Nonnegative doubly periodic solutions for nonlinear telegraph system

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Abstract

This paper deals with the nonnegative doubly periodic solutions for nonlinear telegraph system

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + c_1 \frac{\partial u}{\partial t} + a_{11}(t,x)u + a_{12}(t,x)v &= b_1(t,x)f(t,x,u,v), \\
\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + c_2 \frac{\partial v}{\partial t} + a_{21}(t,x)u + a_{22}(t,x)v &= b_2(t,x)g(t,x,u,v),
\end{align*}
\]

where \(c_i > 0\) is a constant, \(a_{11}, a_{22}, b_1, b_2 \in C(\mathbb{R}^2, \mathbb{R}^+)\), \(a_{12}, a_{21} \in C(\mathbb{R}^2, \mathbb{R}^-)\), \(f, g \in C(\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)\), and \(a_{ij}, b_i, f, g\) are \(2\pi\)-periodic in \(t\) and \(x\). We show the existence and multiplicity results when \(0 \leq a_{ii}(t,x) \leq c_i^2\) and \(f, g\) are superlinear or sublinear on \((u, v)\) by using the fixed point theorem in cones.

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1. Introduction

In this paper we are concerned with the existence and multiplicity of solutions for the nonlinear telegraph system

\[
\begin{align*}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + c_1 \frac{\partial u}{\partial t} + a_{11}(t,x)u + a_{12}(t,x)v &= b_1(t,x)f(t,x,u,v), \\
\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} + c_2 \frac{\partial v}{\partial t} + a_{21}(t,x)u + a_{22}(t,x)v &= b_2(t,x)g(t,x,u,v),
\end{align*}
\]

with doubly periodic boundary conditions

\[
\begin{align*}
u(t + 2\pi, x) &= u(t, x + 2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \\
v(t + 2\pi, x) &= v(t, x + 2\pi) = v(t, x), \quad (t, x) \in \mathbb{R}^2.
\end{align*}
\]

The existence of a doubly periodic solution for a single telegraph equation is studied by many authors when the nonlinearity is bounded or has linear growth, see [1–5]. The first maximum principle for linear telegraph equations...
Doubly $2\pi$-periodic solutions of the linear telegraph equation

$$u_{tt} - u_{xx} + cu_t + \lambda u = h(t, x), \quad (t, x) \in \mathbb{R}^2,$$

holds if and only if $\lambda \in (0, \nu(c))$ and $\nu(c) \in (\frac{c^2}{4}, \frac{c^2}{2} \pm \frac{1}{2})$ is a constant which cannot be concretely determined. This maximum principle on the torus $\mathbb{T}^2$ (here $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ denotes the unit circle) was used in [5] to develop a method of upper and lower solutions for the doubly periodic solutions of the nonlinear telegraph equation

$$u_{tt} - u_{xx} + cu_t + \nu(c)u = F(t, x, u), \quad (t, x) \in \mathbb{R}^2,$$

when the function $u \mapsto F(t, x, u) + \nu(c)u$ is monotonically nondecreasing. Afterwards in [6], Mawhin, Ortega and Robles-Perez built a maximum principle for the solution $u(t, x)$ of the telegraph equation which is bounded and $2\pi$-periodic with respect to $x$. And a similar method of upper and lower solutions was developed when the function $u \mapsto F(t, x, u) + \nu(c)u$ is monotonically nondecreasing. Lately, these authors in [7] have extended their results in [6] to the telegraph equations in space dimensions two or three. Another maximum principle for the telegraph equation can be found in [9]. In [8], by using fixed point theorem in cones, Li obtained the existence results of positive doubly periodic solutions for the nonlinear equation

$$u_{tt} - u_{xx} + a(t, x)u = b(t, x)f(t, x, u), \quad (t, x) \in \mathbb{R}^2,$$

where $c > 0$ is a constant, $a, b \in C(\mathbb{R}^2, \mathbb{R}^+)$, $f \in C(\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$, and $a, b, f$ are $2\pi$-periodic in $t$ and $x$, $0 \leq a(t, x) \leq c^2/4$ and $f$ is either superlinear or sublinear on $u$ on the base of maximum principle in [5].

On the other hand, there are many papers connected with the existence and multiplicity of $t$-periodic solutions for the nonlinear wave systems and telegraph-wave coupled systems, such as [11–13] and the references therein. And also many authors deal with second-order ordinary differential systems and second-order elliptic systems, see [14–16]. Inspire by those papers, our interest here is in the existence and multiplicity of nonnegative doubly periodic solutions for the nonlinear telegraph systems (1).

The paper is organized as follows: In Section 2, we make some preliminaries; in Sections 3, 4, we prove the existence and multiplicity results of (1).

2. Preliminaries

Let $\mathbb{T}^2$ be the torus defined as

$$\mathbb{T}^2 = (\mathbb{R}/2\pi\mathbb{Z}) \times (\mathbb{R}/2\pi\mathbb{Z}).$$

Doubly $2\pi$-periodic functions will be identified to be functions defined on $\mathbb{T}^2$. We use the notations

$$L^p(\mathbb{T}^2), C(\mathbb{T}^2), C^a(\mathbb{T}^2), D(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \ldots$$

to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

By a doubly periodic solution of (1) we mean that a $(u, v) \in L^1(\mathbb{T}^2) \times L^1(\mathbb{T}^2)$ satisfies (1) in the distribution sense, i.e.

$$\begin{cases}
\int_{\mathbb{T}^2} \varphi_{tt} u - \varphi_{xx} - c_1 \varphi_t + a_{11} \varphi + a_{12} \int_{\mathbb{T}^2} v \varphi = \int_{\mathbb{T}^2} b_1 f \varphi, \\
\int_{\mathbb{T}^2} \varphi_{tt} v - \varphi_{xx} - c_2 \varphi_t + a_{22} \varphi + a_{21} \int_{\mathbb{T}^2} u \varphi = \int_{\mathbb{T}^2} b_2 g \varphi,
\end{cases} \quad \forall \varphi \in D(\mathbb{T}^2).$$

First, we consider the linear equation

$$u_{tt} - u_{xx} + c_1 u_t - \lambda_i u = h_i(t, x), \quad \text{in } D'(\mathbb{T}^2), \quad (i = 1, 2),$$

where $c_i > 0$, $\lambda_i \in \mathbb{R}$, $h_i \in L^1(\mathbb{T}^2)$ ($i = 1, 2$).

Let $\mathcal{L}_{\lambda_i}$ be the differential operator

$$\mathcal{L}_{\lambda_i} = u_{tt} - u_{xx} + c_1 u_t - \lambda_i u,$$
acting on functions on $\mathbb{T}^2$. Following the discussion in [5,8], we know that if $\lambda_i < 0$, $\mathcal{L}_{\lambda_i}$ has the resolvent $R_{\lambda_i}$

$$R_{\lambda_i} : L^1(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2), \quad h_i \mapsto u_i,$$

where $u_i$ is the unique solution of (3), and the restriction of $R_{\lambda_i}$ on $L^p(\mathbb{T}^2) (1 < p < \infty)$ or $C(\mathbb{T}^2)$ is compact. In particular, $R_{\lambda_i} : C(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ is a completely continuous operator.

For $\lambda_i = -c_i^2/4$, the Green function of the differential operator $\mathcal{L}_{\lambda_i}$ is explicitly expressed, which has been obtained in [5]. We denote it by $G_i(t, x)$. By Lemma 5.1 in [5], $G_i(t, x) \in L^\infty$, is doubly $2\pi$-periodic. So the unique solution of (3) can be represented by the convolution product

$$u_i(t, x) = (R_{\lambda_i} h_i)(t, x) = \int_{\mathbb{T}^2} G_i(t, x) h_i(s, y) ds dy. \quad (4)$$

The expression of $G_i(t, x)$ will be given in what follows.

Let $D_i = R^2 \setminus \xi_i$, where $\xi_i$ is the family of lines

$$x \pm t = 2k\pi, \quad k \in \mathbb{Z}.$$ 

Let $D_i^{(mn)}$ denote the connected component of $D_i$ with center at $(m\pi, n\pi)$, where $m + n$ is an odd number. By periodicity, the value of $G_i$ on $D_i^{(10)}$ and $D_i^{(01)}$ completely determines the value of $G_i$ on the whole set $D_i$. In $D_i^{(10)}$ and $D_i^{(01)}$, $G_i(t, x)$ is explicitly given by

$$G_i(t, x) = \begin{cases} 
\gamma_i^{(10)} e^{-c_i t/2}, & (t, x) \in D_i^{(10)}, \\
\gamma_i^{(01)} e^{-c_i t/2}, & (t, x) \in D_i^{(01)},
\end{cases}$$

where

$$\gamma_i^{(10)} = (1 + e^{-c_i \pi})/2(1 - e^{-c_i \pi})^2, \quad \gamma_i^{(01)} = e^{-c_i \pi}/(1 - e^{-c_i \pi})^2,$$

see Lemma 5.2 in [5].

From the definition of $G_i(t, x)$, we have

$$G_i := \text{ess inf} G_i(t, x) = e^{-3c_i \pi/2}/(1 - e^{-c_i \pi})^2,$$

$$\overline{G_i} := \text{ess sup} G_i(t, x) = (1 + e^{-c_i \pi})/2(1 - e^{-c_i \pi})^2.$$ 

Let $h \in L^1(\mathbb{T}^2)$ with $h(t, x) \geq 0$ for a.e. $(t, x) \in \mathbb{T}^2$. Then from (4) we have, $R_{\lambda_i} h$ satisfies the positive estimate

$$\|h\|_{L^1(\mathbb{T}^2)} \leq (R_{\lambda_i} h)(t, x) \leq \overline{G_i} \|h\|_{L^1(\mathbb{T}^2)}. \quad (5)$$

Let $X$ denote the Banach space $C(\mathbb{T}^2)$. Then $X$ is an ordered Banach space with cone

$$K_0 = \{ u \in X \mid u(t, x) \geq 0, \forall (t, x) \in \mathbb{T}^2 \}.$$ 

Now, we consider Eq. (3) when $-\lambda_i$ is replaced by $a_{ii}(t, x)$. In [8], the author has proved the following unique existence and positive estimate result.

**Lemma 1.** $h_i \in L^1, X$ is the Banach space $C(\mathbb{T}^2)$, Then Eq. (3) has a unique solution $u_i = P_i h_i$, $P_i : L^1 \rightarrow X$ is a linear bounded operator with the following properties,

(i) $P_i : C(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ is a completely continuous operator;

(ii) If $h_i > 0$, a.e $(t, x) \in \mathbb{T}^2$, $P_i h_i$ has the positive estimate

$$\frac{G_i}{\overline{G_i} \|a_{ii}\|_{L^1}} \|h_i\|_{L^1} \leq (P_i h_i) \leq \frac{\overline{G_i}}{G_i} \|h_i\|_{L^1}. \quad (6)$$

To prove our results, we need the following fixed-point theorem of cone mapping.
Lemma 2. (See Guo and Lakshmikantham [10].) Let $E$ be a Banach space, and let $K \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are open subsets of $E$ with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either

(i) $\|T\tilde{u}\| \leq \|\tilde{u}\|$, $\tilde{u} \in K \cap \partial \Omega_1$ and $\|T\tilde{u}\| \geq \|\tilde{u}\|$, $\tilde{u} \in K \cap \partial \Omega_2$;

(ii) $\|T\tilde{u}\| \geq \|\tilde{u}\|$, $\tilde{u} \in K \cap \partial \Omega_1$ and $\|T\tilde{u}\| \leq \|\tilde{u}\|$, $\tilde{u} \in K \cap \partial \Omega_2$.

Then $T$ has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence result

We assume the following conditions throughout:

(H1) $a_{ii} \in C(T^2)$, $0 \leq a_{ii}(t, x) \leq \frac{\varepsilon^2}{4}$ for $(t, x) \in T^2$, and $\int_{T^2} a_{ii}(t, x) \, dt \, dx > 0$, $a_{12}, a_{21} \in C(T^2, R^-)$;

(H2) $b_i \in C(T^2)$, $0 \leq b_i(t, x)$ for $(t, x) \in T^2$, and $\int_{T^2} b_i(t, x) \, dt \, dx > 0$;

(H3) $f, g \in C(T^2 \times R^+ \times R^+ \times R^+)$.

For convenience, we introduce the following notations:

\[
\begin{align*}
 f_0 &= \lim_{(u, v) \to 0} \left( f(t, x, u, v) / (u + v) \right), & g_0 &= \lim_{(u, v) \to 0} \left( g(t, x, u, v) / (u + v) \right), \\
 f_\infty &= \lim_{(u, v) \to \infty} \left( f(t, x, u, v) / (u + v) \right), & g_\infty &= \lim_{(u, v) \to \infty} \left( g(t, x, u, v) / (u + v) \right), \\
 A_1 &= \frac{G_1 a_{11} L_1}{2G_1 b_1 L_1}, & A_2 &= \frac{G_2 a_{12} L_1}{2G_2 b_2 L_1}, \\
 B_1 &= \frac{(1 + G_1 a_{12} L_1) G_1}{G_3 b_1 L_1}, & B_2 &= \frac{(1 + G_2 a_{21} L_1) G_2}{G_3 b_2 L_1}.
\end{align*}
\]

We have the following existence result.

Theorem 3.1. Assume (H1)–(H3) hold. $G_1 a_{11} L_1 \geq 2G_1 a_{12} L_1, G_2 a_{22} L_1 \geq 2G_2 a_{21} L_1$. Then in each case of the following conditions

(i) $f_0 \leq A_1, g_0 \leq A_2$, and $f_\infty \geq B_1$ or $g_\infty \geq B_2$;

(ii) $f_\infty \leq A_1, g_\infty \leq A_2$ and $f_0 \geq B_1$ or $g_0 \geq B_2$,

system (1) has at least one nonnegative doubly periodic solutions.

Proof. Set

\[
\delta = \min \left\{ \frac{G_2^2 a_{11} L_1}{G_1}, \frac{G_2^2 a_{22} L_1}{G_2} \right\}.
\]

From condition (H1) and the definitions of $G_1$ and $\overline{G}_1$, it can be obtained that $0 < \delta < 1$.

Let $E$ denote the Banach space $C(T^2) \times C(T^2)$ with the norm $\|\tilde{u}\|_E = \|u\| + \|v\|$, $\|u\| = \max_{(t, x) \in T^2} |u(t, x)|$, $\tilde{u} = (u, v) \in E$. The cone is defined as

\[
K = \{ \tilde{u} = (u, v) \in E: u \geq 0, \ v \geq 0, \ u + v \geq \delta \|\tilde{u}\|_E \}.
\]

By $P_i$ $(i = 1, 2) : L^1 \to C(T^2)$, we denote the solution operators as follows, respectively

\[
\begin{align*}
 u_{tt} - u_{xx} + c_1 u_t + a_{11}(t, x) u &= h_1(t, x), \\
 v_{tt} - v_{xx} + c_2 v_t + a_{22}(t, x) v &= h_2(t, x).
\end{align*}
\]
By Lemma 1, (H2) and (H3) we have
\[ I_n \text{ in the same way, we have} \]
so we have
\[ T(\tilde{u}) = T(u, v) = (Q_1(u, v), Q_2(u, v)). \]

By Lemma 1, (H2) and (H3) we have
\[ Q_1(u, v) = P_1(-a_{12}v + b_1(t, x) f(t, x, u, v)) \]
\[ Q_2(u, v) = P_2(-a_{21}u + b_1(t, x) g(t, x, u, v)) \]
\[ T(u, v) = (Q_1(u, v), Q_2(u, v)), \quad K \rightarrow E. \]

We have the conclusion that \( T : E \rightarrow E \) is completely continuous and \( T(K) \subseteq K \). The complete continuity is obvious by Lemma 1, we show that \( T(K) \subseteq K \).

\( \forall \tilde{u} \in K \), we have
\[ T(\tilde{u}) = T(u, v) = (Q_1(u, v), Q_2(u, v)). \]

By Lemma 1, (H2) and (H3) we have
\[ Q_1(u, v) = P_1(-a_{12}v + b_1(t, x) f(t, x, u, v)) \geq G_1 \| -a_{12}v + b_1 f \|_{L^1}. \]
\[ \| Q_1(u, v) \| = \| P_1(-a_{12}v + b_1(t, x) f(t, x, u, v)) \| \leq \frac{G_1}{G_1 \| a_{11} \|_{L^1}} \| -a_{12}v + b_1 f \|_{L^1}. \]

So we get
\[ Q_1(u, v) \geq \frac{G_1^2 \| a_{11} \|_{L^1}}{G_1} \| Q_1(u, v) \| \geq \delta \| Q_1(u, v) \|. \]

In the same way, we also have
\[ Q_2(u, v) \geq \frac{G_2^2 \| a_{22} \|_{L^1}}{G_2} \| Q_2(u, v) \| \geq \delta \| Q_2(u, v) \|. \]

So
\[ Q_1(u, v) + Q_2(u, v) \geq \delta (\| Q_1(u, v) \| + \| Q_2(u, v) \|), \]

namely
\[ T(\tilde{u}) = T(u, v) = (Q_1(u, v), Q_2(u, v)) \in K. \]

Thus, \( T(K) \subseteq K \).

To prove the results we verify in what follows that the conditions of Lemma 2 are satisfied.

Let \( \Omega_i = \{ \tilde{u} = (u, v) \in E: \| \tilde{u} \|_E \leq r_i \} \) \( (i = 1, 2) \), \( 0 \in \Omega_1, \tilde{r}_1 \subseteq \Omega_2 \).

**Case 1.** Assume condition (i) hold. Since \( f_0 \leq A_1, g_0 \leq A_2 \), by the definition of \( f_0, g_0 \), we may choose \( \eta > 0 \) such that \( f(t, x, u, v) \leq A_1(u + v), g(t, x, u, v) \leq A_2(u + v) \forall (t, x) \in \mathbb{T}^2, 0 \leq u + v \leq \eta \).

Choose \( r_1 \in (0, \eta) \), we now prove that
\[ \| T\tilde{u} \|_E \leq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_1. \] (7)

By Lemma 1 and the above inequality, we have
\[ \| Q_1(u, v) \| = \| P_1(-a_{12}v + b_1 f(t, x, u, v)) \| \]
\[ \leq \frac{G_1}{G_1 \| a_{11} \|_{L^1}} \| -a_{12}v + b_1 f \|_{L^1} \]
\[ \leq \frac{G_1}{G_1 \| a_{11} \|_{L^1}} (\| b_1 \|_{L^1} \| f \| + \| a_{12} \|_{L^1} \| v \|) \]
\[ \leq \frac{G_1}{G_1 \| a_{11} \|_{L^1}} (A_1 \| u + v \| \| b_1 \|_{L^1} + \| a_{12} \|_{L^1} \| v \|) \]
\[
\begin{align*}
&\leq \frac{G_1}{\|a_{11}\|_{L^1}} \left[ \|b_1\|_{L^1} A_1 (\|u\| + \|v\|) + \|a_{12}\|_{L^1} (\|u\| + \|v\|) \right] \\
&\leq \frac{G_1}{\|a_{11}\|_{L^1}} \left[ \|b_1\|_{L^1} A_1 + \|a_{12}\|_{L^1} \right] (\|u\| + \|v\|) \\
&\leq \frac{1}{2} (\|u\| + \|v\|),
\end{align*}
\]

namely
\[
\|Q_1(u, v)\| \leq \frac{1}{2} (\|u\| + \|v\|).
\]

In the same way, we also have
\[
\|Q_2(u, v)\| \leq \frac{1}{2} (\|u\| + \|v\|).
\]

So we have
\[
\|T\tilde{u}\|_E = \|Q_1(u, v)\| + \|Q_2(u, v)\| \leq \|u\| + \|v\| = \|\tilde{u}\|_E.
\]

On the other hand, since \(f_\infty \geq B_1\), or \(g_\infty \geq B_2\), by the definition of \(f_\infty, g_\infty\), there exists \(R > 0\) such that
\[
f(t, x, u, v) \geq B_1(u + v) \quad \text{or} \quad g(t, x, u, v) \geq B_2(u + v), \quad \forall (t, x) \in \mathbb{T}^2, \ u + v \geq R.
\]

Choosing \(r_2 = \max\{R/\delta, 2r_1\}\), we now prove
\[
\|T\tilde{u}\|_E \geq \|\tilde{u}\|_E. \tag{8}
\]

If \(\tilde{u} \in K \cap \partial\Omega_2, u + v \geq \delta \|\tilde{u}\|_E \geq R\) and \(f(t, x, u, v) \geq B_1(u + v), \forall (t, x) \in \mathbb{T}^2, \ u + v \geq R\), then
\[
h_1 = -a_{12}v + b_1 f \geq -a_{12}v + B_1 b_1 (u + v) \geq -a_{12}v + B_1 b_1 \delta \|\tilde{u}\|_E, \\
\|h_1\|_{L^1} = -a_{12}v + b_1 f \|_{L^1} \geq B_1 \delta \|b_1\|_{L^1} \|\tilde{u}\|_E - \|a_{12}\|_{L^1} \|\tilde{u}\|_E.
\]

By Lemma 1, we have
\[
\|Q_1(u, v)\| = \|P_1 h_1\| \geq \frac{G_1}{\|a_{11}\|_{L^1}} \geq \frac{G_1}{\|a_{11}\|_{L^1}} (B_1 \delta \|b_1\|_{L^1} - \|a_{12}\|_{L^1}) \|\tilde{u}\|_E,
\]

namely
\[
\|Q_1(u, v)\| \geq \|\tilde{u}\|.
\]

If \(\tilde{u} \in K \cap \partial\Omega_2, u + v \geq \delta \|\tilde{u}\|_E \geq R\) and \(g(t, x, u, v) \geq B_2(u + v), \forall (t, x) \in \mathbb{T}^2, \ u + v \geq R\), in the same way, we also have
\[
\|Q_2(u, v)\| \geq \|\tilde{u}\|.
\]

Thus (8) holds.

By Lemma 2, \(T\) has a fixed point in \(K \cap (\overline{\Omega_2} \setminus \Omega_1)\)
\[
\tilde{u} = (u, v) \in E: \quad u \geq 0, \ v \geq 0, \ u + v \geq \delta \|\tilde{u}\|_E \geq \delta r_1 > 0.
\]

So \(\tilde{u} = (u, v)\) is a negative doubly periodic solution of (1).

**Case 2.** Assume (ii) holds. Since \(f_0 \geq B_1\) or \(g_0 \geq B_2\), there exists \(\tilde{\eta} > 0\) such that
\[
f(t, x, u, v) \geq B_1(u + v) \quad \text{or} \quad g(t, x, u, v) \geq B_2(u + v), \quad \forall (t, x) \in \mathbb{T}^2, \ 0 \leq u + v \leq \tilde{\eta}.
\]

Let \(r_1 \in (0, \tilde{\eta})\), with a similar argument of (8), we can prove that
\[
\|T\tilde{u}\|_E \geq \|\tilde{u}\|_E. \quad \tilde{u} \in K \cap \partial\Omega_1.
\]

On the other hand, since \(f_\infty \leq B_1, g_\infty \leq B_2\), by the definition of \(f_\infty, g_\infty\), there exists \(\overline{R} > 0\) such that
\[ f(t,x,u,v) \leq B_1(u + v), \quad g(t,x,u,v) \leq B_1(u + v), \quad \forall (t,x) \in \mathbb{T}^2, \quad u + v \geq \overline{R}. \]

Choose \( r_2 = \max\{\overline{R}/\delta, 2r_1\} \), we now prove
\[
\|T\tilde{u}\|_E \leq \|	ilde{u}\|_E, \quad \tilde{u} \in K \cap \partial \Omega_2.
\]

If \( \tilde{u} \in K \cap \partial \Omega_2, u + v \geq \delta \|	ilde{u}\|_E \geq \overline{R} \), and therefore
\[
h_1(t,x) = -a_{12} v + b_1(t,x) f(t,x,u,v) \leq -a_{12} v + b_1 A_1(u + v).
\]

By Lemma 1 and the above inequality, we have
\[
\|Q_1(u,v)\| = \|P_1 h_1\| = \|P_1 (-a_{12} v + b_1 f(t,x,u,v))\| \leq \|P_1\|_{L^1} \leq \frac{G_1}{G_1 \|a_{11}\|_{L^1}} \|a_{12} v + b_1 f\|_{L^1} \leq \frac{G_1}{G_1 \|a_{11}\|_{L^1}} \|a_{12} v + b_1 f\|_{L^1} \leq \frac{G_1}{G_1 \|a_{11}\|_{L^1}} \left( A_1 \|u + v\||b_1||_{L^1} + \|a_{12}\|_{L^1} \|v\| \right) \leq \frac{G_1}{G_1 \|a_{11}\|_{L^1}} \left( \|b_1||_{L^1} A_1 + \|a_{12}\|_{L^1} \right) \left( \|u\| + \|v\| \right) \leq \frac{1}{2} \left( \|u\| + \|v\| \right).
\]

In a similar way, we also have
\[
\|Q_2(u,v)\| \leq \frac{1}{2} \left( \|u\| + \|v\| \right).
\]

So
\[
\|T\tilde{u}\|_E = \|Q_1(u,v)\| + \|Q_2(u,v)\| \leq \|u\| + \|v\| = \|	ilde{u}\|_E.
\]

By Lemma 2, \( T \) has a fixed point in \( K \cap (\overline{\Omega}_2 \setminus \Omega_1) \)
\[
\tilde{u} = (u,v) \in E: \quad u \geq 0, \quad v \geq 0, \quad u + v \geq \delta \|	ilde{u}\|_E \geq \delta r_1 > 0.
\]

So \( \tilde{u} = (u,v) \) is a nonnegative doubly periodic solution of (1).

The proof of Theorem 1 is complete. \( \square \)

4. Multiplicity theorems

In this section we consider the multiplicity of solutions. The main results are the following theorems.

**Theorem 4.1.** Assume (H1)–(H3) hold. \( G_1 \|a_{11}\|_{L^1} \geq 2G_1 \|a_{12}\|_{L^1}, \quad G_2 \|a_{22}\|_{L^1} \geq 2G_2 \|a_{21}\|_{L^1} \). In addition, assume that there exist constants \( r > 0, \quad 0 < \sigma < 1 \) such that

(a) \( f_0 = g_0 = f_\infty = g_\infty = 0 \);
(b) \( f(t,x,u,v) \geq Mr \text{ or } g(t,x,u,v) \geq Mr \), for \( \sigma r \leq u + v \leq r \), where \( M = \max \left\{ \frac{1+G_1 \|a_{12}\|_{L^1}}{G_1 \|b_1\|_{L^1}}, \frac{1+G_2 \|a_{21}\|_{L^1}}{G_2 \|b_2\|_{L^1}} \right\} \).

Then (1) has at least two nonnegative doubly periodic solutions.

**Proof.** Let \( A = \min\{A_1, A_2\} \).
Step 1. Since \( f_0 = g_0 = 0 \), there exists \( r_1 \in (0, \sigma r] \) such that
\[
f(t, x, u, v) \leq A(u + v), \quad g(t, x, u, v) \leq A(u + v), \quad \text{for } 0 \leq u + v \leq r_1.
\]
Let \( \Omega_1 = \{ \tilde{u} = (u, v) \in E : \| \tilde{u} \|_E < r_1 \} \). By the proof of Theorem 1, we have
\[
\| T \tilde{u} \|_E \leq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_1.
\] (9)

Step 2. Since \( f_\infty = g_\infty = 0 \), there exists \( R > r_1 \) such that
\[
f(t, x, u, v) \leq A(u + v), \quad g(t, x, u, v) \leq A(u + v), \quad \text{for } u + v \geq R.
\]
Let \( r_2 = R/\delta \), \( \Omega_2 = \{ \tilde{u} = (u, v) \in E : \| \tilde{u} \|_E < r_2 \} \). By the proof of Theorem 1, we have
\[
\| T \tilde{u} \|_E \leq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_2.
\] (10)

Step 3. Let \( \Omega_3 = \{ \tilde{u} = (u, v) \in E : \| \tilde{u} \|_E < r_1 \} \), \( \forall \tilde{u} \in K \cap \partial \Omega_3 \). By Lemma 1, we have
\[
Q_1(u, v) = P_1 h_1 \geq G_1 \| h_1 \|_{L^1}
\[
= G_1 \int_{T^2} (b_1 f - a_{12}v) \, dt \, dx
\[
\geq G_1 \left[ Mr \| b_1 \|_{L^1} - \| a_{12} \|_{L^1} r \right]
\[
\geq r = \| \tilde{u} \|_E,
\]

namely
\[
\| Q_1(u, v) \| \geq \| \tilde{u} \|_E.
\] So we can get
\[
\| T \tilde{u} \|_E \geq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_3.
\] (11)

Consequently, from (9)–(11) and Lemma 2, (1) has at least two nonnegative solutions \( (u_1, v_1) \in K \cap (\overline{\Omega}_3 \setminus \Omega_1), (u_2, v_2) \in K \cap (\overline{\Omega}_2 \setminus \Omega_3) \) with
\[
0 \leq \| (u_1, v_1) \|_E < r < \| (u_2, v_2) \|_E.
\]
Thus, the proof of Theorem 4.1 is complete. \( \square \)

Theorem 4.2. Assume (H1)–(H3) hold. \( G_1 \| a_{11} \|_{L^1} \geq 2G_1 \| a_{12} \|_{L^1}, \quad G_2 \| a_{22} \|_{L^1} \geq 2G_2 \| a_{21} \|_{L^1}. \) In addition, assume that there exist constants \( r > 0, \ 0 < \sigma < 1 \) such that

(a) \( f_\infty = \infty \) or \( g_\infty = \infty \);
(b) \( f(t, x, u, v) \leq M r \) and \( g(t, x, u, v) \leq M r \), for \( \sigma r \leq u + v \leq r \), where \( M = \min \{ A_1, A_2 \} \).

Then (1) has at least two nonnegative doubly periodic solutions.

Proof. Let \( B = \max \{ B_1, B_2 \} \).

Step 1. Since \( f_\infty = \infty \), there exist \( r_1 \in (0, \sigma r] \) such that
\[
f(t, x, u, v) \geq B(u + v), \quad \text{for } 0 < u + v \leq r_1.
\]
Let \( \Omega_1 = \{ \tilde{u} = (u, v) \in E : \| \tilde{u} \|_E < r_1 \} \). Like in the proof of Theorem 1, we have
\[
\| T \tilde{u} \|_E \geq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_1.
\] (12)
Step 2. Since \( f_\infty = \infty \), there exists \( R > r \) such that
\[
f(t, x, u, v) \geq B(u + v), \quad \text{for } u + v \geq R.
\]
Let \( r_2 = R/\delta \), \( \Omega_2 = \{ \tilde{u} = (u, v) \in E: \| \tilde{u} \|_E < r_2 \} \). Like in the proof of Theorem 1, we have
\[
\| T\tilde{u} \|_E \geq \| \tilde{u} \|_E, \quad \tilde{u} \in K \cap \partial \Omega_2.
\]
(13)

Step 3. Let \( \Omega_3 = \{ \tilde{u} = (u, v) \in E: \| \tilde{u} \|_E < r \} \), \( \forall \tilde{u} \in K \cap \partial \Omega_3 \). By Lemma 1, we have
\[
Q_1(u, v) = P_1h_1 \leq \frac{G_1}{G_1\| a_{11} \|_{L^1}} \| -a_{12}v + b_1f \|_{L^1}
\leq \frac{G_1}{G_1\| a_{11} \|_{L^1}} (\| b_1 \|_{L^1}Mr + \| a_{12} \|_{L^1}r)
\leq \frac{1}{2}r,
\]
namely
\[
\| Q_1(u, v) \| \leq \frac{1}{2}r.
\]
In a similar way, we have
\[
Q_2(u, v) = P_2h_2 \leq \frac{G_2}{G_2\| a_{22} \|_{L^1}} \| -a_{21}u + b_2f \|_{L^1}
\leq \frac{G_2}{G_2\| a_{22} \|_{L^1}} (\| b_2 \|_{L^1}Mr + \| a_{21} \|_{L^1}r)
\leq \frac{1}{2}r,
\]
namely
\[
\| Q_2(u, v) \| \leq \frac{1}{2}r.
\]
So we get
\[
\| T\tilde{u} \|_E = \| Q_1(u, v) \| + \| Q_2(u, v) \| \leq r = \| \tilde{u} \|_E.
\]
(14)

Consequently, from (12)–(14) and Lemma 2, (1) has at least two nonnegative solutions \( (u_1, v_1) \in K \cap (\Omega_3 \setminus \Omega_1) \), \( (u_2, v_2) \in K \cap (\Omega_2 \setminus \Omega_3) \) with \( 0 \leq \| (u_1, v_1) \|_E < r < \| (u_2, v_2) \|_E \).

We can have the same result when \( g_0 = g_\infty = \infty \).

Thus, the proof of Theorem 4.2 is complete. \( \square \)

References