A generalization of functions of the first class

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Abstract


A definition of “functions of the first class” is proposed, which for functions on metrizable spaces coincides with the classical one. The definition allows us to generalize classical theorems of Baire on functions of the first class and to prove some results relating various continuity and measurability properties of functions.

Keywords: Point of continuity property, fragmentability, resolvable set, function of the first class, H-complete space, tight measure, t-Baire space.

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1. Introduction

In this paper we study continuity and measurability properties of functions $f$ from a topological space $X$ into a metric space $Y$. We say that (a) $f$ has the point of continuity property (abbreviated PCP) if for every nonempty closed $F \subseteq X$, the restriction $f|_F$ of $f$ to $F$ has a continuity point, and (b) $f$ is fragmented if for every $\varepsilon > 0$ and every nonempty (equivalently, nonempty closed) subset $A$ of $X$, there exists a nonempty relatively open subset $U$ of $A$ such that the diameter $\text{diam} f(U)$ of $f(U)$ is less than $\varepsilon$. It is clear that every function with PCP is fragmented; the
converse holds when \( X \) is hereditarily Baire (i.e., every closed subspace of \( X \) is a Baire space). Functions with PCP are called barely continuous in [16] where some properties of these functions are studied.

The above definitions generalize corresponding concepts in [6] of PCP and fragmentability of a space \( X \) with respect to a metric \( \rho \) on \( X \). More precisely, \( X \) has PCP for the metric \( \rho \) (respectively is fragmented by \( \rho \)) in the sense of [6] if and only if the identity function \( X \to (X, \rho) \) has PCP (respectively is fragmented). These concepts are useful in the theory of Banach spaces from which they have actually been derived. Of particular significance with connections with the Radon-Nikodym property in Banach spaces is the case where \( X \) is a subset of a Banach space with the relative weak topology and \( \rho \) is the metric provided by the norm of the Banach space (see [1, 6]), as well as the case where \( X \) is a compact Hausdorff space and the metric \( \rho \), as a real-valued function on \( X \times X \), is lower semicontinuous (see [5, 19]).

The PCP property is equivalent to a measurability property in a special case according to the following classical theorem of Baire (see [12, § 34, VII]).

**Theorem A.** Let \( f \) be a function from a complete metric space \( X \) into a separable metric space \( Y \). Then \( f \) has PCP if and only if \( f \) is of the first class (i.e., for every open \( G \subset Y \), \( f^{-1}(G) \) is \( F_{\alpha} \) in \( X \)).

Here, PCP can be replaced by the property that for every closed subset \( F \) of \( X \), the set of continuity points of \( f | F \) is dense in \( F \). (In fact, this is the usual statement of the theorem of Baire.) It is also well known that Theorem A remains valid if \( X \) is hereditarily Baire (not necessarily complete).

In view of the significance of PCP in more general situations, it would be interesting to generalize Theorem A when \( X \) is not metrizable (e.g., when \( X \) is a compact Hausdorff space or a Čech-complete space) and when \( Y \) is not separable. We shall prove that Theorem A holds for arbitrary—not necessarily separable—metric spaces \( Y \) (see Theorem 4.12 or Remark following it). We notice, however, that a function with PCP on a nonmetrizable space need not be of the first class (take, for example, the characteristic function of an open set which is not \( F_{\alpha} \)). Thus, for a characterization of PCP as in Theorem A for nonmetrizable \( X \), we seek a more general measurability property of functions than the property of being of the first class.

For this purpose we make use of the following concept whose importance in functions of the first class is well known: a subset \( Z \) of a topological space \( X \) is said to be an \( H \)-set or a resolvable set, if there exists a decreasing transfinite sequence \( \{ F_\alpha : \alpha < \kappa \} \) of closed subsets of \( X \) such that \( Z = \bigcup \{ F_\alpha \setminus F_{\alpha+1} : \alpha < \kappa, \alpha \text{ even ordinal} \} \) (see [12, § 12, I]). An \( H_\alpha \)-set is a countable union of \( H \)-sets. It is clear that every \( F_{\omega} \)-set is \( H_\omega \). Since every \( H \)-set in a metric space is \( F_{\omega} \) [12, § 30, X, Theorem 5], the classes of \( F_{\sigma^+} \) and \( H_\alpha \)-sets in metric spaces are identical. Thus the following definition generalizes functions of the first class and seems to be the appropriate one.
Definition. A function \( f \) from a topological space \( X \) into a metric space \( Y \) is said to be of the first \( H \)-class, if \( f^{-1}(G) \) is \( H \) in \( X \) for every open \( G \subset Y \).

Section 2 contains some characterizations of PCP for functions \( f \) from a hereditarily Baire space into a separable metric space. One such characterization is that \( f \) is of the first \( H \)-class and so the generalized Theorem A (replacing “first class” by “first \( H \)-class”) holds when \( X \) is an arbitrary—not necessarily metrizable—hereditarily Baire space. It is also proved by an example that the generalized Theorem A may fail if \( Y \) is also allowed to be an arbitrary—not necessarily separable—metric space.

In Section 3, using tight (or Radon) measures, we introduce the class of \( t \)-Baire spaces as a class of Baire spaces which contains Čech-complete spaces. The main results are proved in Section 4 and are concerned with functions from a hereditarily \( t \)-Baire space into a metric space. It is proved that the generalized Theorem A holds for such functions assuming that the cardinality of the range space is less than the least \((0, \omega_1)\)-measurable cardinal. Also, the PCP property is characterized in terms of several measurability properties involving \( H \)-sets, Baire property or tight measures. Some of these characterizations generalize results of [5]. In the course of this investigation we prove that every \( t \)-Baire space satisfies the theorem of Namioka [18] on separate and joint continuity.

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2. PCP and related properties

In this section we examine the relationships among some properties which are possessed by all functions with PCP and are equivalent to PCP when the domain of the functions is a hereditarily Baire space and the range is a separable metric space (Theorem 2.3 and Examples 2.4). First we need some properties of \( H \)-sets (or resolvable sets). Since these sets are used extensively in this paper, it will be convenient to adopt the following definition: we say that a subset \( Z \) of a topological space \( X \) is an \( H \)-set, if for every nonempty (equivalently, nonempty closed) subset \( A \) of \( X \) there exists a nonempty relatively open subset \( U \) of \( A \) such that either \( U \subset Z \) or \( U \subset X \setminus Z \). (See [12, §12, V] for the equivalence of this definition and the original one mentioned in the Introduction.) With this definition, the following known elementary properties of \( H \)-sets [12, §12, VI] become immediate. We shall frequently use these properties, sometimes without reference.

Proposition 2.1. Let \( X \) be a topological space.

(i) The family of \( H \)-sets in \( X \) is an algebra of subsets of \( X \) containing the open sets.

(ii) If \( f \) is a continuous function from \( X \) into a space \( Y \), then for every \( H \)-set \( Z \) in \( Y \), \( f^{-1}(Z) \) is an \( H \)-set in \( X \).
Every H-set in X with empty interior is nowhere dense.

(iv) Every H-set in X has the Baire property in the restricted sense.

A subset Z of X has the Baire property if for some open set G, G ∩ Z is of the first category; Z has the Baire property in the restricted sense if for every A ⊂ X, Z ∩ A has the Baire property in A.

We prove only property (iv). Let Z be an H-set in X. Then Z \ Z° is an H-set (by (i)) with empty interior and so nowhere dense (by (iii)). Thus Z = (Z \ Z°) \ Z° has the Baire property. But it is easy to see that for every A ⊂ X, Z ∩ A is an H-set in A. (This follows also from (ii) when f : A → X is the natural injection.) Therefore, Z has the Baire property in the restricted sense.

From the proof of (iv) it follows that if Z is an H-set, then for every A ⊂ X, Z ∩ A is a union of an open set in A and a nowhere dense set in A. It is easy to check that this is a characterization of H-sets. Replacing “nowhere dense” by “first category” we have the following weaker concept. We say that a subset Z of X is an almost H-set, if for every A ⊂ X, Z ∩ A is a union of an open set in A and a first category set in A; equivalently, if for every A ⊂ X such that Z ∩ A has empty interior in A, Z ∩ A is of the first category in A. Again, we can assume that A in this definition is closed in X. From Proposition 2.1(iii) it follows that every Hα-set is an almost H-set.

We now give a characterization of H-sets in terms of semi-open partitions. We say that a partition D of a topological space X is semi-open, if D is expressible as a transfinite sequence \{X_\alpha : \alpha < \kappa\} such that \bigcup_{\alpha < \beta} X_\alpha is open in X for every \beta < \kappa. This concept is essentially equivalent to that of [21, Definition 1.1].

It is easy to see that for any partition \{X_\alpha : \alpha < \kappa\} of a topological space X we have: \bigcup_{\alpha < \beta} X_\alpha is open in X for every \beta < \kappa if and only if \bigcup_{\alpha < \beta} X_\alpha is open in X for every \beta \leq \kappa. Since X_\alpha = (\bigcup_{\beta < \alpha} X_\beta) \setminus (\bigcup_{\beta < \alpha} X_\beta) for every \alpha < \kappa, it follows that each member of a semi-open partition is a difference of two open sets.

**Lemma 2.2.** A subset Z of a topological space X is an H-set if and only if there exists a semi-open partition D of X such that Z = \bigcup D' for some D' \subset D.

**Proof.** Assume that Z is an H-set. Let X_0 be a nonempty open subset of X such that either X_0 \subset Z or X_0 \subset X \setminus Z. Assume that nonempty sets X_\alpha, \alpha < \gamma, have been constructed such that for every \alpha < \gamma either X_\alpha \subset Z or X_\alpha \subset X \setminus Z and \bigcup_{\alpha < \beta} X_\alpha is open in X for every \beta < \gamma. If \bigcup_{\alpha < \gamma} X_\alpha = X, the process terminates. Otherwise, we choose a nonempty relatively open subset X_\gamma of X \setminus \bigcup_{\alpha < \gamma} X_\alpha such that either X_\gamma \subset Z or X_\gamma \subset X \setminus Z. Then \bigcup_{\alpha < \gamma} X_\alpha is open in X. Since this process must terminate, for some \kappa we have \bigcup_{\alpha < \kappa} X_\alpha = X. It is clear that D = \{X_\alpha : \alpha < \kappa\} is the required semi-open partition of X.

Conversely, assume that D is a semi-open partition of X such that Z = \bigcup D' for some D' \subset D. Write D = \{X_\alpha : \alpha < \kappa\} such that \bigcup_{\alpha < \beta} X_\alpha is open in X for every \beta < \kappa. Let A be a nonempty subset of X and let \alpha_0 < \kappa be the least ordinal with
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Then

\[ A \cap X_{\alpha} \neq \emptyset. \]

Therefore \( Z \) is an \( H \)-set. \( \Box \)

Theorem 2.3. Let \( X \) be a topological space, \( Y \) a metric space and \( f : X \to Y \). Then the conditions

(a) \( f \) has PCP,
(b) \( f \) is fragmented,
(c) there exists a sequence \((D_n)_{n=1}^{\infty}\) of semi-open partitions of \( X \) such that the family \( \mathcal{D} = \bigcup_{n=1}^{\infty} D_n \) is a base for \( f \) (i.e., for every open \( G \subset Y \), \( f^{-1}(G) \) is a union of some members of \( \mathcal{D} \)),
(d) for every \( A \subset X \), the set of discontinuity points of \( f|A \) is of the first category in \( A \),
(c') \( f \) is of the first \( H \)-class, and
(d') for every open \( G \subset Y \), \( f^{-1}(G) \) is an almost \( H \)-set in \( X \)

are related as follows:

\[ (a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (d), \]

\[ (c') \Rightarrow (d'). \]

Further, if \( X \) is hereditarily Baire, \( (d) \Rightarrow (a) \) and \( (d') \Rightarrow (c') \) and, if \( Y \) is separable, \( (c') \Rightarrow (c) \) and \( (d') \Rightarrow (d) \).

Remark. If \( X \) is a metric space, the implications \( (c') \Rightarrow (d) \) for separable \( Y \) and \( (a) \Rightarrow (c') \) (where, of course, \( (c') \) says that \( f \) is of the first class) are classical theorems of Baire (see [12, § 31, X]). Theorem A of the Introduction (i.e., the equivalence \( (a) \Leftrightarrow (c') \) for separable \( Y \) and complete \( X \)) is a consequence of the above.

Proof of Theorem 2.3. (a)\( \Rightarrow \)(b) is obvious.

(b)\( \Rightarrow \)(c). For every \( n \) and every nonempty \( A \subset X \), there exists a nonempty relatively open subset \( U \) of \( A \) such that \( \text{diam} \ f(U) < 1/n \). As in the proof of Lemma 2.2, it is easy to see that for every \( n \) we can construct a semi-open partition \( \mathcal{D}_n \) of \( X \) such that \( \text{diam} \ f(D) < 1/n \) for every \( D \in \mathcal{D}_n \). We show that the family \( \mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n \) is a base for \( f \). Let \( G \) be an open subset of \( Y \) and \( x \in f^{-1}(G) \). We choose an \( n \) such that \( G \) contains the open ball with center \( f(x) \) and radius \( 1/n \). Let \( D \in \mathcal{D}_n \) with \( x \in D \). It is clear that \( D \in f^{-1}(G) \) and \( D \in \mathcal{D} \), so \( \mathcal{D} \) is a base for \( f \).

(c)\( \Rightarrow \)(d). Without loss of generality we assume that \( A = X \). Let \( \mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n \) be a base for \( f \), where each \( \mathcal{D}_n \) is a semi-open partition of \( X \). We set \( U_n = \bigcup \{ D^2 : D \in \mathcal{D}_n \} \) for every \( n \). Then each \( U_n \) is open and we show that it is also dense in \( X \). We fix an \( n \) and write \( \mathcal{D}_n = \{ D_{\beta} : \alpha < \kappa \} \) so that \( \bigcup_{\alpha < \beta} D_\alpha \) is open in \( X \) for every \( \beta < \kappa \). Let \( U \) be a nonempty open subset of \( X \) and let \( \alpha_0 < \kappa \) be the least ordinal with \( U \cap D_{\alpha_0} \neq \emptyset \). Now \( U \cap D_{\alpha_0} = U \cap (\bigcup_{\alpha_0} D_{\alpha_0}) \) is open and so \( U \cap D_{\alpha_0} = U \cap (D_{\alpha_0})^0 \subset U \cap U_n \). Therefore \( U \cap U_n \neq \emptyset \) and \( U_n \) is dense in \( X \).

We set \( C = \bigcap_{n=1}^{\infty} U_n \). It follows from the above that \( X \setminus C \) is of the first category in \( X \). Thus it suffices to show that \( f \) is continuous at all points of \( C \). Let \( x \in C \) and
let $G$ be an open subset of $Y$ such that $x \in f^{-1}(G)$. Since $\mathcal{D}$ is a base for $f$, there exist $n \in \mathbb{N}$ and $D \in \mathcal{D}_n$ such that $x \in D \subset f^{-1}(G)$. Since $x \in U_n$, it is clear that $x \in D \subset f^{-1}(G)$ and so $f$ is continuous at $x$.

(c)⇒(c'). Let $\mathcal{D} = \bigcup_{n=1}^{\infty} \mathcal{D}_n$ be a base for $f$, where each $\mathcal{D}_n$ is a semi-open partition of $X$. If $G$ is an open subset of $Y$, then $f^{-1}(G) = \bigcup_{n=1}^{\infty} (\bigcup\{D \in \mathcal{D}_n : D \subset f^{-1}(G)\})$ and Lemma 2.2 implies that $f^{-1}(G)$ is $H_c$. Therefore $f$ is of the first $H$-class.

(d)⇒(d'). Let $G$ be an open subset of $Y$. If $A$ is a subset of $X$ such that $f^{-1}(G) \cap A$ has empty interior in $A$, then $f|A$ is discontinuous at every point of $f^{-1}(G) \cap A = (f|A)^{-1}(G)$ and so, by (d), $f^{-1}(G) \cap A$ is of the first category in $A$. Therefore $f^{-1}(G)$ is an almost $H$-set.

(c')+(d') since every $H_c$-set is an almost $H$-set.

We now assume that $Y$ is separable and prove (c')⇒(c) and (d')⇒(d). Let $(V_n)_{n=1}^{\infty}$ be a countable base for the topology of $Y$.

(c')⇒(c). For every $n$, $f^{-1}(V_n)$ is $H_c$, and so by Lemma 2.2 there exists a countable family $(\mathcal{D}_{n,m})_{m=1}^{\infty}$ of semi-open partitions of $X$ such that $f^{-1}(V_n)$ is a union of some members of $\bigcup_{m=1}^{\infty} \mathcal{D}_{n,m}$. It is clear that $\bigcup_{n=1}^{\infty} \bigcup_{m=1}^{\infty} \mathcal{D}_n$ is a base for $f$.

(d')⇒(d). Without loss of generality we assume $A = X$. For every $n$, $f^{-1}(V_n)$ is an almost $H$-set and so $f^{-1}(V_n) \setminus f^{-1}(V_n)^o$ is of the first category in $X$. Therefore the set $\bigcup_{n=1}^{\infty} (f^{-1}(V_n) \setminus f^{-1}(V_n)^o)$, which is the set of discontinuity points of $f$, is also of the first category.

Finally, we assume that $X$ is hereditarily Baire. Then (d)⇒(a) is obvious and so it remains to prove (d')⇒(c').

(d')⇒(c'). First, let us also assume that $Y$ is separable. Then, by the above, we have (d')⇒(d), (d)⇒(c) and (c)⇒(c'). Therefore (d')⇒(c'). To prove the general case, let $G$ be an open subset of $Y$. We choose a continuous function $\phi : Y \to \mathbb{R}$ such that $G = \phi^{-1}(\mathbb{R} \setminus \{0\})$. If $f$ satisfies condition (d') then so does $\phi \circ f$ and since $\phi \circ f$ has separable range, $\phi \circ f$ satisfies (c'). In particular, the set $(\phi \circ f)^{-1}(\mathbb{R} \setminus \{0\}) = f^{-1}(G)$ is $H_c$ and so $f$ is of the first $H$-class.

Next we collect some examples and remarks concerning the conditions of Theorem 2.3. In some examples we use the following concept: a space $X$ is said to be always of the first category, if every dense in itself subset of $X$ is of the first category in itself. It is known that every uncountable Polish space contains an uncountable subspace which is always of the first category (see [12, § 40, III, Theorems 1 and 2]).

Let $X$ be a space which is always of the first category. Then for every subset $A$ of $X$, if $A_0$ denotes the set of isolated points of $A$, $A \setminus A_0$ is of the first category in $A$. Indeed, $A \setminus A_0 = ((\text{cl}_A A_0) \setminus A_0) \cup (A \setminus \text{cl}_A A_0)$, where $(\text{cl}_A A_0) \setminus A_0$ is closed nowhere dense in $A$ and $A \setminus \text{cl}_A A_0$ is dense in itself and so of the first category in $A$. It follows from this observation that every function on $X$ satisfies (d) of Theorem 2.3 and that every subset of $X$ is an almost $H$-set.

Examples 2.4. The following examples show that in general no other implication between the conditions of Theorem 2.3 is valid. An axiom of set theory is assumed where needed.
(1) \((d) \Rightarrow (c')\) even if \(Y\) is separable.

Let \(X\) be an uncountable subset of \(\mathbb{R}\), which is always of the first category. Assuming the continuum hypothesis (or the weaker assumption \(2^{\omega} > 2^\omega\)), the power set of \(X\) has cardinality \(> 2^\omega\) and so there exists a subset \(A\) of \(X\) which is not \(F_\sigma\).

If \(f : X \to \mathbb{R}\) is the characteristic function of \(A\), then \(f\) is not of the first \(H\)-class, although \(f\) satisfies \((d)\) since \(X\) is always of the first category.

(2) \((c') \Rightarrow (d)\) even if \(X\) is hereditarily Baire.

Assume that the Lebesgue measure on \([0, 1]\) can be extended to a measure \(\mu\) defined on all subsets of \([0, 1]\). (This assumption is equivalent to the existence of a real-valued measurable cardinal \(\leq 2^\omega\), see [7, p. 302].) We shall use some facts from [20, Chapter 22] about the density topology. Let \(\phi : \mathcal{P}([0, 1]) \to \mathcal{P}([0, 1])\) be a lower density for \(\mu\), where \(\mathcal{P}([0, 1])\) denotes the power set of \([0, 1]\), and let \(X\) be \([0, 1]\) equipped with the density topology induced by \(\phi\) (see [20, Theorems 22.4 and 22.5]). Recall that the open sets in \(X\) are of the form \(\phi(A) \setminus N\) for some \(A \subseteq X\) and some \(\mu\)-null set \(N\). Also, the family of \(\mu\)-null sets coincides with the family of nowhere dense sets in \(X\) and every first category subset of \(X\) is closed (see [20, Theorem 22.6]).

For every subset \(A\) of \(X\), we have \(A = (A \cap \phi(A)) \cup (A \setminus \phi(A))\), where \(A \cap \phi(A)\) is open in \(X\) and \(A \setminus \phi(A)\) is a \(\mu\)-null set and so closed in \(X\). It follows that every subset of \(X\) is an \(H\)-set and so every function on \(X\) is of the first \(H\)-class.

Next we show that every nonempty subset of \(X\) is of the second category in itself. Indeed, if \(A\) is a subset of \(X\) of the first category in itself, then every subset of \(A\) is of the first category in \(X\) and so closed in \(X\). Thus \(A\) is a discrete set of the first category in itself and consequently \(A\) is empty. It follows from this observation that \(X\) is hereditarily Baire (in fact, every subspace of \(X\) is a Baire space).

Now let \(f : X \to Y\) be a one to one function, where \(Y\) is a metric space with the discrete metric \(\rho\), i.e., \(\rho(x, y) = 1\) for every \(x, y \in Y, x \neq y\). Since \(\mu\) vanishes on singletons, \(X\) is dense in itself and so \(f\) has no continuity points. Since \(X\) is a Baire space, \((d)\) fails. However, as noted above, every function on \(X\) is of the first \(H\)-class. Thus \((c') \Rightarrow (d)\).

As \(\phi\) is an arbitrary lower density, the study of other topological properties of \(X\) does not seem obvious. However, in the above example we can replace \(X\) by a dense open subspace \(X_0\) which admits a continuous one to one function into \([0, 1]\); in particular \(X_0\) is a Hausdorff space. Indeed, since every subset of \(X\) is an \(H\)-set, the identity function \(h : X \to [0, 1]\) has the Baire property (see Proposition 2.1(iv)) and by [20, Theorem 8.1] there exists a subspace \(X_0\) of \(X\) such that \(X \setminus X_0\) is of the first category (and so \(X_0\) is open dense in \(X\)) and \(h| X_0\) is continuous.

(3) \((d) \land (c') \Rightarrow (c)\).

Assuming Martin’s Axiom and the negation of the continuum hypothesis, there exists an uncountable subset \(X\) of \(\mathbb{R}\) such that every subset of \(X\) is relatively \(F_\sigma\) (see [15, p. 162]). By [12, § 40, III, Theorem 1] \(X\) is always of the first category. Let \(f : X \to Y\) be a one to one function, where \(Y\) is a metric space with the discrete metric. Since every subset of \(X\) is \(F_\sigma\), \(f\) satisfies \((c')\) and since \(X\) is always of the
first category, \( f \) satisfies (d). However, \( f \) does not satisfy (c) because every semi-open partition of \( X \) is countable (since \( X \) is hereditarily Lindelöf) and every base for \( f \) contains all singletons and therefore is uncountable.

(4) (b) \( \Leftrightarrow \) (a) and (c) \( \Leftrightarrow \) (b) even if \( Y \) is separable.

Let \( X \) be the space of rational numbers. Enumerate \( X \) as \( \{x_n : n = 1, 2, \ldots \} \), \( x_n \neq x_m \) for \( n \neq m \), such that \( \{x_{2n} : n = 1, 2, \ldots \} \) and \( \{x_{2n-1} : n = 1, 2, \ldots \} \) are both dense in \( X \). Define \( f \) and \( g : X \to \mathbb{R} \) by \( f(x_n) = 1/n \), \( g(x_n) = 1 \) and \( g(x_{2n-1}) = 0 \) for every \( n \). Then \( f \) is fragmented (if \( A \subset X \) is infinite and \( \varepsilon > 0 \), there exists a finite \( F \subset X \) such that \( \text{diam}(f(A \setminus F)) < \varepsilon \)), but \( f \) has no continuity points. Thus (b) \( \Leftrightarrow \) (a).

For every \( n \), let \( \mathcal{D}_n = \{X \setminus \{x_n\}, \{x_n\}\} \). Then each \( \mathcal{D}_n \) is a semi-open partition of \( X \) and \( \bigcup_{n=1}^{\infty} \mathcal{D}_n \) is a base for \( g \), but \( g \) is not fragmented. Thus (c) \( \Rightarrow \) (b).

Finally, we note that, in view of the implication (d') \( \Rightarrow \) (c') when \( X \) is a hereditarily Baire space, it is natural to ask whether every almost \( H \)-set in a hereditarily Baire space is \( H_\sigma \). However, this is not the case even in \( \mathbb{R} \). Take, for example, an uncountable subset \( Z \) of \( \mathbb{R} \) which is always of the first category. \( Z \) is not \( H_\sigma (= F_\sigma) \) in \( \mathbb{R} \) because \( Z \) does not contain any compact perfect set. To show that \( Z \) is an almost \( H \)-set in \( \mathbb{R} \), let \( A \) be a subset of \( \mathbb{R} \) such that \( Z \cap A \) has empty interior in \( A \). Then the set \( (Z \cap A)_0 \) of isolated points of \( Z \cap A \), as a countable subset of \( Z \cap A \), is of the first category in \( A \) because \( Z \cap A \) does not contain isolated points of \( A \). Also, by the comments before Examples 2.4, \( (Z \cap A) \setminus (Z \cap A)_0 \) is of the first category in \( Z \cap A \), hence also in \( A \). Therefore \( Z \cap A \) is of the first category in \( A \) and \( Z \) is an almost \( H \)-set.

In connection with the above example it should be noted that if \( Z \) is a subset of a hereditarily Baire space \( X \) such that both \( Z \) and \( X \setminus Z \) are almost \( H \)-sets (in particular, \( H_\sigma \)-sets), then \( Z \) is an \( H \)-set. This is easy to see directly, but it also follows from Theorem 2.3, (d') \( \Rightarrow \) (a), when \( f \) is the characteristic function of \( Z \).

Remarks. (1) Conditions (a), (c), (d), (c') and (d') of Theorem 2.3 are not affected if the metric of \( Y \) is replaced by another equivalent metric (i.e., a metric giving the same topology). More generally, it is easy to see that if \( f \) satisfies any of the above conditions and \( \phi \) is a continuous function from \( Y \) into a metric space, then \( \phi \circ f \) satisfies the same condition.

It follows from Theorem 2.3 that the above remark applies also for condition (b) when \( X \) is hereditarily Baire, but not in general. Indeed, let \( f : X \to f(X) \subset \mathbb{R} \) be the fragmented function of Example 2.4(4). If \( f(X) \) is equipped with the discrete metric (which is equivalent to the original metric of \( f(X) \) as subspace of \( \mathbb{R} \)) then \( f \) is not fragmented.

(2) A function \( f \) from a topological space \( X \) into a metric space \( Y \) satisfies condition (c') (respectively (d')) if and only if for every continuous function \( \phi \) from \( Y \) into a separable metric space, \( \phi \circ f \) satisfies condition (c) (respectively (d)). This follows from Remark (1), the equivalences (c') \( \Leftrightarrow \) (c) and (d') \( \Leftrightarrow \) (d) for functions with separable range and an argument used in the proof of (d') \( \Rightarrow \) (c') in Theorem 2.3.
3. \( t \)-Baire spaces

By Example 2.4(2) the separability assumption in Theorem 2.3 is necessary even for a function on a hereditarily Baire space. In this section we shall consider a class of Baire spaces which is needed in the next section in order to drop the separability. The definition of these spaces, called \( t \)-Baire spaces, is suggested by a result from [9], which we now describe.

Let \( X \) be a topological space. We denote by \( M(X) \) the space of nonnegative finite Borel measures on \( X \) (i.e., measures defined on the Borel \( \sigma \)-algebra of \( X \)). \( M(X) \) is endowed with the weak topology, that is, the smallest topology on \( M(X) \) such that the function \( M(X) \ni \mu \to \mu(X) \) is continuous and, for each fixed open \( G \subset X \), the function \( M(X) \ni \mu \to \mu(G) \) is lower semicontinuous (see [25]).

**Theorem 3.1** [9, Theorem 2.1]. Let \( X \) be a Hausdorff space, \( M \) a dense subset of \( M(X) \), \( n \in \mathbb{N} \) and \( R \) a subset of \( X^n \) of the first category. Then \( (\mu \times \cdots \times \mu)^*(R) = 0 \) for all \( \mu \in M \) except for a set of measures of the first category in \( M \).

Here \( \mu \times \cdots \times \mu \) denotes the simple product measure defined on the product \( \sigma \)-algebra of \( X^n \) (i.e., \( n \) times the product of the Borel \( \sigma \)-algebra of \( X \)) and \( (\mu \times \cdots \times \mu)^* \) denotes the outer measure induced by \( \mu \times \cdots \times \mu \) and defined on all subsets of \( X^n \).

It follows from Theorem 3.1 that if \( M \) is also assumed to be of the second category in itself, then there exists a nonzero measure \( \mu \in M \) such that \( (\mu \times \cdots \times \mu)^*(R) = 0 \). Of special importance is the case where \( n = 2 \) and \( M \) is the space \( M_t(X) \) of tight (or Radon) measures on \( X \) (i.e., the space of measures in \( M(X) \) that are inner regular with respect to compact sets). Thus, let us call a Hausdorff space \( X \) a \( t \)-Baire space, if \( M_t(X) \) is of the second category in itself (equivalently, if \( M_t(X) \) is a Baire space; see [9, § 4]). Then, by the above, we have the following.

**Corollary 3.2.** If \( X \) is a \( t \)-Baire space and \( R \) is a subset of \( X \times X \) of the first category, then there exists a nonzero measure \( \mu \in M_t(X) \) such that \( (\mu \times \mu)^*(R) = 0 \).

Compact Hausdorff spaces and, more generally, \( \check{C} \)ech-complete spaces (that is, spaces that are homeomorphic to a \( G_\delta \) subspace of a compact Hausdorff space) are \( t \)-Baire (cf. [26, Part II, Theorem 17]). For the sake of completeness and readability, a proof of Corollary 3.2 at least for \( \check{C} \)ech-complete spaces without isolated points is given in Theorem 3.4 below (see also Remark (2) following that theorem).

We shall also need the following properties of \( t \)-Baire spaces. Recall that a residual set is a complement of a first category set.

**Proposition 3.3.** (i) Every \( t \)-Baire space is a Baire space (cf. [9, Corollary 3.5]).

(ii) Every residual subspace of a \( t \)-Baire space is \( t \)-Baire.

(iii) Every open subspace of a \( t \)-Baire space is \( t \)-Baire.
Proof. We shall use the following fact: if $P$ is a first category subset of a Hausdorff space $X$ then $\{\mu \in M_1(X): \mu^*(P) = 0\}$ is residual in $M_1(X)$. This follows from Theorem 3.1 when $n = 1$ and $M = M_1(X)$, but it is also easy to see directly.

(i) Let $X$ be a $t$-Baire space and $G$ a nonempty open subset of $X$. It is easy to see that $\{\mu \in M_1(X): \mu(G) = 0\}$ is closed nowhere dense in $M_1(X)$ and, since $M_1(X)$ is of the second category in itself, it follows from the above that $G$ is of the second category.

(ii) Let $Y$ be a residual subspace of a $t$-Baire space $X$. It is easy to see that the function $h: M_1(Y) \to M_1(X)$ with $h(\mu)(B) = \mu(B \cap Y)$ for every Borel set $B$ in $X$ is a topological embedding. Thus $M_1(Y)$ is homeomorphic to $h(M_1(Y)) = \{\mu \in M_1(X): \mu^*(X \setminus Y) = 0\}$ which is residual in $M_1(X)$ by the above. Since $M_1(X)$ is of the second category in itself, so is $M_1(Y)$.

(iii) Let $Y$ be an open subspace of a Hausdorff space $X$ and set

$$S = \{\mu \in M_1(X): \mu(\overset{\sim}{Y} \setminus Y) = 0\}.$$ 

Since $\overset{\sim}{Y} \setminus Y$ is closed nowhere dense in $X$, it follows that $S$ is a $G_\delta$ dense in $M_1(X)$. Define $h: M_1(Y) \to M_1(X)$ with $h(\mu)$ the restriction of $\mu$ to the Borel sets in $Y$. First we show that $h$ has the following properties:

1. $h$ is continuous at every point of $S$, and
2. for every dense subset $D$ of $M_1(Y)$, $S \cap h^{-1}(D)$ is dense in $S$ (and so in $M_1(X)$).

For (1), let $\mu_0 \in S$ and let

$$W = \{\nu \in M_1(Y): \nu(Y) < \beta, \nu(U_i) > \alpha_i \text{ for } i = 1, \ldots, n\},$$

where $\beta > 0$, $\alpha_i \geq 0$, $U_i$ is open in $Y$ and $n = 0, 1, \ldots$, be a basic open neighborhood of $h(\mu_0)$ in $M_1(Y)$. Since $Y$ is open in $X$ and $\mu_0 \in S$, the set

$$V = \{\mu \in M_1(X): \mu(\overset{\sim}{Y}) < \beta, \mu(U_i) > \alpha_i \text{ for } i = 1, \ldots, n\}$$

is an open neighborhood of $\mu_0$ in $M_1(X)$ and clearly $h(V) \subseteq W$.

For (2), let

$$V = \{\mu \in S: \mu(X) < \beta, \mu(U_i) > \alpha_i \text{ for } i = 1, \ldots, n\},$$

where $\beta > 0$, $\alpha_i \geq 0$, $U_i$ is open in $X$ and $n = 0, 1, \ldots$, be a nonempty basic open set in $S$. Let $\mu_0 \in V$ and choose $\varepsilon > 0$ such that $\varepsilon < \beta - \mu_0(X)$ and $\varepsilon < \mu_0(U_i) - \alpha_i$ for $i = 1, \ldots, n$. Then the set

$$W = \{\nu \in M_1(Y): \nu(Y) < \mu_0(Y) + \varepsilon, \nu(U_i \cap Y) > \mu_0(U_i \cap Y) - \varepsilon \text{ for } i = 1, \ldots, n\}$$

is open in $M_1(Y)$ and nonempty since $h(\mu_0) \in W$. Thus there exists a $\nu_0 \in W \cap D$. If $\lambda_0 \in M_1(X)$ is given by $\lambda_0(B) = \nu_0(B \cap Y) + \mu_0(B \setminus Y)$, it is easy to check that $\lambda_0 \in (S \cap h^{-1}(D)) \cap V$. Therefore $S \cap h^{-1}(D)$ is dense in $S$. 


Next we show that (1) and (2) imply that for every nowhere dense subset $N$ of $M_t(Y)$, $h^{-1}(N)$ is nowhere dense in $M_t(X)$. Indeed, we have that the set $D = M_t(Y) \setminus M_t(Y) \cap N$ is open and dense in $M_t(Y)$. By (2), $S \cap h^{-1}(D)$ is dense in $M_t(X)$ and, by (1), for every $\mu \in S \cap h^{-1}(D)$ there exists an open neighborhood $V$ of $\mu$ in $M_t(X)$ such that $h(V) \subset D$ and so $V \cap h^{-1}(N) = \emptyset$. Thus $h^{-1}(N)$ is nowhere dense.

It now follows from the above that if $M_t(Y)$ is of the first category in itself, then $M_t(X)$ is of the first category in itself and (iii) follows.

We now define and discuss a subclass of $t$-Baire spaces. We say that a space is an $H$-complete space, if it is homeomorphic to an $H_\kappa$-set (i.e., a countable intersection of $H$-sets) in a compact Hausdorff space. It is clear that $H$-complete spaces generalize Čech complete spaces.

To see that every $H$-complete space $X$ is $t$-Baire, we observe that there exists a compactification $\tilde{X}$ of $X$ such that $X = \cap_{n=1}^\infty Z_n$, where each $Z_n$ is a $H_\kappa$ in $\tilde{X}$. Then $\tilde{X} \setminus X$, as an $H_\kappa$-set with empty interior in $\tilde{X}$, is of the first category (Proposition 2.1(iii)) and so Proposition 3.3(iii) yields that $X$ is $t$-Baire.

More generally, every Hausdorff space $X$ which contains a dense $H$-complete subspace $Y$ is $t$-Baire. This follows from the above since $M_t(Y)$ is homeomorphic to a dense subspace of $M_t(X)$.

We say that a space is hereditarily $t$-Baire, if every closed subspace is $t$-Baire. Since every closed (and every open) subspace of an $H$-complete space is $H$-complete, it follows that $H$-complete spaces are hereditarily $t$-Baire.

In the next theorem we prove Corollary 3.2 for $H$-complete spaces without isolated points.

**Theorem 3.4.** Let $X$ be an $H$-complete space without isolated points and $R$ a subset of $X \times X$ of the first category. Then

(i) there exists a compact subset $L$ of $X$ and a function $\phi : L \to \{0, 1\}^N$ continuous, onto, such that for every $x, y \in L$ with $\phi(x) \neq \phi(y)$, $(x, y) \notin R$; and

(ii) there exists a nonzero measure $\mu \in M_t(X)$ such that $(\mu \times \mu)^*(R) = 0$.

**Proof.** (i) Let $\tilde{X}$ be a compactification of $X$ such that $X = \cap_{n=1}^\infty Z_n$, where each $Z_n$ is a $H$-set in $\tilde{X}$. $R$ is also of the first category in $\tilde{X} \times \tilde{X}$, so $R = \cup_{n=1}^\infty R_n$, where $(R_n)$ is an increasing sequence of nowhere dense subsets of $\tilde{X} \times \tilde{X}$.

Let $\{0, 1\}^{\{N\}}$ be the set of finite sequences of 0's and 1's and for every $s \in \{0, 1\}^{\{N\}}$ and $i = 0$ or 1, let $l(s)$ be the length of $s$ and $(s, i)$ be the sequence extending $s$ with length $l(s) + 1$ and last term $i$. We shall construct a family $V_s$, $s \in \{0, 1\}^{\{N\}}$ of nonempty open subsets of $\tilde{X}$ such that:

1. $\overline{V_s} \setminus V_s = \emptyset$ if $l(s) = l(\tau)$, $s \neq \tau$,
2. $\overline{V_s} \setminus V_s \subset V_s \cap Z_n$ if $l(s) = n - 1$, $i = 0$, 1, and
3. $(\overline{V_s} \times \overline{V_s}) \cap R_n = \emptyset$ if $l(s) = l(\tau) = n$, $s \neq \tau$.

We set $V_s = \tilde{X}$ if $s = \emptyset$. Assume that for some $n \in \mathbb{N}$, we have constructed $V_s$ for all $s \in \{0, 1\}^{\{N\}}$ with $l(s) \leq n - 1$, satisfying (1)–(3). Since $Z_n$ is an $H$-set dense in $\tilde{X}$,
for every $s \in \{0,1\}^{(n)}$ with $l(s) = n - 1$, there exists a nonempty open set $U_s$ in $\tilde{X}$ such that $U_s \subset V_s \cap Z_n$. Since $\tilde{X}$ has no isolated points, there exist two disjoint nonempty open sets $U_0, U_1 \subset U_s$. Now, using that $R_n$ is nowhere dense, it is not hard to see that we can choose nonempty open sets $V_s$ for every $s \in \{0,1\}^{(n)}$ with $l(s) = n - 1$ and $i, j \in \{0,1\}$ (cf. [13, § 3] or [9, Lemma 2.21]). The construction is now completed.

For every $\sigma = (\sigma_m)_{m \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ and every $n \in \mathbb{N}$, let $\sigma|n = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and set

$$L = \bigcup_{\sigma \in \{0,1\}^n} \bigcap_{m=1}^{\infty} \overline{V_{\sigma_m,n}}.$$ 

Using (1) and (2) and the fact that $\tilde{X}$ is compact we see that the sets $\bigcap_{m=1}^{\infty} \overline{V_{\sigma_m,n}} = \bigcap_{n=1}^{\infty} V_{\sigma[n]}$, $\sigma \in \{0,1\}^\mathbb{N}$, are nonempty pairwise disjoint subsets of $X$ and that $L = \bigcap_{n=1}^{\infty} \bigcup_{\sigma \in \{0,1\}^n} \overline{V_{\sigma[n]}}$. It follows that $L$ is a nonempty compact subset of $X$. Also, the function $\phi : L \to \{0,1\}^\mathbb{N}$ given by $\phi|\bigcap_{m=1}^{\infty} \overline{V_{\sigma_m,n}} = \sigma$ is onto and continuous since for every $m \in \mathbb{N}$ and $i = 0, 1$,

$$\phi^{-1}((\sigma_m)_{m \in \mathbb{N}} \in \{0,1\}^{\mathbb{N}} : \sigma_m = i)) = L \cap (\bigcup_{n=1}^{\infty} \overline{V_{\sigma[n]} : s \in \{0,1\}^{m-1}})$$

is closed and open in $L$.

Finally, let $x, y \in L$ with $\phi(x) \neq \phi(y)$. Write $\phi(x) = (\sigma_n)_{n \in \mathbb{N}}$ and $\phi(y) = (\tau_n)_{n \in \mathbb{N}}$ and choose $n_0 \in \mathbb{N}$ such that $\sigma_{n_0} \neq \tau_{n_0}$. By (3), $(\overline{V_{\phi(x)[n]} \times V_{\phi(y)[n]}} \cap R_n = \emptyset$ for all $n \geq n_0$ and so $(x, y) \notin \bigcup_{n=n_0}^{\infty} R_n = \bigcup_{n=1}^{\infty} R_n = R$.

(ii) Let $L$ and $\phi$ be as in (i) and let $\lambda$ be an atomless nonzero tight measure on $\{0,1\}^\mathbb{N}$ (for example, the usual product measure on $\{0,1\}^\mathbb{N}$ is such a measure). It is well known that there exists a measure $\nu \in M_c(L)$ such that $\phi(\nu) = \lambda$, where $\phi(\nu)$ denotes the image measure under $\phi$ given by $\phi(\nu)(B) = \nu(\phi^{-1}(B))$ for every Borel set $B$ in $\{0,1\}^\mathbb{N}$. Define $\mu \in M_c(X)$ by $\mu(A) = \nu(A \cap L)$. Clearly $\mu \neq 0$.

Let $\phi \times \phi : L \times L \to \{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ be given by $(\phi \times \phi)(x, y) = (\phi(x), \phi(y))$ and let $\Delta - \{(\sigma, \sigma) : \sigma \in \{0,1\}^\mathbb{N}\}$ be the diagonal of $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$. Then $R \cap (L \times L) \subset (\phi \times \phi)^{-1}(\Delta)$ and $(\phi \times \phi)(\nu \times \nu) = \lambda \times \lambda$. Therefore

$$(\mu \times \mu)^*(R) = (\nu \times \nu)^*(R \cap (L \times L)) \leq (\nu \times \nu)((\phi \times \phi)^{-1}(\Delta))$$

$$= (\lambda \times \lambda)(\Delta) = 0.$$

Remarks. (1) With an obvious modification of condition (3) in the above proof, we see that Theorem 3.4, like Theorem 3.1, holds when $R$ is a first category subset of $X^n$ for some $n \in \mathbb{N}$. Then the last property of $\phi$ becomes: for every $x_1, \ldots, x_n \in L$ with $\phi(x_i) \neq \phi(x_j)$ for $i \neq j$, $(x_1, \ldots, x_n) \notin R$. We also note that if $X$ is assumed to be a complete metric space, then $\phi$ can be chosen to be one to one (choose $V_s$ such that diam$(V_s \cap X) < 1/2^{l(s)}$). Thus part (i) of Theorem 3.4 becomes: there exists a subset $L$ of $X$ homeomorphic to the Cantor set $\{0,1\}^{\mathbb{N}}$ such that for every $x_1, \ldots, x_n \in L$ with $x_i \neq x_j$ for $i \neq j$, $(x_1, \ldots, x_n) \notin R$. This result is also a consequence of the main theorem of [17].
A generalization of functions of the first class

(2) The proof of Theorem 3.4 describes a method for constructing a nonzero measure \( \mu \in M_*(X) \) such that \((\mu \times \mu)^*(R) = 0\), whereas the proof of Corollary 3.2 is based on the Baire category method and simply ensures the existence of such a measure.

Corollary 3.2 and Proposition 3.3 are needed in the next section for the main results which are concerned with (hereditarily) \( t \)-Baire spaces. If one is interested only in \( H \)-complete spaces, it suffices to use Theorem 3.4(ii) and some properties of \( H \)-complete spaces mentioned before the statement of Theorem 3.4.

4. Main results

In this section we state and prove our main results.

Theorem 4.1. Let \( X \) be a hereditarily \( t \)-Baire space, \( Y \) a metric space with cardinality less than the least \((\{0, 1\},-)\) measurable cardinal and \( f : X \to Y \). Then \( f \) has PCP if and only if \( f \) is of the first \( H \)-class.

In view of Theorem 2.3, the conclusion of Theorem 4.1 actually says that all conditions of Theorem 2.3 are equivalent in this case.

Concerning the cardinality restriction in Theorem 4.1, we recall that a cardinal \( \kappa \) is less than the least measurable cardinal if and only if there does not exist any nonzero \( \{0, 1\} \)-valued measure defined on all subsets of \( \kappa \) and vanishing on singletons (see [7, § 27]). This mild cardinality restriction can be dropped in special cases (see Theorem 4.12 below and the Remark following it), but I don't know if it is necessary in general. The next theorem relates this problem to the product function \( f \times f : X \times X \to Y \times Y \) defined by \((f \times f)(x, x') = (f(x), f(x'))\).

Theorem 4.2. Let \( X \) be a hereditarily \( t \)-Baire space, \( Y \) a metric space and \( f : X \to Y \). Then the conditions

(a) \( f \) has PCP,
(b) \( f \times f \) is of the first \( H \)-class,
(c) \( f \) is of the first \( H \)-class,
(c) \( f \) has the Baire property in the restricted sense,
(d) \( f \) is \( \mu \)-measurable for every \( \mu \in M_*(X) \),
(e) \( f \) has \( \mu \)-almost separable range for every \( \mu \in M_*(X) \),
(f) \( f \) has PCP for every compact subset \( K \) of \( X \), and
(g) for every \( \epsilon > 0 \) the set \( \{(x, x') : \rho(f(x), f(x')) < \epsilon \} \) is \( H_\rho \) in \( X \times X \), where \( \rho \) denotes the metric of \( Y \),

are related as follows: \((a) \iff (b) \iff (c) \land (d)\) for \( i = 1, \ldots, 5 \).

We explain (c), (c) and (c). (c) means that \( f^{-1}(G) \) has the Baire property in the restricted sense for every open \( G \subset Y \); \( f \) is said to be \( \mu \)-measurable if \( f^{-1}(G) \)
is \( \mu \)-measurable for every open \( G \subset Y \); and \( f \) is said to have \( \mu \)-almost separable range if there exists a separable \( S \subset Y \) such that \( f(x) \in S \) for \( \mu \)-almost all \( x \in X \).

From the equivalence (a) \( \iff \) (b) (respectively (a) \( \iff \) (c) \( \land \) (d)) it follows that Theorem 4.1 is valid without cardinality restrictions if and only if \( f \times f \) is of the first \( H \)-class (respectively the weaker condition (d) holds) whenever \( f \) is of the first \( H \)-class. We note that if \( f \) is a function of the first class on an arbitrary metric space, then \( f \times f \) need not be of the first class (cf. [4, Example 6.1]).

In the case where \( \rho \) is a metric on \( X \), \( Y = (X, \rho) \) and \( f : X \rightarrow (X, \rho) \) is the identity, Theorems 4.1 and 4.2 provide necessary and sufficient conditions in order that \( X \) have PCP for the metric \( \rho \) (or \( X \) be fragmented by \( \rho \)) in the sense of [5, 6]. In particular, the equivalences (a) \( \iff \) (c) \( \land \) (d), \( i = 2, \ldots, 5 \), generalize results of [5] where the metric \( \rho \) is assumed to be lower semicontinuous and so (d) automatically holds; in fact the sets in (d) are \( F_\sigma \) in this case.

For the statement of the next theorem we need the following definition. A topological space \( X \) is said to be a Namioka space, if for every compact Hausdorff space \( K \) and every separately continuous function \( \phi : X \times K \rightarrow \mathbb{R} \) there exists a dense \( G_\delta \)-subset \( C \) of \( X \) such that \( \phi \) is (jointly) continuous at every point of \( C \times K \). An equivalent formulation of this concept is as follows: \( X \) is a Namioka space if and only if for every compact Hausdorff space \( K \), every continuous function \( f \) from \( X \) into \( C(K) \) equipped with the topology of pointwise convergence is continuous at every point of a dense \( G_\delta \)-subset of \( X \) when \( C(K) \) is equipped with the supremum norm topology.

The class of completely regular Namioka spaces is a subclass of Baire spaces [24, Theorem 3] and contains Čech-complete spaces [18, Theorem 1.2].

**Theorem 4.3.** Every \( t \)-Baire space is a Namioka space.

Theorems 4.1–4.3 will be proved in a series of lemmas, some of which are of independent interest and are stated and proved without additional effort in a more general setting. For instance, the following lemma is stated for \( \tau \)-additive measures, although tight measures would suffice here. Recall that a measure \( \mu \in M(X) \) is said to be \( \tau \)-additive, if \( \mu \) is inner regular with respect to closed sets and \( \lim_{\tau} \mu(G_\circ) = \mu(G) \) for every net \( \{G_\circ \} \) of open sets filtering up to \( G \). We denote by \( M_\circ(X) \) the space of \( \tau \)-additive measures on \( X \) and it is easy to see that \( M_\circ(X) \subseteq M_\circ(X) \) whenever \( X \) is Hausdorff.

**Lemma 4.4.** Every \( H \)-set in a topological space \( X \) is \( \mu \)-measurable for every \( \mu \in M_\circ(X) \).

**Proof.** First we show that if \( \mu \in M_\circ(X) \) and \( A \) is a locally \( \mu \)-measurable subset of \( X \) (i.e., for every \( x \in A \) there exists an open neighborhood \( G \) of \( x \) such that \( A \cap G \) is \( \mu \)-measurable), then \( A \) is \( \mu \)-measurable. Indeed, there exists a family \( \{G_i\}_{i \in I} \) of open sets such that \( A \subseteq \bigcup_{i \in I} G_i \) and \( A \cap G_i \) is \( \mu \)-measurable for every \( i \in I \). By the \( \tau \)-additivity of \( \mu \), there exists a countable \( J \subseteq I \) such that \( \mu(\bigcup_{i \in J} G_i) = \mu(\bigcup_{i \in J} G_i) \).
We set $B = \bigcup_{i \in J} (A \cap G_i)$. Then $B$ is $\mu$-measurable, $B \subseteq A$ and $A \setminus B = (\bigcup_{i \in I} G_i) \setminus (\bigcup_{i \in J} G_i)$ is of $\mu$-measure zero. Therefore, $A$ is $\mu$-measurable.

Now, let $\mathcal{D}$ be a semi-open partition of $X$. Write $\mathcal{D} = \{X_\alpha : \alpha < \kappa\}$ such that $\bigcup_{\alpha < \beta} X_\alpha$ is open in $X$ for every $\beta < \kappa$ and let $\{D_\alpha : \alpha < \kappa\}$ be a family of subsets of $X$ such that for every $\alpha < \kappa$ either $D_\alpha = X_\alpha$ or $D_\alpha = \emptyset$. Notice that $\{D_\alpha : \alpha < \kappa\}$ is a semi-open partition of $\bigcup_{\alpha < \kappa} D_\alpha$. By Lemma 2.2 it suffices to show that for every $\mu \in \mathcal{M}_r(X)$, $\bigcup_{\alpha < \kappa} D_\alpha$ is $\mu$-measurable. In fact we show by induction that $\bigcup_{\alpha < \beta} D_\alpha$ is $\mu$-measurable for every $\beta \leq \kappa$. Assume that $\bigcup_{\alpha < \gamma} D_\alpha$ is $\mu$-measurable for every $\gamma < \beta$. If $\beta$ is limit, then since $\bigcup_{\alpha < \beta} D_\alpha = \bigcup_{\gamma < \beta} (\bigcup_{\alpha < \gamma} D_\alpha)$ and $\bigcup_{\alpha < \gamma} D_\alpha$ is relatively open in $\bigcup_{\alpha < \beta} D_\alpha$, $\bigcup_{\alpha < \beta} D_\alpha$ is locally $\mu$-measurable and so, by the above, $\mu$-measurable. If $\beta = \gamma + 1$, then $\bigcup_{\alpha < \beta} D_\alpha = (\bigcup_{\alpha < \gamma} D_\alpha) \cup D_\gamma$, where $D_\gamma$ is a difference of two open sets, and so $\bigcup_{\alpha < \beta} D_\alpha$ is $\mu$-measurable.

The next lemma is a special case of some results of Fremlin [2, § 6 and § 7]; see also [10, 11]. We use the term “disjoint”, applied to a family of sets, as short for “pairwise disjoint”.

**Lemma 4.5.** (i) Let $\mu$ be a tight measure on a Hausdorff space $X$. If $\mathcal{E}$ is a disjoint family of $\mu$-null sets such that $\bigcup \mathcal{E}'$ is $\mu$-measurable for every $\mathcal{E}' \subseteq \mathcal{E}$, then $\bigcup \mathcal{E}$ is $\mu$-null.

(ii) Let $X$ be a compact Hausdorff space with the countable chain condition (ccc). If $\mathcal{E}$ is a disjoint family of first category subsets of $X$ such that $\bigcup \mathcal{E}'$ has the Baire property for every $\mathcal{E}' \subseteq \mathcal{E}$, then $\bigcup \mathcal{E}$ is of the first category.

**Remark.** It is clear that Lemma 4.5 also holds when $\mathcal{E}$ is merely $\sigma$-disjoint, i.e., $\mathcal{E} = \bigcup_{n=1}^{\infty} \mathcal{E}_n$ and each $\mathcal{E}_n$ is disjoint.

The conclusion of the next lemma for countable families is easily seen to be equivalent to the Baire category theorem. Thus the lemma extends Proposition 3.3(i) and shows that $t$-Baire spaces satisfy a strong form of the Baire category theorem.

**Lemma 4.6.** Let $X$ be a $t$-Baire space. If $\mathcal{E}$ is a disjoint family of subsets of $X$ with empty interior such that $\bigcup \mathcal{E}'$ is $H_\sigma$ for every $\mathcal{E}' \subseteq \mathcal{E}$ and the cardinality of $\mathcal{E}$ is less than the least measurable cardinal, then $\bigcup \mathcal{E}$ has empty interior.

**Proof.** Assume, to the contrary, that $(\bigcup \mathcal{E})^o \neq \emptyset$. Then we can also assume that $X = \bigcup \mathcal{E}$ since, by Proposition 3.3(iii), we can replace $X$ by $(\bigcup \mathcal{E})^o$ and $\mathcal{E}$ by $\{E \cap (\bigcup \mathcal{E})^o : E \in \mathcal{E}\}$. We set

$$R = \bigcup \{E \times E : E \in \mathcal{E}\}.$$ 

Let $\mu \in M_r(X)$ be such that $(\mu \times \mu)^*(R) = 0$. Then, for every $E \in \mathcal{E}$, $E$ is a $\mu$-null set since $E \times E \subseteq R$. Also, for every $\mathcal{E}' \subseteq \mathcal{E}$, $\bigcup \mathcal{E}'$ as an $H_\sigma$-set is $\mu$-measurable (Lemma 4.4). Thus, by Lemma 4.5(ii), $\bigcup \mathcal{E}$ is $\mu$-null, i.e., $\mu = 0$ since $X = \bigcup \mathcal{E}$. It now follows from Corollary 3.2 that $R$ is of the second category in $X \times X$. In
particular, there are nonempty open subsets \( U \) and \( V \) of \( X \) such that \( R \) is dense in \( U \times V \).

We claim that \( R \) is dense in \( U \times U \). Indeed, let \( W_1 \) and \( W_2 \) be nonempty open subsets of \( U \) and set
\[
A_j = V \cap (\bigcup \{ E \in \mathcal{E}: E \cap W_j = \emptyset \})
\]
for \( j = 1, 2 \). Then \( (W_j \times A_j^0) \cap R = \emptyset \) since \( \mathcal{E} \) is disjoint. But \( W_j \times A_j^0 \) is an open subset of \( U \times V \) and \( R \) is dense in \( U \times V \). Therefore, \( A_j^0 = \emptyset \) and, since \( A_j \) is also an \( H_\rho \)-set, \( A_j \) is of the first category (Proposition 2.1(iii)). As \( X \) is a Baire space (Proposition 3.3(i)), it follows that \( V \setminus (A_1 \cup A_2) \neq \emptyset \). Thus, there exists an \( E \in \mathcal{E} \) such that \( E \cap (V \setminus (A_1 \cup A_2)) \neq \emptyset \) (since \( \bigcup \mathcal{E} = X \)). It is now easy to check that \( E \cap W_j \neq \emptyset \) for \( j = 1, 2 \) and so
\[
\emptyset \neq (E \times E) \cap (W_1 \times W_2) \subset R \cap (W_1 \times W_2).
\]
This proves the claim.

Next we write \( \mathcal{E} = \{ E_i : i \in I \} \) such that \( E_i \neq E_j \) for \( i \neq j \) and set
\[
\mathcal{J} = \left\{ J \subset I : \left( \bigcup_{i \in J} E_i \right)^ supplement \subset U = \emptyset \right\}.
\]
Since \( X \) is a Baire space and the sets of the form \( (\bigcup_{i \in J} E_i) \cap U \) are \( H_\sigma \), for every \( J \subset I \) we have: \( (\bigcup_{i \in J} E_i) \cap U \) is of the first category if and only if it has empty interior, i.e., \( J \in \mathcal{J} \). It now follows that \( \mathcal{J} \) is a \( \sigma \)-ideal of subsets of \( I \) (i.e., \( \mathcal{J} \) is closed under countable unions and a subset of a member of \( \mathcal{J} \) is in \( \mathcal{J} \)). It is also clear that \( \mathcal{J} \) contains all singletons in \( I \) and that \( I \notin \mathcal{J} \). Another property of \( \mathcal{J} \) is the following: if \( J_1 \) and \( J_2 \) are disjoint subsets of \( I \) then either \( J_1 \in \mathcal{J} \) or \( J_2 \in \mathcal{J} \). Indeed, the sets \( W_1 = (\bigcup_{i \in J_1} E_i)^ supplement \cap U \) and \( W_2 = (\bigcup_{i \in J_2} E_i)^ supplement \cap U \) are open subsets of \( U \) and \( R \cap (W_1 \times W_2) = \emptyset \) since \( \mathcal{E} \) is disjoint. But \( R \) is dense in \( U \times U \) and so either \( W_1 \) or \( W_2 \) is empty, i.e., either \( J_1 \notin \mathcal{J} \) or \( J_2 \notin \mathcal{J} \).

Finally we set for every \( J \subset I \), \( \mu(J) = 0 \) if \( J \in \mathcal{J} \) and \( \mu(J) = 1 \) if \( J \notin \mathcal{J} \). From the above properties of \( \mathcal{J} \) it follows that \( \mu \) is a nonzero \( \{0, 1\} \)-valued measure defined on all subsets of \( I \) and vanishing on singletons. This, however, contradicts the hypothesis on the cardinality of \( \mathcal{E} \) which equals the cardinality of \( I \). The proof is now completed. \( \square \)

**Remark.** Lemma 4.6 remains valid when \( \mathcal{E} \) is \( \sigma \)-disjoint since an \( H_\rho \)-set in a Baire space is of the first category if and only if it has empty interior.

**Lemma 4.7.** Let \( X \) be a \( t \)-Baire space, \( Y \) a metric space with cardinality less than the least measurable cardinal and \( f : X \to Y \) a function of the first \( H \)-class. Then \( f \) is continuous at every point of a dense \( G_\delta \)-subset of \( X \).

**Proof.** The oscillation of \( f \) at a point \( x \in X \) is defined by
\[
O_f(x) = \inf \{ \text{diam}(f(U)) : U \text{ a neighborhood of } x \}.
\]
Since $O_f$ is upper semicontinuous the set $\{x \in X : O_f(x) = 0\}$ of continuity points of $f$ is a $G_\delta$-set. Since $X$ is a Baire space (Proposition 3.3(i)), it suffices to show that, for every $\epsilon > 0$, the set $\{x \in X : O_f(x) < \epsilon\}$ is dense in $X$, i.e., for every nonempty open subset $U$ of $X$ there exists a nonempty open subset $V$ of $U$ such that $\text{diam } f(V) < \epsilon$.

We fix an $\epsilon > 0$ and, without loss of generality, we assume that $U = X$ (see Proposition 3.3(iii)). Since $Y$, as a metric space, is paracompact, we can choose a $\sigma$-disjoint open cover $\mathcal{G}$ of $Y$ by nonempty sets of diameter less than $\epsilon$. Then the family $\mathcal{E} = \{f^{-1}(G) : G \in \mathcal{G}\}$ is a $\sigma$-disjoint cover of $X$ and, since $f$ is of the first $\mathcal{H}$-class, $\bigcup \mathcal{E}'$ is $H_n$ for every $\mathcal{E}' \subseteq \mathcal{E}$. Also, by the hypothesis on the cardinality of $Y$, the cardinality of $\mathcal{E}$ is less than the least measurable cardinal. Thus, Lemma 4.6 (for $\sigma$ disjoint families) implies that for some $G \in \mathcal{G}$, $f^{-1}(G) \neq \emptyset$. The required nonempty open set is $V = f^{-1}(G)$ since $\text{diam } f(V) < \epsilon$. 

Proof of Theorem 4.1. Immediate from Lemma 4.7 (applied to every closed subspace of $X$) and Theorem 2.3, (a)$\Rightarrow$(c').

Lemma 4.8. Let $X$ be a $t$-Baire space, $Y$ a metric space and $f : X \to Y$ a function satisfying conditions (c$_a$) and (d) of Theorem 4.2. Then $f$ is continuous at every point of a dense $G_\delta$-subset of $X$.

Proof. As in the proof of Lemma 4.7, it suffices to show that for every $\epsilon > 0$ and every nonempty open subset $U$ of $X$, there exists a nonempty open subset $V$ of $U$ such that $\text{diam } f(V) < \epsilon$.

We fix an $\epsilon > 0$ and, without loss of generality, we assume that $U = X$. We set $R = \{(x, x') : \rho(f(x), f(x')) < \epsilon / 2\}$ and let $\mu \in M(X)$ be such that $\mu(X \setminus R) = 0$. Since $f$ has $\mu$-almost separable range, there exist $y_n \in Y$, $n = 1, 2, \ldots$, such that $f(x) \in \bigcup_{n=1}^{\infty} B(y_n, \epsilon/4)$ for $\mu$-almost all $x \in X$, where in general $B(y, r)$ denotes the open ball with center $y \in Y$ and radius $r > 0$. Thus, $X \setminus \bigcup_{n=1}^{\infty} f^{-1}(B(y_n, \epsilon/4))$ is a $\mu$-null set. But, since $f^{-1}(B(y_n, \epsilon/4)) \times f^{-1}(B(y_n, \epsilon/4)) \subseteq R$, $f^{-1}(B(y_n, \epsilon/4))$ is also a $\mu$-null set for every $n$. Therefore $\mu = 0$ and Corollary 3.2 implies that $R$ is of the second category in $X \times X$. Since $R$ is also $H_n$, the interior of $R$ is nonempty. In particular, there exists $x \in X$ and a nonempty open subset $V$ of $X$ such that $V \subseteq R_x$, where $R_x$ denotes the vertical section of $R$ at $x$. It is clear that $R_x = f^{-1}(B(f(x), \epsilon/2))$, and so $f(V) \subseteq f(R_x) \subseteq B(f(x), \epsilon/2)$ and $\text{diam } f(V) < \epsilon$.

We shall now prove Theorem 4.3 using the above lemma and the following result due to Grothendieck [3].

Lemma 4.9. Let $K$ be a compact Hausdorff space and let $\mathcal{C}_p$ be the topology on $C(K)$ of pointwise convergence. Then every tight measure on $(C(K), \mathcal{C}_p)$ is concentrated on a norm separable set.
Since every tight measure on \((C(K), \mathcal{C}_p)\) is concentrated on a countable union of norm bounded \(\mathcal{C}_p\)-compact subsets of \(C(K)\), Lemma 4.9 is immediate from the following results of Grothendieck: (a) \(\mathcal{C}_p\) coincides with the weak topology on every norm bounded \(\mathcal{C}_p\)-compact subset of \(C(K)\) (cf. [23, Lemma 3.2]); and (b) every tight measure on a weakly compact set in \(C(K)\) (even in any Banach space) is concentrated on a norm separable set (cf. [14, Theorem 4.31, see also [22, Corollary 4.6]).

**Proof of Theorem 4.3.** Let \(X\) be a \(t\)-Baire space, \(K\) a compact Hausdorff space and \(f: X \to C(K)\) a continuous function when \(C(K)\) is equipped with the topology \(\mathcal{C}_p\) of pointwise convergence. By Lemma 4.8, it suffices to show that \(f\) satisfies (c,\(c_p\)) and (d) of Theorem 4.2 when \(C(K)\) is equipped with the metric \(\rho\) provided by the norm.

(d) follows from the fact that \(\rho\) is a lower semicontinuous metric on \((C(K), \mathcal{C}_p)\) (in fact, the sets in (d) are \(F_\rho\)). For (c,\(c_p\)), let \(\mu\) be a tight measure on \(X\). Then the image measure \(f(\mu)\) is a tight measure on \((C(K), \mathcal{C}_p)\). By Lemma 4.9, \(f(\mu)\) is concentrated on a norm separable set and so \(f\) has \(\mu\)-almost separable range. \(\Box\)

Condition (*) of the next lemma is derived from [5].

**Lemma 4.10.** Let \(X\) be a topological space, \(Y\) a metric space, \(f: X \to Y\) and \(\mu \in M(X)\) satisfying the following condition:

For every \(\varepsilon > 0\) and every Borel set \(A\) in \(X\) with \(\mu(A) > 0\), there exists a Borel set \(B\) with \(B = A\), \(\mu(B) > 0\) and \(\text{diam } f(B) < \varepsilon\).

Then \(f\) has \(\mu\)-almost separable range.

**Proof.** First we prove the following (which is in fact equivalent to condition (**)): for every \(\varepsilon > 0\) there exists a countable disjoint family \(\mathcal{B}_\varepsilon\) of Borel sets in \(X\) such that \(\mu(\bigcup \mathcal{B}_\varepsilon) = \mu(X)\) and, for every \(B \in \mathcal{B}_\varepsilon\), \(\mu(B) > 0\) and \(\text{diam } f(B) < \varepsilon\).

Indeed, let \(\mathcal{B}_\varepsilon\) be a maximal disjoint family of Borel sets such that \(\mu(B) > 0\) and \(\text{diam } f(B) < \varepsilon\) for every \(B \in \mathcal{B}_\varepsilon\). It follows from finiteness of \(\mu\) that \(\mathcal{B}_\varepsilon\) is countable and from the maximality of \(\mathcal{B}_\varepsilon\) and (*) that \(\mu(\bigcup \mathcal{B}_\varepsilon) = \mu(X)\).

Next we set \(N = \bigcup_{n=1}^{\infty} (X \setminus \bigcup \mathcal{B}_{1/n})\), where \(\mathcal{B}_{1/n}\) is as above for \(\varepsilon = 1/n\), and \(S = f(X \setminus N)\). Then for every \(n\), \(f(\mathcal{B}_{1/n})\) is a countable covering of \(S\) by sets of diameter less than \(1/n\) and so \(S\) is separable. Since \(\mu(N) = 0\), it follows that \(f\) has \(\mu\)-almost separable range. \(\Box\)

**Lemma 4.11.** Let \(X\) be a Hausdorff space, \(Y\) a metric space and \(f: X \to Y\) a function satisfying (d) of Theorem 4.2. Then conditions (c,\(c_i\)), \(i = 1, \ldots, 5\), of Theorem 4.2 are related as follows: (c,\(c_i\)) \(\Rightarrow\) (c,\(c_2\)) \(\Rightarrow\) (c,\(c_3\)) \(\Leftrightarrow\) (c,\(c_4\)) \(\Leftrightarrow\) (c,\(c_5\)).

**Proof.** (c,\(c_i\)) \(\Rightarrow\) (c,\(c_2\)) follows from Proposition 2.1(iv).

(c,\(c_2\)) \(\Rightarrow\) (c,\(c_4\)). Let \(\mu \in M_f(X)\). By Lemma 4.10 it suffices to show that condition (*) of that lemma holds. So let \(\varepsilon > 0\) and let \(A\) be a Borel set in \(X\) with \(\mu(A) > 0\).
A generalization of functions of the first class

Choose a nonempty compact subset $K$ of $A$ such that $\mu$ is strictly positive on $K$ (i.e., $\mu(U) > 0$ for every nonempty relatively open $U \subset K$); in particular, $K$ has the ccc. Choose also a $\sigma$-disjoint open cover $\mathcal{G}$ of $Y$ by nonempty sets of diameter less than $\varepsilon/3$ and set $\mathcal{E} = \{f^{-1}(G) \cap K: G \in \mathcal{G}\}$. It is clear that $\mathcal{E}$ is a $\sigma$-disjoint cover of $K$ and, since $f$ has the Baire property in the restricted sense, $\bigcup \mathcal{E}'$ has the Baire property in $K$ for every $\mathcal{E}' \subset \mathcal{E}$. Therefore, by Lemma 4.5(ii), for some $G \in \mathcal{G}$ the set $E = f^{-1}(G) \cap K$ is of the second category in $K$.

We fix some $y \in f(E)$. Since $\operatorname{diam} f(E) < \varepsilon/3$ the set $f^{-1}(B(y, \varepsilon/3)) \cap K$ (where $B(y, \varepsilon/3)$ is the open ball with center $y$ and radius $\varepsilon/3$) contains $E$ and so is of the second category in $K$. Also, by (d), $f^{-1}(B(y, \varepsilon/3)) \cap K$ is $H_\mu$ in $K$ since $f^{-1}(B(y, \varepsilon/3))$ is the vertical section of $\{(x, x'): \rho(f(x), f(x')) < \varepsilon/3\}$ at a point $x \in X$ with $f(x) = y$. Therefore, the set $B = \operatorname{int}_K (f^{-1}(B(y, \varepsilon/3)) \cap K)$ is nonempty and, since $\mu$ is strictly positive on $K$, $\mu(B) > 0$. It is also clear that $B$ is a Borel set in $X$, $B = A$ and $\operatorname{diam} f(B) < \varepsilon$. So $B$ is the required set.

It remains to prove that $(c_3)$, $(c_4)$ and $(c_5)$ are equivalent.

$(c_3) \Rightarrow (c_4)$ We shall prove the following stronger fact: if $\mu \in M_1(X)$ and $f$ is $\mu$-measurable, then $f$ has $\mu$-almost separable range (which follows also from [2, § 9] since $f$ is in fact $\mu$-Lusin measurable). As in the proof of $(c_2) \Rightarrow (c_4)$, it suffices to show that condition $(\ast)$ of Lemma 4.10 holds. So let $\varepsilon > 0$ and let $A$ be a Borel set in $X$ with $\mu(A) > 0$. Choose a $\sigma$-disjoint open cover $\mathcal{G}$ of $Y$ by nonempty sets of diameter less than $\varepsilon$ and set $\mathcal{E} = \{f^{-1}(G) \cap A: G \in \mathcal{G}\}$. It is clear that $\mathcal{E}$ is $\sigma$-disjoint, $\mu(\bigcup \mathcal{E}) = \mu(A) > 0$ and, since $f$ is $\mu$-measurable, $\bigcup \mathcal{E}'$ is $\mu$-measurable for every $\mathcal{E}' \subset \mathcal{E}$. Therefore, by Lemma 4.5(i), for some $G \in \mathcal{G}$ the set $f^{-1}(G) \cap A$ is not $\mu$-null and so it contains a Borel set $B$ with $\mu(B) > 0$. Since $\operatorname{diam} f(B) \leq \operatorname{diam} G < \varepsilon$, $B$ is the required set.

$(c_4) \Rightarrow (c_5)$. Let $K$ be a compact subset of $X$. Then $f|K$ satisfies $(c_4)$ and (d) and, since $K$ is hereditarily $t$-Baire, it follows from Lemma 4.8 that $f|K$ has PCP.

$(c_5) \Rightarrow (c_3)$. Let $\mu \in M_1(X)$ and let $K$ be a compact subset of $X$. Then $f|K$ has PCP and so by Theorem 2.3, $(a) \Rightarrow (c')$, and Lemma 4.4, $f|K$ is $\mu$-measurable. Since $\mu$ is concentrated on a countable union of compact sets, $f$ is $\mu$-measurable. □

**Proof of Theorem 4.2.** $(a) \Rightarrow (b)$. In view of the implications $(a) \Rightarrow (c) \Rightarrow (c')$ in Theorem 2.3 (for functions on an arbitrary space), it suffices to show that if $f: X \to Y$ satisfies $(c)$ of Theorem 2.3 then so does $f \times f: X \times X \to Y \times Y$. But this is immediate from the following fact: if $\mathcal{D}$ and $\mathcal{D}'$ are semi-open partitions of $X$, then $\{D \times D': D \in \mathcal{D}, D' \in \mathcal{D}'\}$ is a semi-open partition of $X \times X$ (consider the lexicographic ordering of the well orderings of $\mathcal{D}$ and $\mathcal{D}'$ witnessing that the partitions $\mathcal{D}$ and $\mathcal{D}'$ are semi-open). We also note that a similar proof is possible if, in the above, condition (c) of Theorem 2.3 is replaced by the condition that $f$ is fragmented (cf. also [19, Lemma 2.22]).

$(b) \Rightarrow (c_1) \land (d)$ is obvious.

We now assume (d). Then, by Lemmas 4.11 and 4.8, we have $(c_1) \Rightarrow (c_2) \Rightarrow (c_3) \Rightarrow (c_4) \Rightarrow (a)$ and the proof of the theorem is completed. □
We close with two results that are motivated by the proofs of Theorems 4.1 and 4.2. The first result shows that Theorem 4.1 holds without cardinality restrictions when \( X \) is a hereditarily Baire metric space. It also shows that the separability assumption in Theorem A of the Introduction is not necessary.

**Theorem 4.12.** Let \( X \) be a hereditarily Baire metric space, \( Y \) a metric space and \( f : X \rightarrow Y \). Then \( f \) has PCP if and only if \( f \) is of the first class.

**Proof.** As it is mentioned in the Remark following the statement of Theorem 2.3, the “only if” part is a classical theorem of Baire.

For the “if” part it suffices to show that Lemma 4.6 (hence also Lemma 4.7) holds without cardinality restrictions when \( X \) is a hereditarily Baire metric space. Suppose that this is not the case. Then, as in the proof of Lemma 4.6, we can assume that \( X = \bigcup \mathcal{E} \) where \( \mathcal{E} \) is a partition of \( X \) into sets with empty interior and \( \bigcup \mathcal{E}' \) is \( F_\sigma \) in \( X \) for every \( \mathcal{E}' \subseteq \mathcal{E} \). Since \( X \) is first countable, using an argument in the proof of Lemma 4.1 in [8], it is easy to construct by induction a countable dense in itself subset \( P \) of \( X \) such that each member of \( \mathcal{E} \) contains at most one point of \( P \). We set \( S = P \) and \( \mathcal{E}|S = \{ E \cap S : E \in \mathcal{E} \} \). Then \( S \) is a closed separable subspace of \( X \) and \( \mathcal{E}|S \) is a partition of \( S \) into sets with empty interior in \( S \) such that every union of members of this partition is \( F_\sigma \) in \( S \). We also note that \( F_\sigma \)-sets have the Baire property and are of the first category whenever their interior is empty.

Therefore, by [11, Corollary 4.2] which implies that the analogue of Lemma 4.5(ii) for separable metric spaces holds, it follows that \( S \) is of the first category in itself. This is a contradiction since \( X \) is hereditarily Baire.

**Remark.** Two other classes of spaces \( X \) for which Theorem 4.1 holds without cardinality restrictions are: (i) first countable hereditarily \( \tau \)-Baire spaces and (ii) hereditarily \( \text{ccc} \) \( H \)-complete spaces.

For (i), the proof of the “if” part of Theorem 4.1 follows the lines of the proof of Theorem 4.12 replacing \( F_\sigma \)-sets by \( H_\sigma \)-sets. However, here the desired contradiction follows from Lemma 4.6 since, by the separability of \( S \), the cardinality of \( \mathcal{E}|S \) is of course less than the least measurable cardinal.

For (ii), we show that Lemma 4.6 (hence also Lemma 4.7) holds without cardinality restrictions when \( X \) is a ccc \( H \)-complete space. Indeed, assume that for some disjoint family \( \mathcal{E} \) of subsets of \( X \) with empty interior such that \( \bigcup \mathcal{E}' \) is \( H_\sigma \) in \( X \) for every \( \mathcal{E}' \subseteq \mathcal{E} \), the set \( Z = (\bigcup \mathcal{E})^c \) is nonempty. Then \( Z \) is an \( H \)-complete space and so there exists a compactification \( \tilde{Z} \) of \( Z \) such that \( Z \) is \( H_\delta \) in \( \tilde{Z} \). Now \( \tilde{Z} \) is a ccc compact space and the desired contradiction follows from Lemma 4.5(ii) using the partition \( \{ Z \cap E : E \in \mathcal{E} \} \cup \{ \tilde{Z} \setminus Z \} \) of \( \tilde{Z} \) and the fact that \( Z \) is residual in \( \tilde{Z} \).

The last result contains some characterizations of condition (\#) of Lemma 4.10. Recall that a function \( f : X \rightarrow Y \) is said to be \( \mu \)-Lusin measurable for some \( \mu \in M(X) \), if for every \( \varepsilon > 0 \) there exists a Borel set \( C \) in \( X \) such that \( \mu(X \setminus C) < \varepsilon \) and \( f|C \) is continuous.
Proposition 4.13. Let $X$ be a topological space, $Y$ a metric space, $f: X \to Y$ and $\mu \in M(X)$. Then:

(i) Condition (*) of Lemma 4.10 holds if and only if $f$ is $\mu$-measurable and has $\mu$-almost separable range.

(ii) If $\mu \in M_2(X)$, then condition (*) of Lemma 4.10 holds if and only if $f$ is $\mu$-Lusin measurable. If instead $X$ is Hausdorff and $\mu \in M_1(X)$, the condition holds if and only if $f$ is $\mu$-measurable.

Proof. (i) Assume that condition (*) holds. By Lemma 4.10, $f$ has $\mu$-almost separable range. To prove that $f$ is $\mu$-measurable, let $G$ be a nonempty open subset of $Y$, $G \neq Y$. For every $n = 1, 2, \ldots$, let $F_n = \{ y \in Y : d(y, Y \setminus G) \geq 1/n \}$ where in general $d(A_1, A_2)$ denotes the distance of two nonempty subsets $A_1$ and $A_2$ of $Y$. Let also $\mathcal{B}_{1/n}$ be the family constructed in Lemma 4.10 for $\epsilon = 1/n$. Since $\text{diam}(f(B)) < 1/n$ for every $B \in \mathcal{B}_{1/n}$ and $d(F_n, Y \setminus G) \geq 1/n$ whenever $F_n \neq \emptyset$, it follows that for every $F \in \mathcal{B}_{1/n}$ either $B \subseteq X \setminus f^{-1}(F_n)$ or $B \subseteq f^{-1}(G)$. We set $A_n = \bigcup \{ B \in \mathcal{B}_{1/n} : B \subseteq f^{-1}(G) \}$ and observe that $A_n$ is a Borel set, $A_n \subseteq f^{-1}(G)$ and $f^{-1}(F_n) \supseteq A_n \cup (X \setminus \mathcal{B}_{1/n})$. We also have $f^{-1}(G) = \bigcup_{n=1}^{\infty} f^{-1}(F_n)$, since $G$ is open, and so $\bigcup_{n=1}^{\infty} A_n \subseteq f^{-1}(G) \subseteq \bigcup_{n=1}^{\infty} A_n \cup (X \setminus \mathcal{B}_{1/n})$ where $\mu(\bigcup_{n=1}^{\infty} (X \setminus \mathcal{B}_{1/n})) = 0$. Therefore $f^{-1}(G)$, hence also $f$, is $\mu$-measurable.

Conversely, assume that $f$ is $\mu$-measurable and has $\mu$-almost separable range. Let $\epsilon > 0$ and let $A$ be a Borel set in $X$ with $\mu(A) > 0$. Choose a separable $S \subseteq f(X)$ such that $f(x) \in S$ for $\mu$-almost all $x \in X$. Choose also $y_n \in S$, $n = 1, 2, \ldots$, such that $S \subseteq \bigcup_{n=1}^{\infty} B(y_n, \epsilon/3)$. Then $\mu(X \setminus \bigcup_{n=1}^{\infty} f^{-1}(B(y_n, \epsilon/3))) = 0$ and since $\mu(A) > 0$ there exists an $n_0$ such that $\mu(f^{-1}(B(y_{n_0}, \epsilon/3)) \cap A) > 0$. It is clear that $B = f^{-1}(B(y_{n_0}, \epsilon/3))$ is the required set for condition (*).

(ii) Let $\mu \in M_2(X)$. First we prove that condition (*) implies the following.

Claim. For every $\epsilon > 0$ there exists a closed subset $F$ of $X$ such that $\mu(X \setminus F) < \epsilon$ and $O_{f|_F}(x) < \epsilon$ for every $x \in F$, where $O_{f|_F}(x)$ denotes the oscillation of $f|_F$ at $x$.

Indeed, as in the proof of Lemma 4.10 (using also the inner regularity of $\mu$ with respect to closed sets), we construct a disjoint sequence $(B_n)$ of closed subsets of $X$ such that $\mu(\bigcup_{n=1}^{\infty} B_n) = \mu(X)$ and diam $f(B_n) < \epsilon$ for every $n$. Then we choose an $n_0$ such that $\mu(X \setminus \bigcup_{n=1}^{n_0} B_n) < \epsilon$ and set $F = \bigcup_{n=1}^{n_0} B_n$. It is clear that $F$ satisfies the claim.

Now we fix an $\epsilon > 0$ and, using the claim, we construct a sequence $(F_n)$ of closed sets such that $\mu(X \setminus F_n) < \epsilon/2^n$ and $O_{f|_{F_n}}(x) < \epsilon/2^n$ for every $x \in F_n$ and $n = 1, 2, \ldots$. Then the set $C = \bigcap_{n=1}^{\infty} F_n$ is closed, $\mu(X \setminus C) < \epsilon$ and $f|_C$ is continuous.

Conversely, assume that $f$ is $\mu$-Lusin measurable. Let $\epsilon > 0$ and let $A$ be a Borel set in $X$ with $\mu(A) > 0$. Then there exists a Borel set $C$ in $X$ such that $\mu(X \setminus C) < \mu(A)$ and $f|_C$ is continuous. In particular, we have that $\mu(A \cap C) > 0$ and $f|_A \cap C$ is continuous. Thus, there exists a family $(B_i)_{i \in I}$ of relatively open subsets of $A \cap C$.
such that \( A \cap C = \bigcup_{i \in I} B_i \) and \( \text{diam} f(B_i) < \varepsilon \) for every \( i \in I \). It now follows from the \( \tau \)-additivity of \( \mu \) that there exists an \( i_0 \in I \) such that \( \mu(B_{i_0}) > 0 \). Then the set \( B = B_{i_0} \) is a Borel set, \( B \subset A \), \( \mu(B) > 0 \) and \( \text{diam} f(B) < \varepsilon \).

Finally, the case where \( X \) is a Hausdorff space and \( \mu \in M_c(X) \) follows from (i) since, by the proof of (c_3) \( \Rightarrow \) (c_4) of Lemma 4.11, \( f \) has \( \mu \)-almost separable range whenever it is \( \mu \)-measurable. \( \square \)

**Remark.** The topological setting of part (i) of Proposition 4.13 is not necessary. In fact it follows from the proof that (i) holds for any finite measure space \( (X, \mathcal{A}, \mu) \).

We also remark that by Proposition 4.13(ii) each of the conditions (c_i), \( i = 1, \ldots, 5 \), of Theorem 4.2 can be replaced by

(c_6) Condition (*) of Lemma 4.10 holds for all \( \mu \in M_c(X) \),

as well as by

(c_7) \( f \) is \( \mu \)-Lusin measurable for all \( \mu \in M_c(X) \).

Finally, we note that functions that are measurable with respect to the \( \sigma \)-algebra generated by \( H \)-sets generalize Borel measurable functions and are similarly classified into \( H \)-classes \( \alpha, \alpha < \omega_1 \). Some results dealing with these functions will appear elsewhere.

**References**

A generalization of functions of the first class