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ORIGINAL ARTICLE

The general solutions of singular and non-singular matrix fractional time-varying descriptor systems with constant coefficient matrices in Caputo sense

Zeyad Al-Zhour

Department of Basic Sciences and Humanities, College of Engineering, University of Dammam, P.O. Box1982, Dammam 34151, Saudi Arabia

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Abstract In this paper, we generalize the time-varying descriptor systems to the case of fractional order in matrix forms. Moreover, we present the general exact solutions of the linear singular and non-singular matrix fractional time-varying descriptor systems with constant coefficient matrices in Caputo sense by using a new attractive method. Finally, two illustrated examples are also given to show our new approach.

1. Introduction

Matrix differential equations have been widely used in the stability, observability and controllability theories of differential equations, control theory, communication systems and many other fields of applied mathematics [1–9], and also recently in the following linear time-varying system [10–32]:

$$A(t)y'(t) = B(t)y(t) + C(t)u(t) : y(t_0) = y_0, \quad t \geqslant 0, \tag{1-1}$$

where $A(t) \in M_n$ is a time-varying singular or non-singular matrix function, $B(t) \in M_m$ and $C(t) \in M_{n,m}$ are time-varying analytic matrix functions, $u(t) \in M_{m,1}$ is the output vector function and $y(t) \in M_{n,1}$ is the state function vector to be solved (where $M_{m,n}$ is denoted by the set of all $m \times n$ matrices over the real number R and when $m = n$ we write M_m instead of $M_{m,n}$). This system is usually known as a non-singular (singular) descriptor system or generalized state (semi) system or system of differential-algebraic equations and plays an important role in many applications such as in electrical networks, economics, optimization problems, analysis of control systems, engineering systems, constrained mechanics aircraft and robot dynamics, biology and large-scale systems [10–15]. The linear time-varying descriptor system as in (1-1) has been studied and discussed by many researchers [16–20]. For example, controllability and observability of this system have been studied by Wang and Liao [17], Wang [18] and Campbell and et al. [19]; the linear of matrix differential inequalities of descriptor system was established by Inoue and et al. [20]; the Weierstrass–Kronecker decomposition theorem of the regular pencil was extended to the time-varying discrete-time descriptor system by Kaczorek [32] and finally, the stability of linear time-varying descriptor system has been discussed in [21–27]. Some special cases of the linear time-varying system as in (1-1) have been also investigated in [28–31]. For example, the stability for the special case of system (1-1) when $B(t) = B,\ C(t) = C$ and $A(t) = A$ are constant matrices has been discussed in [27–29]

E-mail addresses: zeyad1968@yahoo.com, zalzhour@uod.edu.sa
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and also the stability analysis for the special case of system (1-1) when $B(t+T)=B(t),\ C(t+T)=C(t)$ are periodically time-varying matrices with period T and $A(t)=A$ is a constant matrix has been studied in [24,29]. Finally, the optimal control of system as in (1-1) has been investigated in [30,31].

In addition, the topic of fractional calculus has attracted many researchers because of its several applications in various fields of applied sciences, physics and economics. For a detail survey with collections of applications in various fields, see for example [33–44] and numerous real-life problems are also modeled mathematically by systems of fractional differential equations [37,39–41,43,45–54]. Since there are many definitions of fractional derivative of order $\alpha>0$ most of them are used an integral or summation or limit form [e.g., 33,37,42,44,48,51–58]. One of the important and familiar definition for fractional derivative is Caputo operator which is defined by the following:

$$y^{\alpha}(t)=D^{\alpha}y(t)=I^{n-\alpha}D^{n}y(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{y^{(n)}(s)}{(t-s)^{\alpha-n+1}}ds, \tag{1-2}$$

where $\alpha>0,\ t>0$ and $n-1<\alpha\leqslant n\ (n\in\mathbb{N})$.

Note that the fractional derivative of $f(x)$ in the Caputo sense is defined for $0<\alpha<1$ as

$$D^{\alpha}y(t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{y'(s)}{(t-s)^{\alpha}}ds. \tag{1-3}$$

Caputo's definition has the advantage of dealing property with initial value problems in which the initial conditions are given in terms of the field variables and their integer order which is the case most physical processes.

In the present paper, we present the general exact solutions of the singular and non-singular matrix fractional time-varying descriptor systems with constant coefficient matrices in Caputo sense based on the Kronecker product and vector-operator with two illustrated examples.

2. Preliminaries and basic concepts

In this section, we study some important basic results related to the Kronecker product and Mittag–Leffler function on matrices, and fractional linear system that will be useful later in our investigation of the solutions of the linear matrix fractional time-varying descriptor systems.

Definition 2.1. Let $A=(a_{ij})\in M_{m,n}$ and $B=(b_{kl})\in M_{p,q}$ be two rectangular matrices. Then the Kronecker product of A and B is defined by [1–8,59–65]:

$$A\otimes B=\left(a_{ij}B\right)_{ij}\in M_{mp,nq}. \tag{2-1}$$

Definition 2.2. Let $A=(a_{ij})\in M_{m,n}$ be a rectangular matrix. Then the vector-operator of A is defined by [1–8,59–65]:

$$Vec(A)=(a_{11}\ a_{21}\ \dots\ a_{m1}\ a_{12}\ a_{22}\ \dots\ a_{m2}\ \dots\ a_{1n}\ a_{2n}\ \dots\ a_{mn})^T\in M_{mn,1}, \tag{2-2}$$

Lemma 2.1. *Let $A,\ B,\ C,\ D$ and X be matrices with compatible orders, and I_n be the identity matrix of order $n\times n$. Then* [1–3,7,59–65].

$$(i)\ Vec(AXB)=(B^T\otimes A)VecX, \tag{2 - 3}$$

$$(ii)\ (A\otimes B)(C\otimes D)=AC\otimes BD, \tag{2-4}$$

(iv) If f is analytic function on the region containing the eigenvalues of $A\in M_m$ such that $f(A)$ exist. Then

$$f(A\otimes I_n)=f(A)\otimes I_n \text{ and } f(I_n\otimes A)=I_n\otimes f(A). \tag{2-5}$$

Definition 2.3. The one-parameter Mittag–Leffler function $E_{\alpha}(t)$ and Mittag–Leffler matrix function $E_{\alpha}(At^{\alpha})$ are defined, respectively, for $\alpha>0$ by [33,42,51,52,56,66]:

$$E_{\alpha}(t)=\sum_{k=0}^{\infty}\frac{t^k}{\Gamma(k\alpha+1)} \text{ and } E_{\alpha}(At^{\alpha})=\sum_{k=0}^{\infty}\frac{A^k t^{\alpha k}}{\Gamma(k\alpha+1)}, \tag{2-6}$$

where $A\in M_n$ is a matrix of order $n\times n$ and $\Gamma(\cdot)$ is the Gamma function.

Lemma 2.2. *Let $A\in M_m$ be a matrix of order $m\times m$ and let $\{x_1,x_2,\dots,x_m\}$ and $\{y_1,y_2,\dots,y_m\}$ be the eigenvectors corresponding to the eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_m\}$ of A and A^T, respectively. Then the spectral decomposition of $E_{\alpha}(A)$ and $E_{\alpha}(At^{\alpha})$ are given, respectively, for $\alpha>0$ by [56]:*

$$E_{\alpha}(A)=\sum_{k=1}^{m}x_k y_k^T E_{\alpha}(\lambda_k) \text{ and } E_{\alpha}(At^{\alpha})=\sum_{k=1}^{m}x_k y_k^T E_{\alpha}(\lambda_k t^{\alpha}), \tag{2-7}$$

The list of nice properties for Mittag–Leffler matrix $E_{\alpha}(A)$ can be found in [56], and the most important properties for Mittag–Leffler matrix $E_{\alpha}(A)$ that will be used in this study are given below [56].

Theorem 2.1. *Let $A,B\in M_m$ and I_n be an identity matrix of order $n\times n$. Then for $\alpha>0$, we have [56]:*

$$(i) \text{ If } A=diag(a_{11},a_{22},\cdots,a_{mm}),\ \text{then } E_{\alpha}(A)=diag(E_{\alpha}(a_{11}),E_{\alpha}(a_{22}),\cdots,E_{\alpha}(a_{mm})), \tag{2-8}$$

$$(ii)\ E_{\alpha}(A+B)=E_{\alpha}(A)E_{\alpha}(B) \text{ if and only if } AB=BA, \tag{2-9}$$

$$(iii)\ E_{\alpha}(A\otimes I_n)=E_{\alpha}(A)\otimes I_n \text{ and } E_{\alpha}(I_n\otimes A)=I_n\otimes E_{\alpha}(A). \tag{2-10}$$

Lemma 2.3. *Let $H\in M_n$ be a given scalar matrix, $u(t)\in M_{n,1}$ be a given vector function, and $y(t)\in M_{n,1}$ be the unknown vector to be solved. Then the unique solution of the following fractional differential system [51,52,56]:*

$$y^{\alpha}(t)=Hy(t)+u(t):\quad y(0)=y_0\in M_{n,1}, \tag{2-11}$$

is given by

$$y(t)=E_{\alpha}(Ht^{\alpha})y_0+\int_0^t(t-z)^{\alpha-1}E_{\alpha}(H(t-z)^{\alpha})u(z)dz. \tag{2-12}$$

3. Main results

In this section, we formulate and present the general exact solutions of the singular and non-singular matrix fractional time-varying descriptor systems in Caputo sense based on the Kronecker product, vector-operator and Lemma 2.3 with two illustrated examples.

Problem 3.1 (*Singular Matrix Fractional Time-Varying Descriptor System*). The linear singular matrix fractional time-varying descriptor system can be formulated by

$$A(t)Y^{\alpha}(t) = B(t)Y(t) + C(t)U(t) : Y(0) = Y_0, \quad t \geqslant 0, \quad \alpha > 0, \tag{3-1}$$

where $A(t) \in M_n$ is a time-varying singular matrix function, $B(t) \in M_n$ and $C(t) \in M_n$ are time-varying analytic matrix functions, $U(t) \in M_n$ is the output matrix function and $Y(t) \in M_n$ is the state function vector to be solved. Here, we will study the general solution of (3-1) when $A(t) = A,\ B(t) = B$ and $C(t) = C$ are constant matrices, as a special case. For this case, suppose that the constant invertible matrices M and $N \in M_n$ such that:

$$A = M^{-1}\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}N^{-1}, \quad B = M^{-1}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}N^{-1},$$
$$C = M^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \text{and } Y(t) = N\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}. \tag{3-2}$$

If we partition n as $n = m + p$, then $Y_1(t) \in M_{m,n}$ and $Y_2(t) \in M_{p,n}$. This system is restricted equivalent to:

$$Y_1^{\alpha}(t) = B_{11}Y_1(t) + B_{12}Y_2(t) + C_1U(t),$$
$$0 = B_{21}Y_1(t) + B_{22}Y_2(t) + C_2U(t). \tag{3-3}$$

Note that the necessary and sufficient condition for the existence of the solution of a system (3-1) is that $B_{22}(t)$ is invertible.

General Solutions of Problem 3.1. Since $B_{22}(t)$ is an invertible matrix and then from the second equation of (3-3) we have:

$$Y_2(t) = -B_{22}^{-1}B_{21}Y_1(t) - B_{22}^{-1}C_2U(t). \tag{3-4}$$

By substituting this equation in the first equation of (3-3), we get:

$$Y_1^{\alpha}(t) = S_{B_{11}}Y_1(t) + RU(t), \tag{3-5}$$

where

$$R = -B_{12}B_{22}^{-1}C_2 + C_1, \tag{3-6}$$

and

$$S_{B_{11}} = B_{11} - B_{12}B_{22}^{-1}B_{21} \tag{3-7}$$

is called the Schur complement of B_{11} in a matrix $\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

Now, by taking $Vec(\cdot)$ of both sides of (3-5), and using (2-3) in Lemma 2.1, we have:

$$\begin{aligned} Vec\left(Y_1^{\alpha}(t)\right) &= Vec(S_{B_{11}}Y_1(t)) + Vec(RU(t)) \\ &= (I_n \otimes S_{B_{11}})\ Vec(Y_1(t)) \\ &\quad + (I_n \otimes R)Vec(U(t)). \end{aligned} \tag{3-8}$$

Now by letting $Vec\left(Y_1^{\alpha}(t)\right) = y_1^{\alpha}(t),\ Vec(Y_1(t)) = y_1(t)$ and $Vec(U(t)) = u(t)$, then (3-8) can be represented as follows:

$$y_1^{\varepsilon}(t) = (I_n \otimes S_{B_{11}})y_1(t) + (I_n \otimes R)u(t). \tag{3-9}$$

Now by using Lemma 2.3, then the vector solution of (3-9) is given by:

$$\begin{aligned} Vec(Y_1(t)) = y_1(t) &= E_{\alpha}((I_n \otimes S_{B_{11}})t^{\alpha})\ y_1(0) \\ &\quad + \int_0^t (t-z)^{\alpha-1}E_{\alpha}((I_n \otimes S_{B_{11}})(t-z)^{\alpha})((I_n \otimes R)\ u(z))\ dz \\ &= E_{\alpha}((I_n \otimes S_{B_{11}})t^{\alpha})\ Vec(Y_1(0) \\ &\quad + \int_0^t (t-z)^{\alpha-1}E_{\alpha}((I_n \otimes S_{B_{11}})(t-z)^{\alpha})((I_n \otimes R)\ Vec(U(z)))\ dz \\ &= E_{\alpha}((I_n \otimes S_{B_{11}})t^{\alpha})\ Vec(Y_1(0) \\ &\quad + \int_0^t (t-z)^{\alpha-1}E_{\alpha}((I_n \otimes S_{B_{11}})(t-z)^{\alpha})(Vec(RU(z)))\ dz, \end{aligned} \tag{3-10}$$

where R and $S_{B_{11}}$ are constant matrices as defined in (3-6) and (3-7), respectively.

Note that the relationship between $Y_1(t) \in M_{m,n}$ and $x = y_1(t) = Vec(Y_1(t)) \in M_{mn,1}$ is given by:

$$Y_1(t) = \left[x^{(1)},\ x^{(2)},\ \cdots\ x^{(n)}\right] = \begin{bmatrix} x_1 & x_{p+1} & \cdots & x_{(n-1)p+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ x_p & x_{2p} & \cdots & x_{np} \end{bmatrix} \in M_{n,p}. \tag{3-11}$$

Hence, the general solution of Problem 3.1 is given by: $Y(t) = N\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$, where $Y_1(t)$ can be easily obtained from (3-10) and (3-11); and $Y_2(t)$ is given as in (3-4).

Problem 3.2 (*Non-Singular Matrix Fractional Time-Varying Descriptor System*). The linear non-singular matrix fractional time-varying descriptor system can be formulated by:

$$A(t)Y^{\alpha}(t) = B(t)Y(t) + C(t)U(t) : Y(0) = Y_0, \quad t \geqslant 0, \quad \alpha > 0, \tag{3-12}$$

where $A(t) \in M_n$ is a time-varying non-singular matrix function, $B(t) \in M_n$ and $C(t) \in M_n$ are time-varying analytic matrix functions, $U(t) \in M_n$ is the output matrix function and $Y(t) \in M_n$ is the state function matrix to be solved. Here, we will study the general solution of (3-12) when $A(t) = A,\ B(t) = B$ and $C(t) = C$ are constant matrices, as a special case. For this case, suppose that the constant invertible matrices M and $N \in M_n$ such that:

$$A = M^{-1}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}N^{-1}, \quad B = M^{-1}\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}N^{-1},$$
$$C = M^{-1}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \quad \text{and } Y(t) = N\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}. \tag{3-13}$$

This system is restricted equivalent to:

$$Y_1^{\alpha}(t) = B_{11}Y_1(t) + B_{12}Y_2 + C_1U(t),$$
$$Y_2^{\alpha}(t) = B_{21}Y_1(t) + B_{22}Y_2(t) + C_2U(t). \tag{3-14}$$

General Solutions of Problem 3.2. By taking $Vec(\cdot)$ of both sides of (3-14), and using Lemma 2.1, we have:

$$\begin{aligned}
&Vec\big(Y_1^\alpha(t)\big) = Vec(B_{11}Y_1(t) + B_{12}Y_2(t) + C_1U(t)) \\
&= (I_n \otimes B_{11})\ Vec(Y_1(t)) + (I_n \otimes B_{12})\ Vec(Y_2(t)) \\
&\quad + (I_n \otimes C_1)\ Vec(U(t)), \\
&Vec\big(Y_2^\alpha(t)\big) = Vec(B_{21}Y_1(t) + B_{22}Y_2(t) + C_2U(t)) \\
&= (I_n \otimes B_{21}(t))\ Vec(Y_1(t)) + (I_n \otimes B_{22}(t))\ Vec(Y_2(t)) \\
&\quad + (I_n \otimes C_2(t))\ Vec(U(t)).
\end{aligned} \tag{3-15}$$

This system can be represented as:

$$\begin{bmatrix} Vec\big(Y_1^\alpha(t)\big) \\ Vec\big(Y_2^\alpha(t)\big) \end{bmatrix} = \begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix} \begin{bmatrix} Vec(Y_1(t)) \\ Vec(Y_2(t)) \end{bmatrix} + \begin{bmatrix} (I_n \otimes C_1)Vec(U(t)) \\ (I_n \otimes C_2)Vec(U(t)) \end{bmatrix}. \tag{3-16}$$

Suppose that

$$T^\alpha(t) = \begin{bmatrix} Vec\big(Y_1^\alpha(t)\big) \\ Vec\big(Y_2^\alpha(t)\big) \end{bmatrix}, \quad H = \begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix},$$

$$T(t) = \begin{bmatrix} Vec(Y_1(t)) \\ Vec(Y_2(t)) \end{bmatrix}, \quad D(t) = \begin{bmatrix} (I_n \otimes C_1)Vec(U(t)) \\ (I_n \otimes C_2)Vec(U(t)) \end{bmatrix}.$$

Now the system as in (3-16) can be rewritten as follows:

$$T^\alpha(t) = HT(t) + D(t). \tag{3-17}$$

Now by using Lemma 2.3, then the solution of (3-17) is given by:

$$T(t) = E_\alpha(H\ t^\alpha)\ T(0) + \int_0^t (t-z)^{\alpha-1} E_\alpha(H(t-z)^\alpha)\ D(z)\ dz. \tag{3-18}$$

This leads to the following general vector solution of Problem 3.2:

$$\begin{aligned}
\begin{bmatrix} Vec(Y_1(t)) \\ Vec(Y_2(t)) \end{bmatrix} &= E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix} t^\alpha\right) N^{-1} \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix} \\
&\quad + \int_0^t (t-z)^{\alpha-1} E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix} (t-z)^\alpha\right) \begin{bmatrix} (I_n \otimes C_1)Vec(U(z)) \\ (I_n \otimes C_2)Vec(U(z)) \end{bmatrix} dz. \\
&= E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix} t^\alpha\right) N^{-1} \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix} \\
&\quad + \int_0^t (t-z)^{\alpha-1} E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix} (t-z)^\alpha\right) \begin{bmatrix} Vec(C_1U(z)) \\ Vec(C_2U(z)) \end{bmatrix} dz.
\end{aligned} \tag{3-19}$$

Another special case of (3-12) is when $A(t) = A$ and $B(t) = B$ are constant matrices and $U(t) = 0$. Then the general solution of this case is given by $Y(t) = E_\alpha((A^{-1}B)t^\alpha)Y_0$.

The main problem in the solution of Problem 3.2 as in (3-19) is how to compute the following Mittag–Leffler matrix:

$$E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix}\right). \tag{3-20}$$

As a special case, if $B_{11}B_{12} = B_{12}B_{22}$ and $B_{21}B_{11} = B_{22}B_{21}$, then by using the same procedure in the proof of Theorem 2 in [56] and Theorem 2.1, we have:

$$\begin{aligned}
&E_\alpha\left(\begin{bmatrix} I_n \otimes B_{11} & I_n \otimes B_{12} \\ I_n \otimes B_{21} & I_n \otimes B_{22} \end{bmatrix}\right) \\
&= \begin{bmatrix} I_n \otimes E_\alpha(B_{11})\left\{\frac{I_n\otimes E_\alpha(B_{12}) - I_n\otimes E_\alpha(B_{21})}{2}\right\} & I_n \otimes E_\alpha(B_{11})\left\{\frac{I_n\otimes E_\alpha(B_{12}) + I_n\otimes E_\alpha(B_{21})}{2}\right\} \\ I_n \otimes E_\alpha(B_{22})\left\{\frac{I_n\otimes E_\alpha(B_{12}) + I_n\otimes E_\alpha(B_{21})}{2}\right\} & I_n \otimes E_\alpha(B_{22})\left\{\frac{I_n\otimes E_\alpha(B_{12}) - I_n\otimes E_\alpha(B_{21})}{2}\right\} \end{bmatrix}
\end{aligned} \tag{3-21}$$

Now, it is easy to get $Y_1(t)$ and $Y_2(t)$ of this case by substituting (3-21) in (3-19) and then the general solution of this problem is given by $Y(t) = N\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix}$.

Example 3.1. Consider the following linear singular matrix fractional time-varying descriptor system:

$$AY^\alpha(t) = BY(t) + CU(t) : Y(0) = Y_0, \quad t \geqslant 0, \quad \alpha > 0, \tag{3-22}$$

where

$$A = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} I_2 & I_2 \\ I_2 & I_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - & - & - & - \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \quad U(t) = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix},$$

$$Y(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - & - & - & - \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$M = N = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Since

$$S_{B_{11}} = I_2 - I_2I_2^{-1}I_2 = I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$R = -I_2 I_2^{-1}\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Then $Y_1(t) \in M_{2,4}$ and $Y_2(t) \in M_{2,4}$ by using (3-4) and (3-10), respectively, are given by:

$$\begin{aligned} Vec(Y_1(t)) &= E_\alpha((I_4 \otimes I_2)t^\alpha)\ Vec(Y_1(0) \\ &+ \int_0^t (t-z)^{\alpha-1} E_\alpha((I_4 \otimes I_2)(t-z)^\alpha)(Vec(RU(z)))\,dz \\ &= E_\alpha((I_8 t^\alpha)\ Vec(Y_1(0) + \int_0^t (t-z)^{\alpha-1} E_\alpha(I_8(t-z)^\alpha)(Vec(RU(z)))\,dz \\ &= \begin{bmatrix} E_\alpha(t^\alpha) \\ 0 \\ E_\alpha(t^\alpha) \\ 0 \\ 0 \\ E_\alpha(t^\alpha) \\ 0 \\ E_\alpha(t^\alpha) \end{bmatrix} + \int_0^t (t-z)^{\alpha-1} \begin{bmatrix} zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \\ zE_\alpha(t-z)^\alpha \end{bmatrix} dz. \end{aligned} \tag{3-23}$$

$$\begin{aligned} Y_2(t) &= -I_2^{-1} I_2 Y_1(t) - I_2 C_2 U(t) = -Y_1(t) - C_2 U(t) \\ &= -Y_1(t) - \begin{bmatrix} 0 & t & 0 & t \\ t & 0 & t & 0 \end{bmatrix}, \end{aligned} \tag{3-24}$$

where $Y_1(t) \in M_{2,4}$ is given as a vector solution as in (3-23).

Finally the general solution of system as in (3-22) is given by:

$$Y(t) = I_4 \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \in M_4. \tag{3-25}$$

As a special case of system (3-22), if $U(t) = 0$, then $Y_1(t) \in M_{2,4}$, $Y_2(t) \in M_{2,4}$ and $Y(t) \in M_4$ are given, respectively, as in 3-26, 3-27, 3-28 below:

$$Vec(Y_1(t)) = \begin{bmatrix} E_\alpha(t^\alpha) \\ 0 \\ E_\alpha(t^\alpha) \\ 0 \\ 0 \\ E_\alpha(t^\alpha) \\ 0 \\ E_\alpha(t^\alpha) \end{bmatrix}.$$

Now from (3-11), we get:

$$Y_1(t) = \begin{bmatrix} E_\alpha(t^\alpha) & 0 & E_\alpha(t^\alpha) & 0 \\ 0 & E_\alpha(t^\alpha) & 0 & E_\alpha(t^\alpha) \end{bmatrix} \in M_{2,4}. \tag{3-26}$$

$$\begin{aligned} Y_2(t) &= -Y_1(t) - \begin{bmatrix} 0 & t & 0 & t \\ t & 0 & t & 0 \end{bmatrix} \\ &= \begin{bmatrix} -E_\alpha(t^\alpha) & -t & -E_\alpha(t^\alpha) & -t \\ -t & -E_\alpha(t^\alpha) & -t & -E_\alpha(t^\alpha) \end{bmatrix} \in M_{2,4}. \end{aligned} \tag{3-27}$$

$$Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} E_\alpha(t^\alpha) & 0 & E_\alpha(t^\alpha) & 0 \\ 0 & E_\alpha(t^\alpha) & 0 & E_\alpha(t^\alpha) \\ -E_\alpha(t^\alpha) & -t & -E_\alpha(t^\alpha) & -t \\ -t & -E_\alpha(t^\alpha) & -t & -E_\alpha(t^\alpha) \end{bmatrix} \in M_4. \tag{3-28}$$

Example 3.2. Consider the following linear non-singular matrix fractional time-varying descriptor system:

$$A(t)Y^\alpha(t) = B(t)Y(t) : Y(0) = Y_0, \quad t \geqslant 0, \quad \alpha > 0, \tag{3-29}$$

where

$$M = N = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, A(t) = \begin{bmatrix} I_2(t) & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} t & 0 & 0 & 0 \\ 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \end{bmatrix},$$

$$B(t) = \begin{bmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{bmatrix} = \begin{bmatrix} -t & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - & - & - & - \\ 1 & 0 & -t & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$Y(0) = \begin{bmatrix} Y_1(0) \\ Y_2(0) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ - & - & - & - \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Since $B_{11}B_{12} = B_{12}B_{22}$ and $B_{21}B_{11} = B_{22}B_{21}$, then by applying (3-19), (3-21) and Theorem 2.1, we get:

$$\begin{aligned} \begin{bmatrix} Vec(Y_1(t)) \\ Vec(Y_2(t)) \end{bmatrix} &= E_\alpha\left(\begin{bmatrix} I_2 \otimes B_{11}(t) & I_2 \otimes I_2 \\ I_2 \otimes I_2 & I_2 \otimes B_{22}(t) \end{bmatrix} t^\alpha\right) \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix} \\ &= \begin{bmatrix} I_2 \otimes E_\alpha(B_{11}(t))\left\{\frac{I_2 \otimes E_\alpha(I_2) - I_2 \otimes E_\alpha(I_2)}{2}\right\} & I_2 \otimes E_\alpha(B_{11}(t))\left\{\frac{I_2 \otimes E_\alpha(I_2)) + I_2 \otimes E_\alpha(I_2)}{2}\right\} \\ I_2 \otimes E_\alpha(B_{22}(t))\left\{\frac{I_2 \otimes E_\alpha(I_2) + I_2 \otimes E_\alpha(I_2)}{2}\right\} & I_2 \otimes E_\alpha(B_{22}(t))\left\{\frac{I_2 \otimes E_\alpha(I_2) - I_2 \otimes E_\alpha(I_2)}{2}\right\} \end{bmatrix} t^\alpha \times \begin{bmatrix} Vec(Y_1(0)) \\ Vec(Y_2(0)) \end{bmatrix}. \end{aligned}$$

Now,

$$Vec(Y_1(t)) = (I_2 \otimes E_\alpha(B_{11}(t)))(I_2 \otimes E_\alpha(I_2 t^\alpha))\ Vec(Y_2(0))$$
$$= (I_2 \otimes E_\alpha(B_{11}(t))E_\alpha(I_2)t^\alpha)\ Vec(Y_2(0))$$
$$= Vec\{(E_\alpha(B_{11}(t))E_\alpha(I_2)t^\alpha)\ Y_2(0)\}$$

That is by using (2-9), we have

$$\begin{aligned} Y_1(t) &= (E_\alpha(B_{11}(t))E_\alpha(I_2)t^\alpha)Y_2(0) \\ &= \{E_\alpha((B_{11}(t)+E_\alpha(I_2))t^\alpha)\}\ Y_2(0) \\ &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) \end{bmatrix} Y_2(0) \\ &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 & E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) & 0 & E_\alpha(2t^\alpha) \end{bmatrix} \in M_{2,4} \end{aligned} \tag{3-30}$$

Similarly, we have

$$\begin{aligned} Y_2(t) &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) \end{bmatrix} Y_1(0) \\ &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 & E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) & 0 & E_\alpha(2t^\alpha) \end{bmatrix} \in M_{2,4}. \end{aligned} \tag{3-31}$$

Hence, the general solutions of system (3-29) are given by:

$$\begin{aligned} Y(t) &= I_4 \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} \\ &= \begin{bmatrix} E_\alpha(t^\alpha - t^{\alpha+1}) & 0 & E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) & 0 & E_\alpha(2t^\alpha) \\ E_\alpha(t^\alpha - t^{\alpha+1}) & 0 & E_\alpha(t^\alpha - t^{\alpha+1}) & 0 \\ 0 & E_\alpha(2t^\alpha) & 0 & E_\alpha(2t^\alpha) \end{bmatrix} \in M_4. \end{aligned} \tag{3-32}$$

Note that if $A(t) = A$ and $B(t) = B$ are constant matrices in Example 3.2, then the general solution given by $Y(t) = E_\alpha((A^{-1}B)t^\alpha)Y_0$.

4. Conclusion

The general exact solutions of the singular and non-singular matrix fractional time-varying descriptor systems in Caputo sense with constant coefficient matrices are presented by a new attractive method with two illustrated examples. How to find the general solutions of these problems with non-constant coefficient matrices and also how to find the sufficient conditions, stability, controllability and observability of these problems still require further research.

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