\rho_{1,1}c_{1,0}+6\rho_{1,1}^2\right. \\
& -4\rho_{1,0}(t)\rho_{1,1}+\frac43 c_{1,0}\rho_{1,0}(t)+\frac32\rho_{1,0}^2(t)+\frac14\partial_{tt}^2\rho_{1,0}(t)+\left(\frac43\rho_{1,1}c_{1,0}+\rho_{1,1}\right. \\
& \left.\left.-4\rho_{1,1}^2+3\rho_{1,0}(t)\rho_{1,1}\right)\ln y+\frac32\rho_{1,1}^2\ln^2 y\right)+\mathcal{O}(y^3\ln^3 y)\,, \\
\cosh 2\chi = & 1+y\,c_{1,0}+y^2\left(c_{2,0}(t)+\left(\frac12 c_{1,0}+\frac23 c_{1,0}^2\right)\ln y\right)+\mathcal{O}(y^3\ln^2 y)\,,
\end{aligned} \tag{4.9}
$$

[8] In fact, the non-singularity of $A(t,x)$ in this limit automatically solves (4.6).

where we explicitly indicated time-dependence, *i.e.,*

$$\frac{d}{dt}c_{1,0}=0\,,\qquad \frac{d}{dt}\rho_{1,1}=0\,. \tag{4.10}$$

Asymptotic expansion (4.9) is completely characterized by two constants[9] $\{\rho_{1,1},c_{1,0}\}$ and three time-dependent functions

$$\{a_{2,0}\,,\rho_{1,0}\,,c_{2,0}\}\,, \tag{4.11}$$

constraint by (4.6) to satisfy

$$0=\frac{d}{dt}\left(a_{2,0}-8\rho_{1,0}^2(t)-16\rho_{1,0}(t)\rho_{1,1}-\frac{2}{3}c_{2,0}(t)\right). \tag{4.12}$$

The non-normalizable coefficients $\rho_{1,1}$ and $c_{1,0}$ are related to the mass deformation parameters of the dual gauge theory. Following [21], the precise relation can be established by matching the asymptotics (4.9) with the supersymmetric PW RG flow (3.6),

$$\{\rho_{1,1},c_{1,0}\}\bigg|_{PW}=k^2\left\{\frac{1}{48},\frac{1}{8}\right\}=m^2\left\{\frac{1}{12},\frac{1}{2}\right\}. \tag{4.13}$$

A specific relation between the non-normalizable coefficients of the bulk scalars e^{α} and $\cosh 2\chi$, *i.e.,*

$$c_{1,0}=6\rho_{1,1}\,, \tag{4.14}$$

realizes $\mathcal{N}=2$ supersymmetry of the boundary gauge theory in the UV. As in [21], it is possible to study the theory with explicitly broken supersymmetry, *i.e.,*

$$\rho_{1,1}\equiv\frac{1}{48}\,(m_bL)^2\qquad\neq\qquad\frac{1}{6}\,\times\,c_{1,0}\equiv\frac{1}{6}\,\times\,\frac{1}{8}\,(m_fL)^2\,, \tag{4.15}$$

where m_b and m_f are the masses of the bosonic and the fermionic components of the $\mathcal{N}=2$ hypermultiplet of the boundary gauge theory.

A non-equilibrium state of the gauge theory can be specified with the following initial/boundary conditions:

$$\begin{aligned}&\alpha(0,x)=\alpha^{init}(x)\,,\qquad \chi(0,x)=\chi^{init}(x)\,,\qquad \Phi_\alpha(0,x)=\Phi_\alpha^{init}=\frac{d\alpha^{init}}{dx}\,,\\ &\Phi_\chi(0,x)=\Phi_\chi^{init}=\frac{d\chi^{init}}{dx}\,,\qquad \Pi_\alpha(0,x)=\Pi_\alpha^{init}(x)\,,\qquad \Pi_\chi(0,x)=\Pi_\chi^{init}(x)\,,\end{aligned} \tag{4.16}$$

and as $y\equiv\cos^2 x\to 0$,

$$\begin{aligned}&\alpha^{init}(y)=\rho_{1,1}\;y\ln y+\mathcal{O}(y)\,,\qquad \cosh\left(2\chi^{init}(y)\right)=1+y\;c_{1,0}+\mathcal{O}(y^2\ln y)\,,\\ &\Pi_\alpha^{init}(y)=\mathcal{O}(y)\,,\qquad \Pi_\chi^{init}(y)=\mathcal{O}(y^{3/2})\,,\\ &A(0,x)=1+\frac{\cos^4 x}{\sin^2 x}\,\exp\Biggl(-\frac{2}{3}\int_0^x d\xi\;\sin(2\xi)\biggl(\left(\Pi_c^{init}(\xi)\right)^2+\left(\Phi_c^{init}(\xi)\right)^2\\ &\qquad\qquad +3\left(\Pi_\alpha^{init}(\xi)\right)^2+3\left(\Phi_\alpha^{init}(\xi)\right)^2\biggr)\Biggr)\;\times\;g(x)\,,\end{aligned} \tag{4.17}$$

9 Prescribing time dependence to these coefficients amounts to study quantum quenches in $\mathcal{N}=2^*$ gauge theory [20].

$$
\begin{aligned}
g(x) = -\frac{4}{3}\int_0^x d\xi \tan^3\xi \exp\Bigg(\frac{2}{3}\int_0^\xi d\eta\ \sin(2\eta)\Bigg(\left(\Pi_c^{init}(\eta)\right)^2 + \left(\Phi_c^{init}(\eta)\right)^2 \\
+ 3\left(\Pi_\alpha^{init}(\eta)\right)^2 + 3\left(\Phi_\alpha^{init}(\eta)\right)^2\Bigg)\Bigg) \times \Bigg(\frac{4\mathcal{P}^{init}(\xi)+3}{\cos^2\xi} + \left(\Pi_c^{init}(\xi)\right)^2 \\
+ \left(\Phi_c^{init}(\xi)\right)^2 + 3\left(\Pi_\alpha^{init}(\xi)\right)^2 + 3\left(\Phi_\alpha^{init}(\xi)\right)^2\Bigg) \\
\mathcal{P}^{init}(\xi) = \mathcal{P}\left(\alpha^{init}(\xi), \chi^{init}(\xi)\right), \qquad (4.18)
\end{aligned}
$$

$$
\delta(0,x) = -\frac{2}{3}\int_0^x d\xi\ \sin(2\xi)\Bigg(\left(\Pi_c^{init}(\xi)\right)^2 + \left(\Phi_c^{init}(\xi)\right)^2 + 3\left(\Pi_\alpha^{init}(\xi)\right)^2 + 3\left(\Phi_\alpha^{init}(\xi)\right)^2\Bigg), \qquad (4.19)
$$

where we explicitly solved for $A(0,x)$ and $\delta(0,x)$ using constraint equations (4.5). Notice that while $A(0,x)$ and $\delta(0,x)$ are free from the singularities given arbitrary profiles (4.16), a large amplitude initial conditions might cause $A(0,x)$ to vanish for some $0 < x_0 < \frac{\pi}{2}$, *i.e.,* $A(0,x_0) = 0$, — this corresponds to 'putting a black hole in the initial data'. Clearly, initial conditions arbitrarily small perturbed about static gravitational solutions without a horizon (see below) are well defined. In particular one can consider perturbations with

$$
\alpha^{init} = \alpha^v, \qquad \chi^{init} = \chi^v, \qquad \Pi^{init}_{\alpha,\chi} = \lambda\, \pi_{\alpha,\chi}(x), \qquad \lambda \to 0, \qquad (4.20)
$$

where the superscript ^{v} stands for a static (vacuum) solution and λ characterizes an overall amplitude of the perturbation with given initial profiles π_α and π_χ.

The $SO(4)$-invariant vacua of strongly coupled $\mathcal{N} = 2^*$ gauge theory correspond to static solutions of (4.4)–(4.6). To avoid unnecessary cluttering of the formulas, we omit the superscript ^{v}, use a radial coordinate $y \equiv \cos^2 x$, and introduce

$$
\begin{aligned}
&A(t,y) = a(y), \qquad \delta(t,y) = d(y), \qquad e^{\alpha(t,y)} = \rho(y), \\
&\cosh(2\chi(t,y)) = c(y). \qquad (4.21)
\end{aligned}
$$

We find then

$$
\begin{aligned}
0 &= c'' - \frac{c(c')^2}{c^2-1} + c'\left(\frac{a'}{a} - d'\right) - \frac{(y+1)c'}{y(1-y)} - \frac{\rho^2(c^2-1)(\rho^6 c - 4)}{4(1-y)y^2 a}, \\
0 &= \rho'' - \frac{(\rho')^2}{\rho} + \rho'\left(\frac{a'}{a} - d'\right) - \frac{(y+1)\rho'}{y(1-y)} - \frac{(c^2-1)\rho^9}{12(1-y)y^2 a} - \frac{1-\rho^6 c}{6\rho^3 y^2 a(1-y)}, \\
0 &= d' - \frac{2y(1-y)(c')^2}{3(c^2-1)} - \frac{8(1-y)y(\rho')^2}{\rho^2}, \\
0 &= a' - (y-y^2)a\left(\frac{8(\rho')^2}{\rho^2} + \frac{2(c')^2}{3(c^2-1)}\right) + \frac{(y-2)a+y}{y(1-y)} - \frac{(c^2-1)\rho^8 - 8\rho^2 c}{6y} \\
&\quad + \frac{2}{3y\rho^4}, \qquad (4.22)
\end{aligned}
$$

where $' = \frac{d}{dy}$. The boundary conditions as $y \to 0$ are as in (4.9), once we neglect the time dependence. At the origin, using $z \equiv 1 - y$ we have

$$
\begin{aligned}
a &= 1 + \left(-1 + \frac{1}{3(\rho_0^h)^4} - \frac{(\rho_0^h)^8}{12} \left((c_0^h)^2 - 1 \right) + \frac{2c_0^h (\rho_0^h)^2}{3} \right) z + \mathcal{O}(z^2)\,, \\
d &= d_0^h + \mathcal{O}(z^2)\,, \\
\rho &= \rho_0^h + \left(\frac{(\rho_0^h)^9}{24} \left((c_0^h)^2 - 1 \right) + \frac{1 - (\rho_0^h)^6 c_0^h}{12(\rho_0^h)^3} \right) z + \mathcal{O}(z^2)\,, \\
c &= c_0^h + \frac{1}{8} (\rho_0^h)^2 \left((c_0^h)^2 - 1 \right) \left(c_0^h (\rho_0^h)^6 - 4 \right) z + \mathcal{O}(z^2)\,.
\end{aligned}
\tag{4.23}
$$

We consider geometries with $\mathcal{N} = 2$ supersymmetry in the ultraviolet, so we impose the constraint (4.13). Having fixed m, the complete set of normalizable coefficients in the UV/IR is given by:

$$
\{a_{2,0}\,,\ \rho_{1,0}\,,\ c_{2,0}\,,\ \rho_0^h\,,\ c_0^h\,,\ d_0^h\}\,. \tag{4.24}
$$

Note that the six integration constants (4.24) is exactly what is needed to uniquely fix a solution of a coupled system of two second-order and two first-order ODEs.

4.2. Holographic renormalization and the vacuum energy

Holographic renormalization of RG flows in PW geometry was discussed in [10]. Here we apply the analysis for the gravitational solutions dual to vacua of $\mathcal{N} = 2^*$ gauge theory on S^3.

The gravitational action (3.1) evaluated on a static solution (4.22) diverges — this divergence is a gravitational reflection of a standard UV divergence of the free energy in the interacting boundary gauge theory. It is regulated by cutting off the radial coordinate integration at $y = y_c \ll 1$. It is straightforward to verify that the regularized Euclidean gravitational Lagrangian, $\mathcal{L}^E_{reg}$, is a total derivative,

$$
\begin{aligned}
\mathcal{L}^E_{reg} &= \frac{1}{4\pi G_5} \mathrm{vol}(\Omega_3) \int_1^{y_c} dy\, \frac{d}{dy} \left(\frac{4(1-y)^2 e^{-d}}{y^2} \left(a + 2yad' - ya' \right) \right) \\
&= \frac{\mathrm{vol}(\Omega_3)}{4\pi G_5} \left[\frac{4(1-y)^2 e^{-d}}{y^2} \left(a + 2yad' - ya' \right) \right] \Bigg|^{y_c}\,,
\end{aligned}
\tag{4.25}
$$

where in the second equality, using (4.23), we observe that the only contribution comes from the upper limit of integration. Regularized Lagrangian (4.25) has to be supplemented with contributions coming from the familiar Gibbons–Hawking term, $\mathcal{L}^E_{GH}$,

$$
\begin{aligned}
S^E_{GH} &= -\frac{1}{8\pi G_5} \int_{\partial \mathcal{M}_5} d\xi^4 \sqrt{h_E} \nabla_\mu n^\mu \equiv \int dt_E \mathcal{L}^E_{GH}\,, \\
\mathcal{L}^E_{GH} &= \frac{\mathrm{vol}(\Omega_3)}{4\pi G_5} \left[\frac{4(1-y)e^{-d}}{y^2} \left(a(y-4) - 2d'y(1-y)a + a'y(1-y) \right) \right] \Bigg|^{y_c}\,,
\end{aligned}
\tag{4.26}
$$

and the counterterm Lagrangian,[10] $\mathcal{L}^E_{counter}$,

[10] We keep only the counterterms relevant for the $R \times S^3$ background geometry of the gauge theory.

$$
\begin{aligned}
S^E_{counter} &\equiv \int dt_E \mathcal{L}^E_{counter}\,,\\
\mathcal{L}^E_{counter} &= \frac{\text{vol}\Omega_3}{4\pi G_5}\sqrt{h_E}\biggl[\frac34+\frac14 R_4+\frac12\chi^2+3\alpha^2-\frac32\frac{\alpha^2}{\ln\epsilon_c}\\
&\qquad +\ln\epsilon_c\left(-\frac13\chi^2R_4-\frac23\chi^4\right)+\frac16\chi^4\biggr]\biggg|^{y_c}\,,
\end{aligned}
\tag{4.27}
$$

where $R_4 \equiv R_4(h_E)$ is the Ricci scalar constructed from h_E, and ϵ_c parameterizes conformal anomaly terms in terms of the $g_{t_Et_E}$ metric component,

$$
R_4 = \frac{3y}{2(1-y)}\,, \qquad \epsilon_c \equiv \sqrt{g_{t_Et_E}} = \frac{2\sqrt{a}e^{-d}}{\sqrt{y}}\,. \tag{4.28}
$$

The renormalized Lagrangian $\mathcal{L}^E_{renom}$, finite in the limit $y_c \to 0$, is identified with the free energy $\mathcal{F}$ of the boundary gauge theory,

$$
\begin{aligned}
\mathcal{F} &= \mathcal{L}^E_{renom} = \lim_{y_c\to 0}\left(\mathcal{L}^E_{reg}+\mathcal{L}^E_{GH}+\mathcal{L}^E_{counter}\right)\\
&= \frac{\text{vol}\Omega_3}{4\pi G_5}\,\frac32\,\biggl(1+c_{1,0}^2\left(\frac49-\frac{16}{9}\ln 2\right)+c_{1,0}\left(-\frac43-\frac83\ln 2\right)+64\rho_{1,1}^2\ln 2\\
&\qquad +\left\{64\rho_{1,1}\rho_{1,0}+\frac83 c_{2,0}+32\rho_{1,0}^2-4a_{2,0}\right\}\biggr)\\
&= \frac{3N^2}{16\ell}\biggl(1+\frac{(m\ell)^4}{9}-\frac23(1+2\ln 2)(m\ell)^2\\
&\qquad +\left\{32\rho_{1,0}^2+\frac{16}{3}(m\ell)^2\rho_{1,0}+\frac83 c_{2,0}-4a_{2,0}\right\}\biggr)\,,
\end{aligned}
\tag{4.29}
$$

where in the second line we used the asymptotic expansion (4.9) and expressed the last line in terms of gauge theory variables using (3.4) and (4.13) and restoring the size ℓ of the S^3. Several comments are in order:

- For static gravitational solutions without Schwarzschild horizon (as discussed here), the free energy $\mathcal{F}$ must coincide with the energy E of the boundary stress–energy tensor. We explicitly verified that, indeed,

$$
\mathcal{F} = E \equiv E_{vacuum}(m\ell)\,. \tag{4.30}
$$

The latter is identified with the vacuum energy of $\mathcal{N}=2^*$ gauge theory on S^3.

- In a limit when all the (non-)normalizable coefficients vanish we recover the vacuum energy of the $\mathcal{N}=4$ SYM (4.1).
- It is easy to extend discussion for general $SO(4)$-invariant non-equilibrium states of $\mathcal{N}=2^*$ gauge theory — the final answer is as (4.29), except with $\{\rho_{1,0}\,, c_{2,0}\,, a_{2,0}\}$ now being functions of time. Note that

$$
\frac{d\mathcal{E}}{dt} \propto \frac{d}{dt}\left(4\left\{16\rho_{1,1}\rho_{1,0}(t)+\frac23 c_{2,0}(t)+8\rho_{1,0}^2(t)-a_{2,0}(t)\right\}\right) = 0\,, \tag{4.31}
$$

according to (4.12). That is, the boundary gauge theory energy conservation is enforced by the bulk momentum constraint (4.6).

4.3. Vacuum states for $m\ell \ll 1$

In preparation to the full numerical solution of (4.22), we discuss here its perturbative solution for $\rho_{1,1} \ll 1$. We introduce

$$
\begin{aligned}
&c = \cosh(2\lambda\chi_1(y) + \mathcal{O}(\lambda^3))\,, \qquad \rho = e^{\lambda^2\alpha_2(y)+\mathcal{O}(\lambda^4)}\,,\\
&a = 1 + \lambda^2 a_2(y) + \mathcal{O}(\lambda^4)\,, \qquad d = \lambda^2 d_2(y) + \mathcal{O}(\lambda^2)\,,
\end{aligned}
\tag{4.32}
$$

where λ is a small parameter. Substituting (4.32) into (4.22) we find

$$
\begin{aligned}
&0 = \chi_1'' - \frac{1+y}{y(1-y)}\chi_1' + \frac{3}{4y^2(1-y)}\chi_1\,,\\
&0 = \alpha_2'' - \frac{1+y}{y(1-y)}\alpha_2' + \frac{1}{y^2(1-y)}\alpha_2\,,\\
&0 = a_2' - \frac{2-y}{y(1-y)}a_2 - \frac{8}{3}y(1-y)(\chi_1')^2 + \frac{2}{y}(\chi_1)^2\,,\\
&0 = d_2' - \frac{8}{3}y(1-y)(\chi_1')^2\,.
\end{aligned}
\tag{4.33}
$$

Solutions to (4.33) must satisfy boundary conditions corresponding to (4.9) and (4.23). We can solve equation for α_2 analytically,

$$
\alpha_2 = \rho_{1,1,(2)}\,\frac{y\ln y}{1-y}\,,
\tag{4.34}
$$

where $\rho_{1,1,(2)}$ is the non-normalizable integration coefficient. The remaining equations in (4.33) are solved with "shooting method" developed in [22]. In particular, given the asymptotic expansions in the UV, $y \to 0_+$,

$$
\begin{aligned}
&\chi_1 = y^{1/2}\left(1 + y\left(\chi_{1,0,(1)} + \frac{1}{4}\ln y\right) + \mathcal{O}(y^2\ln y)\right)\,,\\
&a_2 = \frac{4}{3}y + y^2\left(a_{2,0,(2)} + \frac{4}{3}\ln y\right) + \mathcal{O}(y^3\ln^2 y)\,,\\
&d_2 = \frac{2}{3}y + y^2\left(-\frac{1}{4} + 2\chi_{1,0,(1)} + \frac{1}{2}\ln y\right) + \mathcal{O}(y^3\ln^2 y)\,,
\end{aligned}
\tag{4.35}
$$

and in the IR, $z \to 0_+$,

$$
\begin{aligned}
&\chi_1 = \chi^h_{0,(1)}\left(1 - \frac{3}{8}z + \mathcal{O}(z^2)\right)\,,\\
&a_2 = (\chi^h_{0,(1)})^2\left(z - \frac{5}{8}z^2 + \mathcal{O}(z^3)\right)\,,\\
&d_2 = d^h_{0,(2)} - \frac{3}{16}(\chi^h_{0,(1)})^2 z^2 + \mathcal{O}(z^3)\,,
\end{aligned}
\tag{4.36}
$$

we find numerically,

$\chi_{1,0,(1)}$	$a_{2,0,(2)}$	$\chi^h_{0,(1)}$	$d^h_{0,(2)}$
0.0568528	−0.363452	0.785398	0.199266

. (4.37)

To compare with the full numerical solution, we identify, to order $\mathcal{O}(\lambda^2)$,

$$
\begin{aligned}
&\rho_{1,1}=\rho_{1,1,(2)}\lambda^2\,, \qquad c_{1,0}=2\lambda^2\,, \qquad \rho_{1,0}=0\,, \qquad c_{2,0}=4\chi_{1,0,(1)}\lambda^2\,,\\
&a_{2,0}=a_{2,0,(2)}\lambda^2\,, \qquad \rho_0^h=1-\rho_{1,1,(2)}\lambda^2\,, \qquad c_0^h=1+2(\chi^h_{0,(1)})^2\lambda^2\,,\\
&d_0^h=d^h_{0,(2)}\lambda^2\,.
\end{aligned}
\tag{4.38}
$$

Note that $\mathcal{N}=2$ supersymmetry in the UV at $\mathcal{O}(\lambda^2)$ leads to (see (4.14))

$$
\rho_{1,1,(2)}=\frac{1}{3}\,. \tag{4.39}
$$

From (4.29),

$$
\begin{aligned}
\epsilon\equiv\frac{E_{vacuum}}{E^{\mathcal{N}=4}_{vacuum}}&=1+\left(\frac{32}{3}\chi_{1,0,(1)}-4a_{2,0,(2)}-\frac{8}{3}(1+2\ln 2)\right)\lambda^2+\mathcal{O}(\lambda^4)\\
&=1+\left(\frac{8}{3}\chi_{1,0,(1)}-a_{2,0,(2)}-\frac{2}{3}(1+2\ln 2)\right)(m\ell)^2+\mathcal{O}((m\ell)^4)\,.
\end{aligned}
\tag{4.40}
$$

4.4. Gravitational solution and E_{vacuum} for general $m\ell$

Using the shooting method of [22], we solve (4.22) and determine the normalizable coefficients (4.24) as a function of $m\ell\equiv(12\rho_{1,1})^{1/2}$. The results of the computations for small values of $\rho_{1,1}$ are collected for numerical test in Fig. 1. The solid curves are obtained from numerical solution of full nonlinear equations (4.22), and the dashed lines represent perturbative prediction (4.38) with (4.37).

In full nonlinear numerical analysis we constructed vacua for $0<m\ell\lesssim 8.5$. The vacuum energy of the $\mathcal{N}=2^*$ gauge theory on S^3 relative to $\mathcal{N}=4$ SYM Casimir energy is given by

$$
\begin{aligned}
\epsilon\equiv\frac{E_{vacuum(m\ell)}}{E^{\mathcal{N}=4}_{vacuum}}&=1+\frac{(m\ell)^4}{9}-\frac{2}{3}(1+2\ln 2)(m\ell)^2]\\
&\qquad+\left\{32\rho_{1,0}^2+\frac{16}{3}(m\ell)^2\rho_{1,0}+\frac{8}{3}c_{2,0}-4a_{2,0}\right\}.
\end{aligned}
\tag{4.41}
$$

It is presented in Fig. 2. The vertical red line indicates the mass scale $m_0\ell$,

$$
\epsilon(m_0\ell)=0 \qquad\Longrightarrow\qquad m_0\ell\approx 0.87031\,, \tag{4.42}
$$

at which the vacuum energy of the $\mathcal{N}=2^*$ gauge theory vanishes and becomes negative for even larger value of $m\ell$.

5. Stability of $\mathcal{N}=2^*$ vacuum states within BEFP

In the previous section we constructed gravitational solutions within PW effective action, identified as vacua of the $\mathcal{N}=2^*$ gauge theory on S^3. While the complete stability analysis of these solutions is beyond the scope of this paper, here we would like to analyze their stability within BEFP effective action.

Effective action describing the fluctuations of an arbitrary PW static solution within BEFP has been constructed in [12],

$$
\begin{aligned}
&\delta\mathcal{L}\equiv\mathcal{L}_{BEFP}-\mathcal{L}_{PW}+\mathcal{O}(X_i^4)\equiv\delta\mathcal{L}_2+\delta\mathcal{L}_V\,,\\
&\delta\mathcal{L}_2=-(1+c)^2(\partial X_2)^2-\frac{1+c}{4}\left((c^2+c)\rho_6^{4/3}-4(1+c)\rho_6^{1/3}+\frac{4(\partial c)^2}{c^2-1}\right)(X_2)^2\,,
\end{aligned}
$$

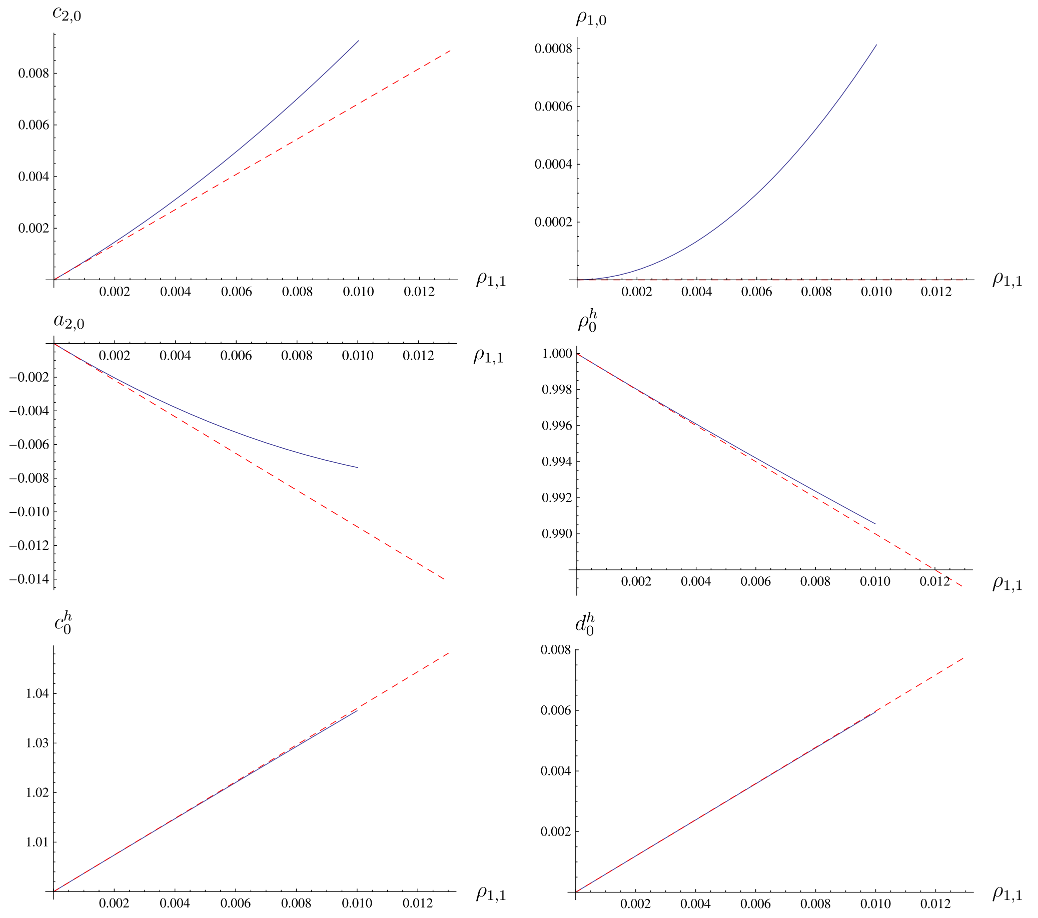

Fig. 1. Normalizable coefficients (4.24) as functions of $\rho_{1,1}$. The dashed lines represent perturbative predictions (4.38) with (4.37).

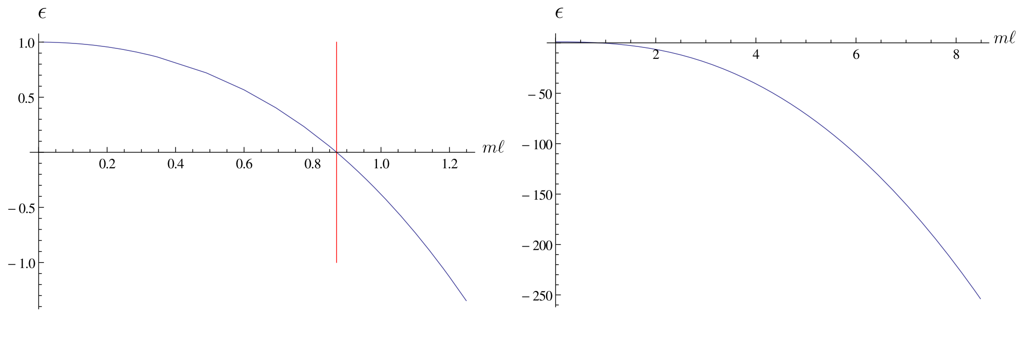

Fig. 2. Vacuum energy of the $\mathcal{N} = 2^*$ gauge theory on S^3 relative to $\mathcal{N} = 4$ SYM Casimir energy, see (4.41). The vertical red line marks vanishing of ϵ, see (4.42). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\delta\mathcal{L}_V = -(1+c)^2(\partial\vec{X}_V)^2 - \frac{1+c}{4}\left((c^2-1)\rho_6^{4/3} - 4(1+c)\rho_6^{1/3} + \frac{4(\partial c)^2}{c^2-1}\right)(\vec{X}_V)^2\,, \tag{5.1}$$

where $\rho_6=\rho^6$ and $\vec{X}_V=(X_3,X_4,X_5)$ (see Section 3 for more details). Note that $\delta\mathcal{L}$ is $SU(2)_V$ invariant; as a result it is enough to consider a spectrum of only one of $\vec{X}_V$ components. In what follows we choose the latter to be X_3.

Introducing

$$X_2 = e^{-i\omega t}F_2(y)\Omega_s(S^3)\,,\qquad X_3(t,y) = e^{-i\omega t}F_3(y)\Omega_s(S^3)\,, \tag{5.2}$$

where $\Omega_s(S^3)$ are S^3 Laplace–Beltrami operator eigenfunctions with eigenvalues $s=l(l+2)$ for integer l,

$$\Delta_{S^3}\,\Omega_s(S^3) = -s\,\Omega_s(S^3) = -l(l+2)\,\Omega_s(S^3)\,, \tag{5.3}$$

we find from (5.1) the following equations of motion

$$\begin{aligned}
0 = {} & F_2'' + F_2'\left(\frac{2cc'}{c+1} + \frac{(c^2-1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{2y-1}{y(y-1)} + \frac{1}{a(y-1)} - \frac{2}{3a\rho^4 y}\right) \\
& + \frac{F_2}{4y(1-y)a}\left(\frac{e^{2d}\omega^2}{a} - \frac{s}{1-y}\right) + F_2\left(\frac{(c')^2}{(1-c^2)(c+1)} + \frac{\rho^2(\rho^6 c-4)}{4ay^2(y-1)}\right),
\end{aligned} \tag{5.4}$$

$$\begin{aligned}
0 = {} & F_3'' + F_3'\left(\frac{2cc'}{c+1} + \frac{(c^2-1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{2y-1}{y(y-1)} + \frac{1}{a(y-1)} - \frac{2}{3a\rho^4 y}\right) \\
& + \frac{F_3}{4y(1-y)a}\left(\frac{e^{2d}\omega^2}{a} - \frac{s}{1-y}\right) + F_3\left(\frac{(c')^2}{(1-c^2)(c+1)} + \frac{\rho^2(\rho^6(c-1)-4)}{4ay^2(y-1)}\right).
\end{aligned} \tag{5.5}$$

The radial wavefunctions $F_{2,3}$ must be regular at the origin, *i.e.,* $z\to 0_+$,

$$F_2 = z^{l/2}\, f_2^h\,(1+\mathcal{O}(z))\,,\qquad F_3 = z^{l/2}\, f_3^h\,(1+\mathcal{O}(z))\,, \tag{5.6}$$

and normalizable as $y\to 0_+$,

$$\begin{aligned}
F_2 &= y^{3/2}\left(1 + y\left(\frac{s}{8} - \frac{1}{2}c_{1,0} + \frac{9-\omega^2}{8}\right) + \mathcal{O}(y^2\ln y)\right), \\
F_3 &= y\left(1 + y\left(\frac{s}{4} + \frac{4-\omega^2}{4} + 4\rho_{1,1} - 2\rho_{1,0} - \frac{1}{6}c_{1,0} - 2\rho_{1,1}\ln y\right) + \mathcal{O}(y^2\ln y)\right).
\end{aligned} \tag{5.7}$$

Note that we set the normalizable coefficient of $F_{2,3}$ in the UV to one.

When both scalars of the PW flow are set to zero, (5.4)–(5.7) corresponds to fluctuations of gravitational modes dual to dimension-3 (for F_2) and dimension-2 (for F_3) operators of the $\mathcal{N}=4$ SYM on S^3. In this case the equations can be solved analytically. We find,

$$\begin{aligned}
F^{SYM}_{2,\{n,l\}} &= y^{3/2}(1-y)^{l/2}{}_2F_1\Big(-n\,,3+n+l\,;l+2\,;1-y\Big), \\
\omega^{SYM}_{2,\{n,l\}} &= 3+2n+l\,,
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
F^{SYM}_{3,\{n,l\}} &= y(1-y)^{l/2}{}_2F_1\Big(-n\,,2+n+l\,;l+2\,;1-y\Big), \\
\omega^{SYM}_{3,\{n,l\}} &= 2+2n+l\,,
\end{aligned} \tag{5.9}$$

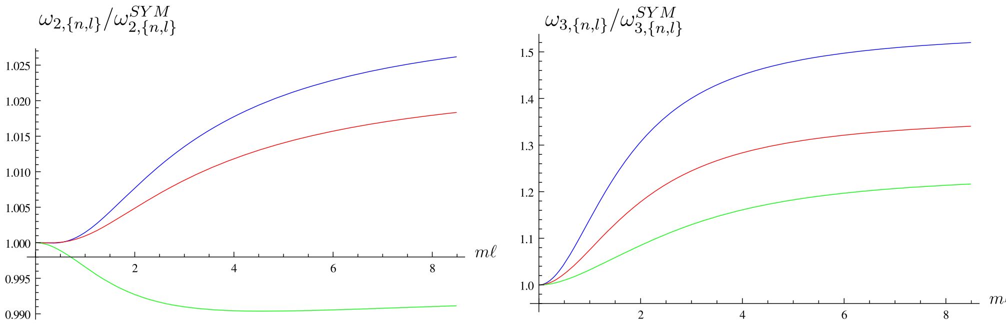

Fig. 3. Low energy states in the spectrum of BEFP fluctuations about PW vacua: $\{n,l\} = \{(0,0)\,;\ (0,1)\,;\ (1,0)\}$ (blue, red, green). See Section 5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

where $\{n,l\}$ are non-negative integers. For supersymmetric PW flows (4.14) we have to resort to numerics. The results of the numerical analysis are presented in Fig. 3. We look at the states with $\{n,l\} = \{(0,0)\,;\ (0,1)\,;\ (1,0)\}$ for both F_2 and F_3 radial functions. Over the range of parameters discussed, the embedding of PW flows within BEFP effective action is stable.

6. Black hole spectrum in PW effective action

We begin with the metric ansatz and the boundary conditions representing regular Schwarzschild black hole solutions in PW effective action with the S^3 horizon. We explain how the normalizable coefficients of the gravitational solution encode the thermodynamic properties of the black holes: the temperature T_{BH}, the energy E_{BH}, the entropy S_{BH} and the free energy $\mathcal{F}_{BH}$. We define the size ℓ_{BH} of a black hole as

$$\left(\frac{\ell_{BH}}{L}\right)^3 \equiv \frac{A_{horizon}}{L^3}\,. \tag{6.1}$$

We compute excitation energy $\Delta(\ell_{BH}/L\,,(m\ell))$,

$$\Delta(\ell_{BH}/L\,,(m\ell)) = \frac{E_{BH}(\ell_{BH}/L\,,m\ell) - E_{vacuum}(m\ell)}{E^{\mathcal{N}=4}_{vacuum}}\,, \tag{6.2}$$

as a function of ℓ_{BH}/L, but for select values of $m\ell$:

- perturbatively in $m\ell$, to order $\mathcal{O}((m\ell)^2)$;
- for $\rho_{1,1} = \frac{1}{12}(m\ell)^2 = \{1, 1.5, 2, \cdots 5, 5.5, 5.8\}$ (the last value corresponds to the largest value of $m\ell$ for which we computed E_{vacuum});

and present a strong numerical evidence that

$$\lim_{\ell_{BH}/L\to 0} \Delta(\ell_{BH}/L\,,(m\ell)) = 0\,. \tag{6.3}$$

Thus, we conclude that there is no gap in the spectrum of black holes in PW geometry; correspondingly, there is no gap in $SO(4)$-invariant equilibrium states of the $\mathcal{N} = 2^*$ gauge theory on S^3 in the planar limit and for large 't Hooft coupling, as there is no energy gap for generic $SO(4)$-invariant excitations in this theory.

6.1. Metric ansatz and the boundary conditions for black holes in PW

Recall that the vacuum solutions of Section 4 were obtained within metric ansatz (4.2),

$$
\begin{aligned}
ds_5^2\Big|_{vacuum} &= \frac{4}{\cos^2 x}\left(-ae^{-2d}(dt)^2 + \frac{(dx)^2}{a} + \sin^2 x (d\Omega_3)^2\right) \\
&= \frac{4}{y}\left(-ae^{-2d}(dt)^2 + \frac{(dy)^2}{4y(1-y)a} + (1-y)(d\Omega_3)^2\right),
\end{aligned}
\tag{6.4}
$$

where in the second line we recalled the radial coordinate $y = \cos^2 x$, $y \in [0, 1]$. Regularity at the origin ($y \to 1_-$) required that the metric functions a and d remain finite and non-zero. Notice that the three-sphere shrinks to zero size in this limit.

In close analogy to (6.4), to describe regular horizon black holes, we reparameterize the radial coordinate $y \to y_h y$, with a constant $0 < y_h < 1$, while keeping $y \in [0, 1]$. We further require that a has a simple zero and d remains finite as $y \to 1_-$:

$$
\begin{aligned}
&ds_5^2\Big|_{BH} = \frac{4}{y_h y}\left(-ae^{-2d}(dt)^2 + \frac{y_h(dy)^2}{4y(1-yy_h)a} + (1-yy_h)(d\Omega_3)^2\right), \\
&0 < y_h < 1, \qquad y \in [0, 1], \qquad \lim_{y\to 1_-} a = 0, \\
&\lim_{y\to 1_-} a' = \text{finite} \neq 0, \qquad \lim_{y\to 1_-} d = \text{finite}.
\end{aligned}
\tag{6.5}
$$

Given (6.5),

$$
A_{horizon} = 16\pi^2 \frac{(1-y_h)^{3/2}}{y_h^{3/2}} \qquad \Longrightarrow \qquad \frac{\ell_{BH}}{L} \equiv \frac{A_{horizon}^{1/3}}{L} = (2\pi^2)^{1/3}\frac{(1-y_h)^{1/2}}{y_h^{1/2}}. \tag{6.6}
$$

The equations of motion describing black holes (6.5) can be obtained from (4.22) with the simple change of variables[11] $y \to yy_h$,

$$
\begin{aligned}
0 &= c'' - \frac{c(c')^2}{c^2-1} + c'\left(\frac{(c^2-1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{a(2yy_h-1)+yy_h}{ya(yy_h-1)} - \frac{2}{3ya\rho^4}\right) \\
&\quad - \frac{\rho^2(c^2-1)(\rho^6 c-4)}{4(1-yy_h)y^2 a}, \\
0 &= \rho'' - \frac{(\rho')^2}{\rho} + \rho'\left(\frac{(c^2-1)\rho^8}{6ay} - \frac{4c\rho^2}{3ay} + \frac{a(2yy_h-1)+yy_h}{ya(yy_h-1)} - \frac{2}{3ya\rho^4}\right) \\
&\quad - \frac{(c^2-1)\rho^9}{12(1-yy_h)y^2 a} - \frac{1-\rho^6 c}{6\rho^3 y^2 a(1-yy_h)}, \\
0 &= d' - \frac{2y(1-yy_h)(c')^2}{3(c^2-1)} - \frac{8(1-yy_h)y(\rho')^2}{\rho^2}, \\
0 &= a' - (y-y^2y_h)a\left(\frac{8(\rho')^2}{\rho^2} + \frac{2(c')^2}{3(c^2-1)}\right) + \frac{(yy_h-2)a+yy_h}{y(1-yy_h)} - \frac{(c^2-1)\rho^8-8\rho^2 c}{6y} \\
&\quad + \frac{2}{3y\rho^4}.
\end{aligned}
\tag{6.7}
$$

The boundary conditions in the UV, *i.e.,* $y \to 0_+$, specify the asymptotic expansion

[11] We used the last two equations to algebraically eliminate a' and d' from the first two.

$$
\begin{aligned}
a &= 1 + y\,\frac{2}{3}\hat{c}_{1,0} + y^2\left(\hat{a}_{2,0} + \left(\frac{2}{3}\hat{c}_{1,0}(\hat{c}_{1,0}+y_h) + 8\hat{\rho}_{1,1}^2 + 16\hat{\rho}_{1,1}\hat{\rho}_{1,0}\right)\ln y\right.\\
&\quad \left. + 8\hat{\rho}_{1,1}^2\ln^2 y\right) + \mathcal{O}(y^3\ln^3 y)\,,\\
d &= y\,\frac{1}{3}\hat{c}_{1,0} + y^2\left(\frac{1}{2}\hat{c}_{2,0} - \frac{1}{36}\hat{c}_{1,0}^2 + 4\hat{\rho}_{1,0}^2 - \frac{1}{8}\hat{c}_{1,0}y_h + 2\hat{\rho}_{1,1}^2 + 4\hat{\rho}_{1,0}\hat{\rho}_{1,1}\right.\\
&\quad \left. + \left(\frac{1}{4}\hat{c}_{1,0}y_h + \frac{1}{3}\hat{c}_{1,0}^2 + 4\hat{\rho}_{1,1}^2 + 8\hat{\rho}_{1,0}\hat{\rho}_{1,1}\right)\ln y + 4\hat{\rho}_{1,1}^2\ln^2 y\right) + \mathcal{O}(y^3\ln^3 y)\,,\\
\rho &= 1 + y\,\left(\hat{\rho}_{1,0} + \hat{\rho}_{1,1}\ln y\right) + y^2\left(\frac{1}{12}\hat{c}_{1,0}^2 + \hat{\rho}_{1,0}y_h - 3\hat{\rho}_{1,1}\hat{c}_{1,0} + 6\hat{\rho}_{1,1}^2\right.\\
&\quad - 4\hat{\rho}_{1,0}\hat{\rho}_{1,1} + \frac{4}{3}\hat{c}_{1,0}\hat{\rho}_{1,0} + \frac{3}{2}\hat{\rho}_{1,0}^2 + \left(\frac{4}{3}\hat{\rho}_{1,1}\hat{c}_{1,0} + \hat{\rho}_{1,1}y_h - 4\hat{\rho}_{1,1}^2\right.\\
&\quad \left.\left. + 3\hat{\rho}_{1,0}\hat{\rho}_{1,1}\right)\ln y + \frac{3}{2}\hat{\rho}_{1,1}^2\ln^2 y\right) + \mathcal{O}(y^3\ln^3 y)\,,\\
c &= 1 + y\,\hat{c}_{1,0} + y^2\left(\hat{c}_{2,0} + \left(\frac{1}{2}\hat{c}_{1,0}y_h + \frac{2}{3}\hat{c}_{1,0}^2\right)\ln y\right) + \mathcal{O}(y^3\ln^2 y)\,.
\end{aligned}
\tag{6.8}
$$

In (6.8) the non-normalizable coefficients $\hat{\rho}_{1,1}$ and $\hat{c}_{1,0}$ are related to corresponding coefficients of the vacuum solution as

$$
\hat{\rho}_{1,1} = y_h\rho_{1,1}\,, \qquad \hat{c}_{1,0} = y_h c_{1,0}\,,
\tag{6.9}
$$

to be further matched with the mass parameters $\{m_b, m_f\}$ of the dual gauge theory as in (4.15). The rest of the coefficients in (6.8) are normalizable. The asymptotic expansion in the IR, *i.e.,* as $z = (1-y) \to 0_+$ is different from the one in (4.23) — here it reflects the presence of a regular horizon (see (6.5)),

$$
\begin{aligned}
a &= \frac{z}{6}\left(\left(1-(\hat{c}_0^h)^2\right)(\hat{\rho}_0^h)^8 + 8\hat{c}_0^h(\hat{\rho}_0^h)^2 + \frac{4}{(\hat{\rho}_0^h)^4} + \frac{6y_h}{1-y_h}\right) + \mathcal{O}(z^2)\,,\\
d &= \hat{d}_0^h + \mathcal{O}(z)\,,\\
\rho &= \hat{\rho}_0^h + \mathcal{O}(z)\,,\\
c &= \hat{c}_0^h + \mathcal{O}(z)\,.
\end{aligned}
\tag{6.10}
$$

The full set of the non-normalizable coefficients is

$$
\{\hat{a}_{2,0}\,,\ \hat{\rho}_{1,0}\,,\ \hat{c}_{2,0}\,,\ \hat{\rho}_0^h\,,\ \hat{c}_0^h\,,\ \hat{d}_0^h\}\,.
\tag{6.11}
$$

Note that we have the correct number of non-normalizable coefficients to uniquely specify a solution of two second-order and two first-order ODEs given a choice of (6.9).

6.1.1. *Perturbative black holes solutions*

As in Section 4.3, we can construct solutions to (6.7)–(6.10) perturbatively in $m\ell$ to order $\mathcal{O}((m\ell)^2)$.

We introduce

$$
\begin{aligned}
c &= \cosh(2\lambda\hat{\chi}_1(y) + \mathcal{O}(\lambda^3))\,, \qquad \rho = e^{\lambda^2\hat{\alpha}_2(y)+\mathcal{O}(\lambda^4)}\,,\\
a &= \frac{(1-y)(1+y(1-y_h))}{1-yy_h} + \lambda^2\hat{a}_2(y) + \mathcal{O}(\lambda^4)\,, \qquad d = \lambda^2\hat{d}_2(y) + \mathcal{O}(\lambda^2)\,,
\end{aligned}
\tag{6.12}
$$

where λ is a small parameter. Substituting (4.32) into (4.22) we find

$$
\begin{aligned}
0 &= \hat{\chi}_1'' - \frac{\hat{\chi}_1'}{y(1-y)}\left(1+y+\frac{y(1-y_h)((2-y)yy_h-2)}{(1-yy_h)(1+y(1-y_h))}\right) \\
&\quad + \frac{3\hat{\chi}_1}{4y^2(1-y)(1+y(1-y_h))}, \\
0 &= \hat{\alpha}_2'' - \frac{\hat{\alpha}_2'}{y(1-y)}\left(1+y+\frac{y(1-y_h)((2-y)yy_h-2)}{(1-yy_h)(1+y(1-y_h))}\right) + \frac{\hat{\alpha}_2}{y^2(1-y)(1+y(1-y_h))}, \\
0 &= \hat{a}_2' - \frac{2-yy_h}{y(1-yy_h)}\hat{a}_2 - \frac{8}{3}y(1-y)(1+y(1-y_h))(\hat{\chi}_1')^2 + \frac{2}{y}(\hat{\chi}_1)^2, \\
0 &= \hat{d}_2' - \frac{8}{3}y(1-yy_h)(\hat{\chi}_1')^2. \qquad (6.13)
\end{aligned}
$$

For the asymptotic expansions we have:

- as $y \to 0_+$,

$$
\begin{aligned}
\hat{\chi}_1 &= y^{1/2}\left(1+y\left(\hat{\chi}_{1,0,(1)}+\frac{y_h}{4}\ln y\right)+\mathcal{O}(y^2\ln y)\right), \\
\hat{\alpha}_2 &= \hat{\rho}_{1,1,(2)}\left(\left(\hat{\alpha}_{1,0,(2)}+\ln y\right)y+\mathcal{O}(y^2\ln y)\right), \\
\hat{a}_2 &= \frac{4}{3}y+y^2\left(\hat{a}_{2,0,(2)}+\frac{4y_h}{3}\ln y\right)+\mathcal{O}(y^3\ln^2 y), \\
\hat{d}_2 &= \frac{2}{3}y+y^2\left(-\frac{y_h}{4}+2\hat{\chi}_{1,0,(1)}+\frac{y_h}{2}\ln y\right)+\mathcal{O}(y^3\ln^2 y), \qquad (6.14)
\end{aligned}
$$

- as $z \to 0_+$

$$
\begin{aligned}
\hat{\chi}_1 &= \hat{\chi}^h_{0,(1)}\left(1-\frac{3}{4(2-y_h)}z+\mathcal{O}(z^2)\right), \\
\hat{\alpha}_2 &= \hat{\rho}_{1,1,(2)}\left(\hat{\alpha}^h_{0,(2)}\left(1-\frac{1}{(2-y_h)}z+\mathcal{O}(z^2)\right)\right), \\
\hat{a}_2 &= 2(\hat{\chi}^h_{0,(1)})^2 z+\mathcal{O}(z^2), \\
\hat{d}_2 &= \hat{d}^h_{0,(2)}-\frac{3(1-y_h)}{2(2-y_h)^2}(\hat{\chi}^h_{0,(1)})^2 z+\mathcal{O}(z^2). \qquad (6.15)
\end{aligned}
$$

Eqs. (6.13)–(6.14) have to be solved numerically for different values of y_h.

To compare with the full numerical solution, we identify, to order $\mathcal{O}(\lambda^2)$,

$$
\begin{aligned}
&\hat{\rho}_{1,1} = \hat{\rho}_{1,1,(2)}\lambda^2, \qquad \hat{c}_{1,0} = 2\lambda^2, \qquad \hat{\rho}_{1,0} = \hat{\rho}_{1,1,(2)}\hat{\alpha}_{1,0,(2)}\lambda^2, \qquad \hat{c}_{2,0} = 4\hat{\chi}_{1,0,(1)}\lambda^2, \\
&\hat{a}_{2,0} = y_h - 1 + \hat{a}_{2,0,(2)}\lambda^2, \qquad \hat{\rho}^h_0 = 1+\hat{\rho}_{1,1,(2)}\hat{\alpha}^h_{0,(2)}\lambda^2, \\
&\hat{c}^h_0 = 1+2(\hat{\chi}^h_{0,(1)})^2\lambda^2, \qquad \hat{d}^h_0 = \hat{d}^h_{0,(2)}\lambda^2. \qquad (6.16)
\end{aligned}
$$

Note that $\mathcal{N}=2$ supersymmetry in the UV at $\mathcal{O}(\lambda^2)$ leads to (see (4.14))

$$
\hat{\rho}_{1,1,(2)} = \frac{1}{3}. \qquad (6.17)
$$

6.2. Thermodynamic properties of black holes in PW

Requiring that there is no conical singularity in the analytical continuation $t \to it_E$ of the metric (6.5) as $y \to 1_-$ we compute the Hawking temperature T_{BH} of the black hole using (6.10),

$$T_{BH} = \frac{e^{-\hat{d}_0^h}}{12\pi y_h^{1/2}(1-y_h)^{1/2}}\biggl((1-y_h)(1-(\hat{c}_0^h)^2)(\hat{\rho}_0^h)^8 + 8\hat{c}_0^h(1-y_h)(\hat{\rho}_0^h)^2 + 6y_h + \frac{4(1-y_h)}{(\hat{\rho}_0^h)^4}\biggr). \tag{6.18}$$

The Bekenstein–Hawking entropy of the black hole is given by

$$S_{BH} = \frac{A_{horizon}}{4G_5} = \frac{4\pi^2}{G_5}\,\frac{(1-y_h)^{3/2}}{y_h^{3/2}}. \tag{6.19}$$

The free energy $\mathcal{F}_{BH}$ can be computed following holographic renormalization procedure discussed in Section 4.2. We find

$$\begin{aligned}\mathcal{F}_{BH} = &\frac{3\pi}{4G_5}\left(1+\frac{\hat{c}_{1,0}^2}{y_h^2}\left(\frac{4}{9}-\frac{16}{9}\ln 2+\frac{8}{9}\ln y_h\right)+\frac{\hat{c}_{1,0}}{y_h}\left(-\frac{4}{3}-\frac{8}{3}\ln 2+\frac{4}{3}\ln y_h\right)\right.\\ &\left.+32\frac{\hat{\rho}_{1,1}^2}{y_h^2}(2\ln 2-\ln y_h)+\frac{1}{y_h^2}\left\{64\hat{\rho}_{1,1}\hat{\rho}_{1,0}+\frac{8}{3}\hat{c}_{2,0}+32\hat{\rho}_{1,0}^2-4\hat{a}_{2,0}\right\}\right)\\ &-\frac{(1-y_h)\pi e^{-\hat{d}_0^h}}{3y_h^2 G_5}\left((1-y_h)(1-(\hat{c}_0^h)^2)(\hat{\rho}_0^h)^8+8\hat{c}_0^h(1-y_h)(\hat{\rho}_0^h)^2+6y_h+\frac{4(1-y_h)}{(\hat{\rho}_0^h)^4}\right).\end{aligned} \tag{6.20}$$

The contribution in the last line in (6.20) comes from the lower limit of integration of the bulk contribution to the regularized free energy, (4.25); it equals precisely to $(-S_{BH}T_{BH})$. Computing the holographic stress–energy tensor, as described in [10] we find

$$\begin{aligned}E_{BH} = &\frac{3\pi}{4G_5}\left(1+\frac{\hat{c}_{1,0}^2}{y_h^2}\left(\frac{4}{9}-\frac{16}{9}\ln 2+\frac{8}{9}\ln y_h\right)+\frac{\hat{c}_{1,0}}{y_h}\left(-\frac{4}{3}-\frac{8}{3}\ln 2+\frac{4}{3}\ln y_h\right)\right.\\ &\left.+32\frac{\hat{\rho}_{1,1}^2}{y_h^2}(2\ln 2-\ln y_h)+\frac{1}{y_h^2}\left\{64\hat{\rho}_{1,1}\hat{\rho}_{1,0}+\frac{8}{3}\hat{c}_{2,0}+32\hat{\rho}_{1,0}^2-4\hat{a}_{2,0}\right\}\right)\\ = &\frac{3N^2}{16\ell}\left(1+\frac{(m\ell)^4}{9}-\frac{2}{3}(1+2\ln 2-\ln y_h)(m\ell)^2\right.\\ &\left.+\frac{1}{y_h^2}\left\{32\hat{\rho}_{1,0}^2+\frac{16}{3}(m\ell)^2 y_h\hat{\rho}_{1,0}+\frac{8}{3}\hat{c}_{2,0}-4\hat{a}_{2,0}\right\}\right),\end{aligned} \tag{6.21}$$

where in the last line we expressed the energy in terms of the dual gauge theory variables using (6.9) and (4.13). Notice that the basic thermodynamic relation,

$$\mathcal{F}_{BH} = E_{BH} - S_{BH}T_{BH}, \tag{6.22}$$

is satisfied automatically.

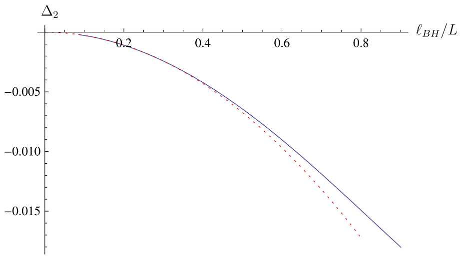

Fig. 4. Solid line represents Δ_2 as defined in (6.24). The dotted red line represents the best quadratic fit to the first 10% of data points, see (6.25). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Using (6.16), from (6.21) we have

$$\frac{E_{BH}}{E^{\mathcal{N}=4}_{vacuum}} = 1 + \frac{4(1-y_h)}{y_h^2} + \left(\frac{8}{3y_h}\hat{\chi}_{1,0,(1)} - \frac{1}{y_h}\hat{a}_{2,0,(2)} - \frac{2}{3}(1+2\ln 2 - \ln y_h)\right)(m\ell)^2 + \mathcal{O}((m\ell)^4)\,. \tag{6.23}$$

6.3. $\Delta(\ell_{BH}/L\,,(m\ell))$

We are now ready to present results for $\Delta(\ell_{BH}/L\,,(m\ell))$ as defined by (6.2). To order $\mathcal{O}((m\ell)^2)$, using (4.40) and (6.23), we find

$$\Delta = \frac{4(1-y_h)}{y_h^2} + \Delta_2\,(m\ell)^2 + \mathcal{O}((m\ell)^4)\,,$$
$$\Delta_2 = \Delta_2(y_h) = \frac{8}{3}\left(\frac{\hat{\chi}_{1,0,(1)}}{y_h} - \chi_{1,0,(1)}\right) - \left(\frac{\hat{a}_{2,0,(2)}}{y_h} - a_{2,0,(2)}\right) + \frac{2}{3}\ln y_h\,. \tag{6.24}$$

Results of numerical computations of Δ_2 are presented in Fig. 4. A solid line represents the data points, and the red dotted line is the best quadratic fit using the first 10% of data points:

$$\Delta_2\bigg|_{fit} = -0.0269118\left(\frac{\ell_{BH}}{L}\right)^2. \tag{6.25}$$

Our numerical results present a strong evidence that

$$\lim_{\ell_{BH}/L\to 0}\Delta_2 = 0\,, \tag{6.26}$$

as a result, we see that Δ vanishes in this limit to order $\mathcal{O}((m\ell)^2)$.

Using (4.29) and (6.21) we compute Δ for $\rho_{1,1} = \frac{1}{12}(m\ell)^2 = \{1, 1.5, 2, \cdots 5, 5.5, 5.8\}$. The results are presented in the left panel of Fig. 5 (the top-to-bottom blue curves correspond to $\rho_{1,1}$ variation $1 \to 5.8$). The green curve represents $\Delta(m\ell = 0)$:

$$\Delta(m\ell = 0) = \frac{2^{4/3}}{\pi^{4/3}}\left(\frac{\ell_{BH}}{L}\right)^2 + \frac{2^{2/3}}{\pi^{8/3}}\left(\frac{\ell_{BH}}{L}\right)^4. \tag{6.27}$$

The right panel represents Δ for the largest value of $m\ell$ computed: $m\ell = 8.34266$, with the red dotted line indicating the best quadratic fit to the first 10% of data points:

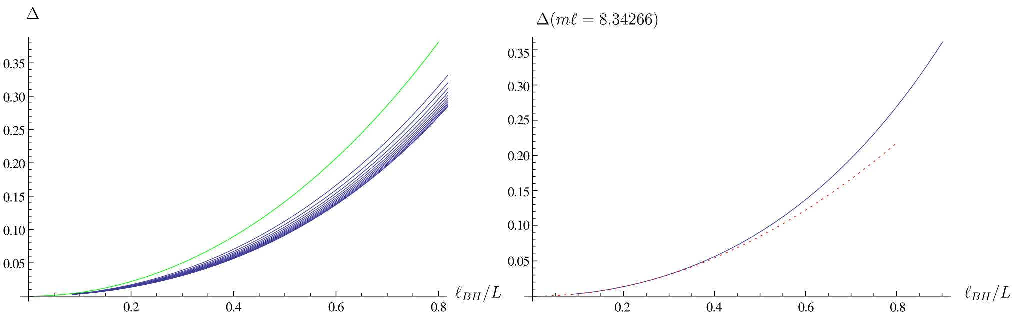

Fig. 5. Left panel: Black hole mass gap relative to $E^{\mathcal{N}=4}_{vacuum}$, see (6.2), as a function of ℓ_{BH}/L for select values of $m\ell$. The green curve represents $\Delta(m\ell = 0)$. Right panel: Δ for the largest value of $m\ell$ computed, $m\ell = 8.34266$; the dotted red line represents the best quadratic fit to the first 10% of data points, see (6.28). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

$$\Delta(m\ell = 8.34266)\bigg|_{fit} = 0.339765\left(\frac{\ell_{BH}}{L}\right)^2 . \tag{6.28}$$

Note that for $m\ell = 8.34266$, $\epsilon = -243.785$, implying that for the smallest size black hole studied, $\ell_{BH}/L = 0.0855056$,

$$\frac{E_{BH} - E_{vacuum}}{E_{vacuum}} = 1.04285 \times 10^{-5} . \tag{6.29}$$

We conclude that numerical results strongly suggest (6.3).

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