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Boundary data smoothness for solutions of nonlocal boundary value problems for second order differential equations

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Abstract

Under certain conditions, solutions of the boundary value problem, $y'' = f(x, y, y')$, $y(x_1) = y_1$, and $y(x_2) - \sum_{i=1}^{m} r_i y(\eta_i) = y_2$, are differentiated with respect to boundary conditions, where $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, $r_1, \ldots, r_m \in \mathbb{R}$, and $y_1, y_2 \in \mathbb{R}$.

Keywords: Nonlinear boundary value problem; Ordinary differential equation; Nonlocal boundary condition; Existence; Uniqueness

1. Introduction

In this paper, we will be concerned with differentiating solutions of certain nonlocal boundary value problems with respect to boundary data for the second order ordinary differential equation,

$$y'' = f(x, y, y'), \quad a < x < b, \tag{1.1}$$

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satisfying

$$y(x_1) = y_1, \qquad y(x_2) - \sum_{i=1}^{m} r_i y(\eta_i) = y_2, \tag{1.2}$$

where $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and $y_1, y_2, r_1, \ldots, r_m \in \mathbb{R}$, and where we assume:

(i) $f(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ is continuous,
(ii) $\frac{\partial f}{\partial u_i}(x, u_1, u_2) : (a, b) \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, $i = 1, 2$, and
(iii) solutions of initial value problems for (1.1) extend to (a, b).

We remark that condition (iii) is not necessary for the spirit of this work's results, however, by assuming (iii), we avoid continually making statements in terms of solutions' maximal intervals of existence.

Under uniqueness assumptions on solutions of (1.1), (1.2), we will establish analogues of a result that Hartman [8] attributes to Peano concerning differentiation of solutions of (1.1) with respect to initial conditions. For our differentiation with respect to boundary conditions results, given a solution $y(x)$ of (1.1), we will give much attention to the *variational equation for* (1.1) *along* $y(x)$, which is defined by

$$z'' = \frac{\partial f}{\partial u_1}\big(x, y(x), y'(x)\big)z + \frac{\partial f}{\partial u_2}\big(x, y(x), y'(x)\big)z'. \tag{1.3}$$

Interest in multipoint boundary value problems for second order ordinary differential equations has been ongoing for several years, with much attention given to positive solutions. To see only few of these papers, we refer the reader to papers by Bai and Fang [1], Gupta and Trofimchuk [7], Ma [15,16] and Yang [23].

Likewise, many papers have been devoted to smoothness of solutions of boundary value problems in regard to smoothness of the differential equation's nonlinearity, as well as the smoothness of the boundary conditions. For a view of how this work has evolved, involving not only boundary value problems for ordinary differential equations, but also discrete versions, functional differential equations versions and smoothness versions concerning solutions of dynamic equations on time scales, we suggest the manifold results in the papers [2–6,8–12,14,17–22]. In fact, smoothness results have been given some consideration for (1.1), (1.2) when $m = 1$, $r_1 = 1$; see [13].

The theorem for which we seek an analogue and attributed to Peano by Hartman can be stated in the context of (1.1) as follows:

Theorem 1.1 *(Peano). Assume that with respect of* (1.1)*, conditions* (i)–(iii) *are satisfied. Let* $x_0 \in (a, b)$ *and* $y(x) \equiv y(x, x_0, c_1, c_2)$ *denote the solution of* (1.1) *satisfying the initial conditions* $y(x_0) = c_1$, $y'(x_0) = c_2$. *Then,*

(a) $\frac{\partial y}{\partial c_1}$ *and* $\frac{\partial y}{\partial c_2}$ *exist on* (a, b), *and* $\alpha_i \equiv \frac{\partial y}{\partial c_i}$, $i = 1, 2$, *are solutions of the variational equation* (1.3) *along* $y(x)$ *satisfying the respective initial conditions,*

$$\alpha_1(x_0) = 1, \qquad \alpha_1'(x_0) = 0,$$
$$\alpha_2(x_0) = 0, \qquad \alpha_2'(x_0) = 1.$$

(b) $\frac{\partial y}{\partial x_0}$ *exists on* (a, b), *and* $\beta \equiv \frac{\partial y}{\partial x_0}$ *is the solution of the variational equation* (1.3) *along* $y(x)$ *satisfying the initial conditions,*

$$\beta(x_0) = -y'(x_0),$$
$$\beta'(x_0) = -y''(x_0).$$

(c) $\frac{\partial y}{\partial x_0}(x) = -y'(x_0)\frac{\partial y}{\partial c_1}(x) - y''(x_0)\frac{\partial y}{\partial c_2}(x)$.

In addition, our analogue of Theorem 1.1 depends on uniqueness of solutions of (1.1), (1.2), a condition we list as an assumption:

(iv) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, if $y(x_1) = z(x_1)$ and $y(x_2) - \sum_{i=1}^m r_i y(\eta_i) = z(x_2) - \sum_{i=1}^m r_i z(\eta_i)$, where $y(x)$ and $z(x)$ are solutions of (1.1), then $y(x) \equiv z(x)$.

We will also make extensive use of a similar uniqueness condition on (1.3) along solutions $y(x)$ of (1.1).

(v) Given $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$, and a solution $y(x)$ of (1.1), if $u(x_1) = 0$ and $u(x_2) - \sum_{i=1}^m r_i u(\eta_i) = 0$, where $u(x)$ is a solution of (1.3) along $y(x)$, then $u(x) \equiv 0$.

2. An analogue of Peano's Theorem for (1.1), (1.2)

In this section, we derive our analogue of Theorem 1.1 for boundary value problem (1.1), (1.2). For such a differentiation result, we need continuous dependence of solutions on boundary conditions. Such continuity was established recently in [11], which we state here.

Theorem 2.1. *Assume* (i)–(iv) *are satisfied with respect to* (1.1). *Let $u(x)$ be a solution of* (1.1) *on (a, b), and let $a < c < x_1 < \eta_1 < \cdots < \eta_m < x_2 < d < b$ be given. Then, there exists a $\delta > 0$ such that, for $|x_i - t_i| < \delta$, $i = 1, 2$, $|\eta_i - \tau_i| < \delta$, $i = 1, \ldots, m$, $|r_i - \rho_i| < \delta$, $i = 1, \ldots, m$, and $|u(x_1) - y_1| < \delta$, $|u(x_2) - \sum_{i=1}^m r_i u(\eta_i) - y_2| < \delta$, there exists a unique solution $u_\delta(x)$ of* (1.1) *such that $u_\delta(t_1) = y_1$, $u_\delta(t_2) - \sum_{i=1}^m \rho_i u_\delta(\tau_i) = y_2$, and $\{u_\delta^{(j)}(x)\}$ converges uniformly to $u^{(j)}(x)$, as $\delta \to 0$, on $[c, d]$, for $j = 0, 1$.*

We now present the result of the paper.

Theorem 2.2. *Assume conditions* (i)–(v) *are satisfied. Let $u(x)$ be a solution* (1.1) *on (a, b). Let $a < x_1 < \eta_1 < \cdots < \eta_m < x_2 < b$ be given, so that $u(x) = u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$, where $u(x_1) = u_1$ and $u(x_2) - \sum_{i=1}^m r_i u(\eta_i) = u_2$. Then,*

(a) *$\frac{\partial u}{\partial u_1}$ and $\frac{\partial u}{\partial u_2}$ exist on (a, b), and $y_i \equiv \frac{\partial u}{\partial u_i}$, $i = 1, 2$, are solutions of* (1.3) *along $u(x)$ and satisfy the respective boundary conditions,*

$$y_1(x_1) = 1, \qquad y_1(x_2) - \sum_{i=1}^m r_i y_1(\eta_i) = 0,$$
$$y_2(x_1) = 0, \qquad y_2(x_2) - \sum_{i=1}^m r_i y_2(\eta_i) = 1.$$

(b) *$\frac{\partial u}{\partial x_1}$ and $\frac{\partial u}{\partial x_2}$ exist on (a, b), and $z_i \equiv \frac{\partial u}{\partial x_i}$, $i = 1, 2$, are solutions of* (1.3) *along $u(x)$ and satisfy the respective boundary conditions,*

$$z_1(x_1) = -u'(x_1), \qquad z_1(x_2) - \sum_{i=1}^{m} r_i z_1(\eta_i) = 0,$$

$$z_2(x_1) = 0, \qquad z_2(x_2) - \sum_{i=1}^{m} r_i z_2(\eta_i) = -u'(x_2).$$

(c) *For* $1 \leqslant j \leqslant m$, $\frac{\partial u}{\partial \eta_j}$ *exists on* (a, b), *and* $w_j \equiv \frac{\partial u}{\partial \eta_j}$, $j = 1, \ldots, m$, *is a solution of* (1.3) *along* $u(x)$ *and satisfies*

$$w_j(x_1) = 0, \qquad w_j(x_2) - \sum_{i=1}^{m} r_i w_j(\eta_i) = r_j u'(\eta_j).$$

(d) *For* $1 \leqslant j \leqslant m$, $\frac{\partial u}{\partial r_j}$ *exists on* (a, b), *and* $v_j \equiv \frac{\partial u}{\partial r_j}$, $j = 1, \ldots, m$, *is a solution of* (1.3) *along* $u(x)$ *and satisfies*

$$v_j(x_1) = 0, \qquad v_j(x_2) - \sum_{i=1}^{m} r_i v_j(\eta_i) = u(\eta_j).$$

Proof. For part (a) we will give the argument for $\frac{\partial u}{\partial u_1}$, since the argument for $\frac{\partial u}{\partial u_2}$ is somewhat similar. In this case we designate, for brevity, $u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, u_1)$.

Let $\delta > 0$ be as in Theorem 2.1. Let $0 < |h| < \delta$ be given and define

$$y_{1h}(x) = \frac{1}{h}\big[u(x, u_1 + h) - u(x, u_1)\big].$$

Note that $u(x_1, u_1 + h) = u_1 + h$, and $u(x_1, u_1) = u_1$, so that, for every $h \neq 0$,

$$y_{1h}(x_1) = \frac{1}{h}[u_1 + h - u_1] = 1.$$

In addition, for every $h \neq 0$,

$$y_{1h}(x_2) - \sum_{i=1}^{m} r_i y_{1h}(\eta_i) = \frac{1}{h}[u_2 - u_2] = 0.$$

Let

$$\beta_2 = u'(x_1, u_1),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, u_1 + h) - \beta_2.$$

By Theorem 2.1, $\epsilon_2 = \epsilon_2(h) \to 0$, as $h \to 0$. Using the notation of Theorem 1.1 for solutions of initial value problems for (1.1) and viewing the solutions u as solutions of initial value problems, we have

$$y_{1h}(x) = \frac{1}{h}\big[y(x, x_1, u_1 + h, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\big].$$

Then, by utilizing a telescoping sum, we have

$$y_{1h}(x) = \frac{1}{h}\big[\big\{y(x, x_1, u_1 + h, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2 + \epsilon_2)\big\} \\ + \big\{y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\big\}\big].$$

By Theorem 1.1 and the Mean Value Theorem, we obtain

$$y_{1h}(x) = \frac{1}{h}\alpha_1\big(x, y(x, x_1, u_1 + \bar{h}, \beta_2 + \epsilon_2)\big)(u_1 + h - u_1) \\ + \frac{1}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big)(\beta_2 + \epsilon_2 - \beta_2),$$

where $\alpha_i(x, y(\cdot))$, $i = 1, 2$, is the solution of the variational equation (1.3) along $y(\cdot)$ and satisfies in each case,

$$\alpha_1(x_1) = 1, \qquad \alpha_1'(x_1) = 0,$$
$$\alpha_2(x_1) = 0, \qquad \alpha_2'(x_1) = 1.$$

Furthermore, $u_1 + \bar{h}$ is between u_1 and $u_1 + h$, and $\beta_2 + \bar{\epsilon}_2$ is between β_2 and $\beta_2 + \epsilon_2$. Now simplifying,

$$y_{1h}(x) = \alpha_1\big(x, y(x, x_1, u_1 + \bar{h}, \beta_2 + \epsilon_2)\big) + \frac{\epsilon_2}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big).$$

Thus, to show $\lim_{h\to 0} y_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists.

Now $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$, and $\alpha_2(x_1, y(\cdot)) = 0$. So, by assumption (v),

$$\alpha_2\big(x_2, y(\cdot)\big) - \sum_{i=1}^{m} r_i \alpha_2\big(\eta_i, y(\cdot)\big) \neq 0.$$

However, we observed that $y_{1h}(x_2) - \sum_{i=1}^{m} r_i y_{1h}(\eta_i) = 0$, from which we obtain

$$\frac{\epsilon_2}{h} = \frac{\sum_{i=1}^{m} r_i \alpha_1(\eta_i, y(x, x_1, u_1 + \bar{h}, \beta_2 + \epsilon_2)) - \alpha_1(x_2, y(x, x_1, u_1 + \bar{h}, \beta_2 + \epsilon_2))}{[\alpha_2(x_2, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))]}.$$

As a consequence of continuous dependence, we can let $h \to 0$, so that

$$\lim_{h\to 0}\frac{\epsilon_2}{h} = \frac{-[\alpha_1(x_2, y(x, x_1, u_1, \beta_2)) - \sum_{i=1}^{m} r_i \alpha_1(\eta_i, y(x, x_1, u_1, \beta_2))]}{[\alpha_2(x_2, y(x, x_1, u_1, \beta_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1 \beta_2))]} \\ = \frac{-[\alpha_1(x_2, u(\cdot)) - \sum_{i=1}^{m} r_i \alpha_1(\eta_i, u(\cdot))]}{[\alpha_2(x_2, u(\cdot)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, u(\cdot))]} \\ := D.$$

Let $y_1(x) = \lim_{h\to 0} y_{1h}(x)$, and note by construction of $y_{1h}(x)$,

$$y_1(x) = \frac{\partial u}{\partial u_1}(x, u_1).$$

Furthermore,

$$y_1(x) = \lim_{h\to 0} y_{1h}(x) = \alpha_1\big(x, y(x, x_1, u_1, \beta_2)\big) + D\alpha_2\big(x, u(x, x_1)\big),$$

which is a solution of the variational equation (1.3) along $u(x)$. In addition because of the boundary conditions satisfied by $y_{1h}(x)$, we also have

$$y_1(x_1) = 1, \qquad y_1(x_2) - \sum_{i=1}^{m} r_i y_1(\eta_i) = 0.$$

This completes the argument for $\frac{\partial u}{\partial u_1}$.

In part (b) of the theorem, we will produce the details for $\frac{\partial u}{\partial x_1}$, with the arguments for $\frac{\partial u}{\partial x_2}$ being somewhat along the same lines. This time, we designate $u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, x_1)$.

So, let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$z_{1h}(x) = \frac{1}{h}\big[u(x, x_1 + h) - u(x, x_1)\big].$$

Note that

$$\begin{aligned} z_{1h}(x_1) &= \frac{1}{h}\big[u(x_1, x_1 + h) - u(x_1, x_1)\big] \\ &= \frac{1}{h}\big[u(x_1, x_1 + h) - u(x_1 + h, x_1 + h)\big] \\ &= -\frac{1}{h}\big[u'(c_{x_1,h}, x_1 + h) \cdot h\big] \\ &= -u'(c_{x_1,h}, x_1 + h), \end{aligned}$$

where $c_{x_1,h}$ lies between x_1 and $x_1 + h$. In addition, we note that

$$\begin{aligned} &z_{1h}(x_2) - \sum_{i=1}^{m} r_i z_{1h}(\eta_i) \\ &\quad = \frac{1}{h}\left[u(x_2, x_1 + h) - \sum_{i=1}^{m} r_i u(\eta_i, x_1 + h) - \left\{u(x_2, x_1) - \sum_{i=1}^{m} r_i u(\eta_i, x_1)\right\}\right] \\ &\quad = \frac{1}{h}[u_2 - u_2] \\ &\quad = 0, \end{aligned}$$

for every $h \neq 0$. Next, let

$$\begin{aligned} \beta_2 &= u'(x_1, x_1), \\ \epsilon_1 &= \epsilon_1(h) = u(x_1, x_1 + h) - u_1, \end{aligned}$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, x_1 + h) - \beta_2.$$

Let us note at this point that

$$\frac{\epsilon_1}{h} = z_{1h}(x_1) = -u'(c_{x_1,h}, x_1 + h).$$

By Theorem 2.1, both $\epsilon_1 \to 0$ and $\epsilon_2 \to 0$, as $h \to 0$. As in part (a), we employ the notation of Theorem 1.1 for solutions of initial value problems for (1.1), and viewing the solutions u as solutions of initial value problems, we have

$$\begin{aligned} z_{1h}(x) &= \frac{1}{h}\big[y(x,x_1,u_1+\epsilon_1,\beta_2+\epsilon_2)-y(x,x_1,u_1,\beta_2)\big] \\ &= \frac{1}{h}\big[y(x,x_1,u_1+\epsilon_1,\beta_2+\epsilon_2)-y(x,x_1,u_1,\beta_2+\epsilon_2) \\ &\quad + y(x,x_1,u_1,\beta_2+\epsilon_2)-y(x,x_1,u_1,\beta_2)\big]. \end{aligned}$$

By the Mean Value Theorem,

$$z_{1h}(x) = \frac{1}{h}\big[\epsilon_1\alpha_1\big(x, y(x,x_1,u_1+\bar{\epsilon}_1,\beta_2+\epsilon_2)\big)+\epsilon_2\alpha_2\big(x, y(x,x_1,u_1,\beta_2+\bar{\epsilon}_2)\big)\big],$$

where $u_1+\bar{\epsilon}_1$ lies between u_1 and $u_1+\epsilon_1$, $\beta_2+\bar{\epsilon}_2$ lies between β_2 and $\beta_2+\epsilon_2$, and $\alpha_1(x,y(\cdot))$ and $\alpha_2(x,y(\cdot))$ are the solutions of (1.3) along $y(\cdot)$ and satisfy, respectively,

$$\begin{aligned} &\alpha_1(x_1)=1, \qquad \alpha_1'(x_1)=0, \\ &\alpha_2(x_1)=0, \qquad \alpha_2'(x_1)=1. \end{aligned}$$

As before, to show $\lim_{h\to 0} z_{1h}(x)$ exists, it suffices to show $\lim_{h\to 0}\frac{\epsilon_1}{h}$ and $\lim_{h\to 0}\frac{\epsilon_2}{h}$ exist. Now, from above

$$\lim_{h\to 0}\frac{\epsilon_1}{h} = \lim_{h\to 0} z_{1h}(x_1) = -\lim_{h\to 0} u'(c_{x_1,h}, x_1+h) = -u'(x_1).$$

Since $\alpha_2(x,y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$ and since $\alpha_2(x_1,y(\cdot))=0$, it follows from assumption (v) that

$$\alpha_2\big(x_2, y(\cdot)\big)-\sum_{i=1}^{m} r_i\alpha_2\big(\eta_i, y(\cdot)\big)\neq 0.$$

From $z_{1h}(x_2)-\sum_{i=1}^{m} r_i z_{1h}(\eta_i)=0$, we have

$$\frac{\epsilon_2}{h} = \left(\frac{-\epsilon_1}{h}\right)\frac{A}{\alpha_2(x_2, y(x,x_1,u_1,\beta_2+\bar{\epsilon}_2))-\sum_{i=1}^{m} r_i\alpha_2(\eta_i, y(x,x_1,u_1,\beta_2+\bar{\epsilon}_2))},$$

where

$$A = \alpha_1\big(x_2, y(x,x_1,u_1+\bar{\epsilon}_1,\beta_2+\epsilon_2)\big)-\sum_{i=1}^{m} r_i\alpha_i\big(\eta_i, y(x,x_1,u_1+\bar{\epsilon}_1,\beta_2+\epsilon_2)\big).$$

And so,

$$\begin{aligned} \lim_{h\to 0}\frac{\epsilon_2}{h} &= \frac{u'(x_1)[\alpha_1(x_2, y(x_1,x_1,u_1,\beta_2))-\sum_{i=1}^{m} r_i\alpha_i(\eta_i, y(x,x_1,u_1,\beta_2))]}{\alpha_2(x_2, y(x_1,u_1,\beta_2))-\sum_{i=1}^{m} r_i\alpha_2(\eta_i, y(x,x_1,u_1,\beta_2))} \\ &= \frac{u'(x_1)[\alpha_1(x_2,u(\cdot))-\sum_{i=1}^{m} r_i\alpha_i(\eta_i,u(\cdot))]}{\alpha_2(x_2,u(\cdot))-\sum_{i=1}^{m} r_i\alpha_2(\eta_i,u(\cdot))} \\ &:= E. \end{aligned}$$

From the above expression,

$$z_{1h}(x) = \frac{\epsilon_1}{h}\alpha_1\big(x, y(x_1,x_1,u_1+\bar{\epsilon}_1,\beta_2+\epsilon_2)\big)+\frac{\epsilon_2}{h}\alpha_2\big(x, y(x_1,x_1,u_1,\beta_2+\bar{\epsilon}_2)\big),$$

and we can evaluate the limit as $h\to 0$. If we let $z_1(x)=\lim_{h\to 0} z_{1h}(x)$, then $z_1(x)=\frac{\partial u}{\partial x_1}$, and

$$\begin{aligned} z_1(x) &= \lim_{h\to 0} z_{1h}(x) \\ &= -u'(x_1)\alpha_1\big(x, y(x, x_1, u_1, \beta_2)\big) + E\alpha_2\big(x, y(x, x_1, u_1, \beta_2)\big) \\ &= -u'(x_1)\alpha_2\big(x, u(x, x_1)\big) + E\alpha_2\big(x, u(x, x_1)\big), \end{aligned}$$

which is a solution of (1.3) along $u(x)$. In addition, from above observations, $z_2(x)$ satisfies the boundary conditions,

$$z_1(x_1) = \lim_{h\to 0} z_{1h}(x_1) = -u'(x_1),$$

$$z_2(x_2) - \sum_{i=1}^{m} r_i z_2(\eta_i) = \lim_{h\to 0}\left(z_{1h}(x_2) - \sum_{i=1}^{m} r_i z_{1h}(\eta_i)\right) = 0.$$

This completes the proof for $\frac{\partial u}{\partial x_1}$.

The proofs of (c) and (d) are in very much the same spirit.

For (c), we fix $1 \leqslant j \leqslant m$, and this time we designate $u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, \eta_j)$. Let $\delta > 0$ be as in Theorem 2.1, let $0 < |h| < \delta$ be given, and define

$$w_{jh}(x) = \frac{1}{h}\big[u(x, \eta_j + h) - u(x, \eta_j)\big].$$

Note that for every $h \neq 0$, $w_{jh}(x_1) = 0$. Next, let

$$\beta_2 = u'(x_1, \eta_j),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, \eta_j + h).$$

By Theorem 2.1, $\epsilon_2 \to 0$, as $h \to 0$. Again, we use the notation of Theorem 1.1 for solutions of initial value problems for (1.1), and viewing the solutions u as solutions of initial value problems, we have

$$w_{jh}(x) = \frac{1}{h}\big[y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\big].$$

By the Mean Value Theorem,

$$w_{jh}(x) = \frac{\epsilon_2}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big),$$

where $\alpha_2(x, y(\cdot))$ is the solution of (1.3) along $y(\cdot)$ and satisfies

$$\alpha_2(x_1) = 0, \qquad \alpha_2'(x_1) = 1,$$

and $\beta_2 + \bar{\epsilon}_2$ lies between β_2 and $\beta_2 + \epsilon_2$. Once again, to show $\lim_{h\to 0} w_{jh}(x)$ exists, it suffices to show $\lim_{h\to 0} \frac{\epsilon_2}{h}$ exists.

Since $\alpha_2(x, y(\cdot))$ is a nontrivial solution of (1.3) along $y(\cdot)$ and since $\alpha_2(x_1, y(\cdot)) = 0$, it follows from assumption (v) that

$$\alpha_2\big(x_2, y(\cdot)\big) - \sum_{i=1}^{m} r_i \alpha_2\big(\eta_i, y(\cdot)\big) \neq 0.$$

Hence,

$$\frac{\epsilon_2}{h} = \frac{w_{jh}(x_2) - \sum_{i=1}^{m} r_i w_{jh}(\eta_i)}{\alpha_2(x_2, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))}.$$

We look in more detail at the numerator of this quotient. Consider

$$\begin{aligned}
&w_{jh}(x_2) - \sum_{i=1}^{m} r_i w_{jh}(\eta_i) \\
&\quad = \frac{1}{h}\left[u(x_2, \eta_j + h) - \sum_{i=1}^{m} r_i u(\eta_i, \eta_j + h)\left[u(x_2, \eta_j) - \sum_{i=1}^{m} r_i u(\eta_i, \eta_j)\right]\right] \\
&\quad = \frac{1}{h}\Bigg[u(x_2, \eta_j + h) - \sum_{i\in\{1,\dots,m\}\setminus\{j\}} r_i u(\eta_i, \eta_j + h) - r_j u(\eta_j + h, \eta_j + h) \\
&\qquad + r_j u(\eta_j + h, \eta_j + h) - r_j u(\eta_j, \eta_j + h)\Bigg] - \frac{u_2}{h} \\
&\quad = \frac{u_2}{h} - \frac{u_2}{h} + \frac{r_j u(\eta_j + h, \eta_j + h) - r_j u(\eta_j, \eta_j + h)}{h} \\
&\quad = \frac{r_j}{h}\big[u(\eta_j + h, \eta_j + h) - u(\eta_j, \eta_j + h)\big] \\
&\quad = \frac{r_j}{h}\int_{\eta_j}^{\eta_j + h} u'(s, \eta_j + h)\,ds \\
&\quad = \frac{r_j}{h} u'(c_{j,h}, \eta_j + h)(\eta_j + h - \eta_j) \\
&\quad = r_j u'(c_{j,h}, \eta_j + h),
\end{aligned}$$

where $c_{j,h}$ is between η_j and $\eta_j + h$. So, as $h \to 0$ we obtain

$$r_j u'(c_h, \eta_j + h) \to r_j u'(\eta_j, \eta_j) = u'(\eta_j).$$

When we return to the quotient defining $\frac{\epsilon_2}{h}$, we compute the limit,

$$\begin{aligned}
\lim_{h\to 0} \frac{\epsilon_2}{h} &= \frac{r_j u'(\eta_j)}{\alpha_2(x_2, y(x, x_1, u_1, \beta_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1, \beta_2))} \\
&= \frac{r_j u'(\eta_j)}{\alpha_2(x_2, u(\cdot)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, u(\cdot))} \\
&:= E_j.
\end{aligned}$$

From $w_{jh}(x) = \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))$, if we let $w_j(x) = \lim_{h\to 0} w_{jh}(x)$, then $w_j(x) = r_j \frac{\partial u}{\partial \eta_j}$, and

$$\begin{aligned}
w_j(x) &= \lim_{h\to 0} w_{jh}(x) \\
&= E_j \alpha_2\big(x, y(x, x_1, u_1, \beta_2)\big) \\
&= E_j \alpha_2\big(x, u(x, \eta_j)\big),
\end{aligned}$$

which is a solution of (1.3) along $u(x)$. In addition, from above observations, $w_j(x)$ satisfies the boundary conditions,

$$w_j(x_1) = \lim_{h\to 0} w_{jh}(x_1) = 0,$$

and

$$w_j(x_2) - \sum_{i=1}^{m} r_i w_j(\eta_i) = r_j u'(\eta_j).$$

This concludes the proof of (c).

It remains to verify part (d). Fix $1 \leqslant j \leqslant m$ as before. We consider $\frac{\partial u}{\partial r_j}$. To this end, let $\delta > 0$ be as in Theorem 2.1 and let $0 < |h| < \delta$. Define

$$v_{jh}(x) = \frac{1}{h}\big[u(x, r_j + h) - u(x, r_j)\big],$$

where, for brevity, we designate $u(x, x_1, x_2, u_1, u_2, \eta_1, \ldots, \eta_m, r_1, \ldots, r_m)$ by $u(x, r_j)$. Note that

$$v_{jh} = \frac{1}{h}(u_1 - u_1) = 0,$$

for every $h \neq 0$. Also, we see that

$$\begin{aligned}
&v_{jh}(x_2) - \sum_{i=1}^{m} r_i v_{jh}(\eta_i) \\
&= \frac{1}{h}\left[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{i=1}^{m} r_i\big(u(\eta_i, r_j + h) - u(\eta_i, r_j)\big)\right] \\
&= \frac{1}{h}\left[u(x_2, r_j + h) - u(x_2, r_j) - \sum_{i=1}^{m} r_i u(\eta_i, r_j + h) + \sum_{i=1}^{m} r_i u(\eta_i, r_j)\right] \\
&= \frac{1}{h} u(x_2, r_j + h) - \frac{1}{h}\sum_{i=1}^{m} r_i u(\eta_i, r_j + h) - \frac{u_2}{h} \\
&= \frac{1}{h}\Bigg[u(x_2, r_j + h) - \sum_{i \in \{1,\ldots,m\}\setminus\{j\}} r_i u(\eta_i, r_j + h) - r_j u(\eta_j, r_j + h) \\
&\qquad - h u(\eta_j, r_j + h) + h u(\eta_j, r_j + h)\Bigg] - \frac{u_2}{h} \\
&= \frac{1}{h}\left[u(x_2, r_j + h) - \sum_{i \in \{1,\ldots,m\}\setminus\{j\}} r_i u(\eta_i, r_j + h) - (r_j + h) u(\eta_j, r_j + h)\right] \\
&\qquad + u(\eta_j, r_j + h) - \frac{u_2}{h} \\
&= \frac{u_2}{h} + u(\eta_j, r_j + h) - \frac{u_2}{h} \\
&= u(\eta_j, r_j + h).
\end{aligned}$$

And so by Theorem 2.1,

$$v_{jh}(x_2) - \sum_{i=1}^{m} r_i v_{jh}(\eta_i) \to u(\eta_j, r_j), \quad h \to 0.$$

Now recall that, $u(x_1, r_j) = u_1$, and define

$$\beta_2 = u'(x_1, r_j),$$

and

$$\epsilon_2 = \epsilon_2(h) = u'(x_1, r_j + h) - \beta_2.$$

As usual, $\epsilon_2 \to 0$ as $h \to 0$. Once again, using the notation for solutions of initial value problems for (1.1), we have

$$v_{jh}(x) = \frac{1}{h}\big[y(x, x_1, u_1, \beta_2 + \epsilon_2) - y(x, x_1, u_1, \beta_2)\big].$$

By the Mean Value Theorem,

$$\begin{aligned} v_{jh}(x) &= \frac{1}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big)(\beta_2 + \epsilon_2 - \beta_2) \\ &= \frac{\epsilon_2}{h}\alpha_2\big(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)\big), \end{aligned}$$

where $\alpha_2(x, y(\cdot))$ is the solution of (1.3) along $y(\cdot)$ and satisfies

$$\alpha_2(x_1) = 0, \qquad \alpha_2'(x_1) = 1,$$

and $\beta_2 + \bar{\epsilon}_2$ lies between β_2 and $\beta_2 + \epsilon$. As in previous cases considered, it follows from assumption (v) that

$$\alpha_2\big(x_2, y(\cdot)\big) - \sum_{i=1}^{m} r_i \alpha_2\big(\eta_i, y(\cdot)\big) \neq 0.$$

Hence,

$$\frac{\epsilon_2}{h} = \frac{v_{jh}(x_2) - \sum_{i=1}^{m} r_i v_{jh}(\eta_i)}{\alpha_2(x_2, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))},$$

and so from above,

$$\begin{aligned} \lim_{h\to 0} \frac{\epsilon_2}{h} &= \frac{r_j u(\eta_j, r_j)}{\alpha_2(x_2, y(x, x_1, u_1, \beta_2)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, y(x, x_1, u_1, \beta_2))} \\ &= \frac{r_j u(\eta_j, r_j)}{\alpha_2(x_2, u(\cdot)) - \sum_{i=1}^{m} r_i \alpha_2(\eta_i, u(\cdot))} \\ &:= E_j. \end{aligned}$$

From $v_{jh}(x) = \frac{\epsilon_2}{h}\alpha_2(x, y(x, x_1, u_1, \beta_2 + \bar{\epsilon}_2))$, if we set $v_j(x) = \lim_{h\to 0} v_{jh}(x)$, we obtain $v_j(x) = \frac{\partial u}{\partial r_j}$, and in particular,

$$\begin{aligned} v_j(x) &= \lim_{h\to 0} v_{jh}(x) \\ &= E_j \alpha_2\big(x, y(x, x_1, u_1, \beta_2)\big) \\ &= E_j \alpha_2\big(x, u(x, \eta_j)\big), \end{aligned}$$

which is a solution of (1.3) along $u(x)$. In addition, $v_j(x)$ satisfies the boundary conditions,

$$v_j(x_1) = \lim_{h\to 0} w_{jh}(x_1) = 0,$$

and

$$v_j(x_2) - \sum_{i=1}^{m} r_i v_j(\eta_i) = u(\eta_j).$$

This completes case (d), which in turn completes the proof of the theorem. □

We conclude the paper with a corollary to Theorem 2.2, whose verification is a consequence of the two-dimensionality of the solution space for the variational equation (1.3). In addition, this corollary establishes an analogue of part (c) of Theorem 1.1.

Corollary 2.3. *Assume the conditions of Theorem* 2.2. *Then, for* $i = 1, 2$,

$$\frac{\partial u}{\partial x_i} = -u'(x_i)\frac{\partial u}{\partial u_i},$$

and for $1 \leqslant j \leqslant m$,

$$\frac{\partial u}{\partial \eta_j} = r_j \frac{u'(\eta_j)}{u(\eta_j)} \frac{\partial u}{\partial r_j}.$$

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