Reachability realization and stabilizability
of switched linear discrete-time systems

Guangming Xie * and Long Wang

Center for Systems and Control, Department of Mechanics and Engineering Science,
Peking University, Beijing 100871, China

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Abstract

In this paper, the reachability realization of a switched linear discrete-time system, which is a collection of linear time-invariant discrete-time systems along with some maps for “switching” among them, is addressed. The main contribution of this paper is to prove that for a switched linear discrete-time system, there exists a basic switching sequence such that the reachable (controllable) state set of this basic switching sequence is equal to the reachable (controllable) state set of the system. Hence, the reachability (controllability) can be realized by using only one switching sequence. We also discuss the stabilizability of switched systems, and obtain a sufficient condition for stabilizability. Two numeric examples are given to illustrate the results.

Keywords: Switched linear discrete-time system; Switching sequence; Reachable state set; Reachability realization; Controllable state set; Controllability realization; Stabilizability

1. Introduction

Switched systems are an important class of hybrid systems. Motivated by practical considerations, there have been a lot of studies for switched systems recently, primarily on stability analysis and design [2–6]. Since controllability is a fundamental concept...
in modern control theory, there is also some work on the definition and determination of controllability of switched systems [1,7–13]. Necessary and sufficient conditions for controllability of general (periodically or arbitrarily) switched linear continuous-time systems with/without time delays were established in [11–13]. For discrete-time case, a sufficient and necessary condition for the controllability based on both switching path and control input were given in [10]. In this paper, we prove that for a switched linear discrete-time system, there exists a basic switching sequence such that the reachable (controllable) state set of this basic switching sequence is equal to the reachable (controllable) state set of the system. Hence, the reachability (controllability) can be realized by using only one switching sequence. We also discuss the stabilizability of switched systems, and obtain a sufficient condition for stabilizability. Two numeric examples are given to illustrate the results.

This paper is organized as follows. Section 2 formulates the problem and presents the preliminary results. Section 3 is the main result of this paper. Section 4 contains two numeric examples. Finally, we provide the conclusion in Section 5.

2. Preliminaries

Consider a switched linear discrete-time control system given by

\[ x(k + 1) = A_{r(k)}x(k) + B_{r(k)}u(k) \]  \hspace{1cm} (1)

where \( x(k) \in \mathbb{R}^n \) is the state, \( u(k) \in \mathbb{R}^p \) is the input, the piecewise constant scalar function \( r(k) : \{0, 1, \ldots\} \rightarrow \{1, 2, \ldots, N\} \) is the switching path to be designed. Moreover, \( r(k) = i \) implies that the pair \( (A_i, B_i) \) is chosen as the system realization, \( i = 1, 2, \ldots, N \).

In this paper we assume that system (1) is reversible, i.e., \( \forall i = 1, \ldots, N, \ A_i \) is nonsingular. For clarity, for any integer \( M > 0 \), set \( M = \{0, 1, \ldots, M - 1\} \) and \( \infty = \{0, 1, \ldots\} \).

Definition 1 [10]. For system (1), state \( x \) is reachable (controllable), if there exists a time instant \( M > 0 \), a switching path \( r(m) : M \rightarrow \{1, 2, \ldots, N\} \), and inputs \( u(m) : M \rightarrow \mathbb{R}^p \), such that \( x(0) = 0 \) and \( x(M) = x \) (\( x(0) = x \) and \( x(M) = 0 \)).

Definition 2 [10]. System (1) is said to be reachable (controllable) if any state \( x \) is reachable (controllable).

Now, we introduce some mathematical preliminaries as the basic tools for the discussion in the remainder of the paper.

Definition 3 (Column space). Given a matrix \( B_{n \times p} = [b_1, \ldots, b_p] \), the column space \( \mathcal{R}(B) \) is defined as

\[ \mathcal{R}(B) \overset{\text{def}}{=} \text{span}[b_1, \ldots, b_p]. \]  \hspace{1cm} (2)
Definition 4 (Invariant subspace). Given a matrix $A_{n \times n}$ and a linear subspace $W \subseteq \mathbb{R}^n$, the invariant subspace $\langle A | W \rangle$ is defined as
\[
\langle A | W \rangle \overset{\text{def}}{=} \sum_{i=1}^{n} A_i^{-1} W.
\]
(3)

For notational simplicity, let $\langle A | B \rangle = \langle A | R(B) \rangle$, where $A, B$ are $(n \times n)$-dimensional matrix and $(n \times p)$-dimensional matrix, respectively.

For system (1), [7–10] defined a subspace sequence as follows:
\[
\begin{align*}
W_1 &= \sum_{i=1}^{N} \langle A_i | B_i \rangle, \\
W_2 &= \sum_{i=1}^{N} \langle A_i | W_1 \rangle, \\
& \quad \vdots \\
W_n &= \sum_{i=1}^{N} \langle A_i | W_{n-1} \rangle.
\end{align*}
\]
(4)

Let $T, C$ denote the set of all reachable states of system (1) and the set of all controllable states of system (1), respectively. Ge et al. [10] gave the following proposition.

Proposition 1 [10]. For system (1), $T \equiv C \equiv W_n$.

3. Main result

3.1. Reachable state set

The purpose of this subsection is to introduce the concept of reachable state set of a switching sequence and describe its characteristics.

For system (1), a switching sequence is to specify when and to which realization one should switch at each instant of time.

Definition 5 (Switching sequence). A switching sequence $\pi$ is a set with finite scalars
\[
\pi \overset{\text{def}}{=} \{i_0, \ldots, i_{M-1}\}
\]
(5)

where $M \leq \infty$ is the length of $\pi$, $i_m \in \{1, \ldots, N\}$ is the index of the $m$th realization $(A_{im}, B_{im})$, for $m \in \mathbb{M}$.

Given a switching sequence $\pi = \{i_0, \ldots, i_{M-1}\}$, an associated switching path $r(m) : \mathbb{M} \rightarrow \{1, \ldots, N\}$ can be determined as
\[
\begin{align*}
r(m) &= i_m, \\
& \quad m \in \mathbb{M}.
\end{align*}
\]
(6)

Definition 6 (Reachable state set). Given a switching sequence $\pi = \{i_0, \ldots, i_{M-1}\}$, the reachable state set of $\pi$ is defined as
\[
T(\pi) \overset{\text{def}}{=} \{x \mid \exists \text{ inputs } u(m) : \mathbb{M} \rightarrow \mathbb{R}^p \text{ such that } x(0) = 0 \text{ and } x(M) = x\}.
\]
(7)
It can be calculated that
\[
x(M) = \sum_{m=0}^{M-2} \prod_{j=M-1}^{m+1} A_{ij} B_{im} u(m) + B_{iM-1} u(M-1)
\]  
(8)

where the product notation is to be read left-to-right, i.e., in general, \(\prod_{j=1}^{N} X_j\) means \(X_1 X_2 \cdots X_N\). In the following it is similar.

Thus we can redefine the reachable state set as follows:
\[
T(\pi) = \left\{ x \left| x = \sum_{m=0}^{M-2} \prod_{j=M-1}^{m+1} A_{ij} B_{im} u(m) + B_{iM-1} u(M-1), \forall u(m) \right. \right\}
\]  
(9)

It is easy to prove that
\[
T(\pi) = R\left(\left[ \prod_{m=M-1}^{1} A_{im} \right] B_{i0} \left[ \prod_{m=M-1}^{2} A_{im} \right] B_{i1} \cdots B_{iM-1} \right)
\]  
(10)

The above analysis is summarized in the following proposition.

**Proposition 2.** Given a switching sequence \(\pi = \{i_0, \ldots, i_{M-1}\}\), the reachable state set of \(\pi\) is a linear space described as (10).

It is easy to see that
\[
T = \bigcup_{\forall \pi} T(\pi).
\]  
(11)

**Proposition 3.** Consider the switching sequence \(\pi = \{i, \ldots, i\}\); it follows that
\[
T(\pi) = \langle A_i | B_i \rangle.
\]  
(12)

**Proof.** \(T(\pi) = R([A_i^{n-1} B_i, A_i^{n-2} B_i, \ldots, B_i]) = \langle A_i | B_i \rangle.\) \(\square\)

In the following, we define two operations on switching sequence and discuss the associated reachable state sets.

**Definition 7** (Product of switching sequences). Given two switching sequences \(\pi_1 = \{i_0, \ldots, i_{M-1}\}\) and \(\pi_2 = \{j_0, \ldots, j_{L-1}\}\). The product of \(\pi_1\) and \(\pi_2\) is defined as
\[
\pi_1 \land \pi_2 \overset{\text{def}}{=} \{i_0, \ldots, i_{M-1}, j_0, \ldots, j_{L-1}\}.
\]  
(13)

Since it is easy to verify that \((\pi_1 \land \pi_2) \land \pi_3 = \pi_1 \land (\pi_2 \land \pi_3)\), we just denote it by \(\pi_1 \land \pi_2 \land \pi_3\).
**Definition 8** (Power of switching sequences). Given a switching sequence \( \pi \), the power of \( \pi \) is defined as
\[
\pi \wedge n \overset{\text{n times}}{=} \overbrace{\pi \wedge \cdots \wedge \pi}^{n}. \tag{14}
\]
Given a switching sequence \( \pi = \{ i_0, \ldots, i_{M-1} \} \), denote
\[
A_{\pi} = \prod_{m=M-1}^{0} A_{i_m}. \tag{15}
\]

**Theorem 1.** Given switching sequence \( \pi_1 \) and \( \pi_2 \),
\[
T(\pi_1 \wedge \pi_2) = A_{\pi_2} T(\pi_1) + T(\pi_2). \tag{16}
\]

**Proof.** It is a direct consequence of the definition of reachable state set. \( \square \)

**Theorem 2.** Given switching sequence \( \pi \), it follows that
\[
T(\pi \wedge n) = \langle A_{\pi} \mid T(\pi) \rangle. \tag{17}
\]

**Proof.** \( T(\pi \wedge n) = A_{\pi} T(\pi \wedge (n-1)) + T(\pi) = \cdots = \sum_{l=1}^{n} (A_{\pi})^{l-1} T(\pi) = \langle A_{\pi} \mid T(\pi) \rangle \). \( \square \)

**Corollary 1.** For any switching sequence \( \pi \),
\[
A_{\pi} \wedge T(\pi \wedge n) = T(\pi \wedge n). \tag{18}
\]

**Proof.** From (17), using the property of invariant subspace, we have \( A_{\pi} \wedge T(\pi \wedge n) = (A_{\pi})^{n} \langle A_{\pi} \mid T(\pi) \rangle \subseteq \langle A_{\pi} \mid T(\pi) \rangle \). Since \( A_{\pi} \) is nonsingular, we have \( \dim((A_{\pi})^{n} \langle A_{\pi} \mid T(\pi) \rangle) = \dim(\langle A_{\pi} \mid T(\pi) \rangle) \). It follows that \( (A_{\pi})^{n} \langle A_{\pi} \mid T(\pi) \rangle = \langle A_{\pi} \mid T(\pi) \rangle \). \( \square \)

### 3.2. Reachability realization

In this subsection, we will prove that reachability can be realized by a single switching sequence. First, we give the following important theorem.

**Theorem 3.** For system (1), there exists a basic switching sequence \( \pi_b \), such that \( T(\pi_b) = \mathcal{W}_n \).

**Proof.** Suppose \( \dim(\mathcal{W}_n) = d \). By (4), there must exist subspaces \( \mathcal{V}_1, \ldots, \mathcal{V}_d \) such that
\[
\mathcal{W}_n = \sum_{l=1}^{d} \mathcal{V}_l \tag{19}
\]
and each \( \mathcal{V}_l \) has the following form:
\[
\prod_{m=1}^{M-1} A_{i_m} \langle A_{j} \mid B_{j} \rangle \tag{20}
\]
where \( i_1, \ldots, i_{M-1}, j \in [1, \ldots, N], 0 \leq M < \infty \).

Consider the subspace which has the form (20); we can select two switching sequences

\[
\pi_{\alpha} = \{j, \ldots, j\}, \quad \pi_{\beta} = \{i_1, \ldots, i_{M-1}\}
\]

(21)
such that

\[
\prod_{m=1}^{M-1} A_{i_m} (A_j | B_j) = A_{\pi_{\alpha}} T(\pi_{\alpha}) \subseteq T(\pi_{\alpha} \land \pi_{\beta}).
\]

(22)

Thus, we can select switching sequences \( \pi_1, \ldots, \pi_d \) such that \( \forall_1 \subseteq T(\pi_1) \), for \( l = 1, \ldots, d \).

By (19), we have

\[
W_n = \sum_{l=1}^{d} V_l \subseteq \sum_{l=1}^{d} T(\pi_l).
\]

(23)

Now we construct the switching sequence \( \pi_b \) as follows.

First, if \( T(\pi_1^{\land n}) = W_n \), we can take \( \pi_b = \pi_1^{\land n} \). If not, there must exist \( k \in \{2, \ldots, d\} \) such that (without loss of generality, let \( k = 2 \))

\[
T(\pi_2) \not\subseteq T(\pi_1^{\land n}).
\]

(24)

Consider

\[
T(\pi_2 \land \pi_1^{\land n}) = A_{\pi_1^{\land n}} T(\pi_2) + T(\pi_1^{\land n}).
\]

(25)

By (18), we have

\[
T(\pi_2 \land \pi_1^{\land n}) = A_{\pi_1^{\land n}} (T(\pi_2) + T(\pi_1^{\land n})).
\]

(26)

This implies that

\[
\dim(T(\pi_2 \land \pi_1^{\land n})) = \dim(A_{\pi_1^{\land n}} (T(\pi_2) + T(\pi_1^{\land n}))) = \dim(T(\pi_2) + T(\pi_1^{\land n})).
\]

(27)

By (24), we have

\[
T(\pi_2) + T(\pi_1^{\land n}) \supseteq T(\pi_1^{\land n}).
\]

Thus,

\[
\dim(T(\pi_2 \land \pi_1^{\land n})) > \dim(T(\pi_1^{\land n})) \geq 2.
\]

Similarly, we construct switching sequences

\[
\pi_1 = \pi_1,
\]

\[
\pi_2 = \pi_2 \land (\pi_1)^{\land n},
\]

\[
\vdots
\]

\[
\pi_d = \pi_d \land (\pi_{d-1})^{\land n},
\]

\[
\pi_{d+1} = \pi_{d+1} \land (\pi_d)^{\land n},
\]

\[
\vdots
\]

(28)
and 
\[ \pi_b = \mathbb{P}_d. \]

By the similar analysis, we have \( \dim(T(\pi_l)) \geq l \), for \( l = 1, \ldots, d \). Since \( T(\pi_b) \subseteq \mathcal{W}_n \), we have \( \dim(T(\pi_b)) \leq d \), thus, we have \( \dim(T(\pi_b)) = d \). Hence, \( T(\pi_b) = \mathcal{W}_n \). This completes the proof of Theorem 3. \( \square \)

**Remark 1.** For system (1), since \( T \subseteq \mathcal{W}_n \) and \( T(\pi_b) \subseteq T \), we have that \( T(\pi_b) = T = \mathcal{W}_n \). This implies Proposition 1.

**Remark 2.** For system (1), we can use only one switching sequence to realize reachability.

**Remark 3.** By the proof of Theorem 3, \( \pi_b \) is not unique because \( V_1, \ldots, V_d \) are not unique.

**Remark 4.** The proof of Theorem 3 provides a method to construct \( \pi_b \).

### 3.3. Controllable state set and controllability realization

For controllability of switched linear systems, similar results can be established as follows.

**Definition 9 (Controllable state set).** Given a switching sequence \( \pi = \{i_0, i_1, \ldots, i_{M-1}\} \), the controllable state set of \( \pi \) is defined as
\[
C(\pi) \overset{\text{def}}{=} \{ x \mid \exists \text{ inputs } u(m), m \in M \rightarrow \mathbb{R}^p \text{ such that } x(0) = x \text{ and } x(M) = 0 \}. \tag{27}
\]

**Proposition 4.** Given a switching sequence \( \pi = \{i_0, \ldots, i_{M-1}\} \), the controllable state set of \( \pi \) is a linear space described as
\[
C(\pi) = \mathbb{R}\left( \begin{bmatrix}
B_{i_0}, A_{i_0}^{-1}B_{i_1}, \ldots, M-2 \prod_{m=0}^{M-2} A_{i_m}^{-1}B_{i_{M-1}}
\end{bmatrix} \right). \tag{28}
\]

**Proof.** It can be calculated that
\[
0 = x(M) = \prod_{m=M-1}^{0} A_{i_m} x(0) + \sum_{m=0}^{M-2} \prod_{j=M-1}^{m+1} A_{i_j} B_{i_m} u(m) + B_{i_{M-1}} u(M-1). \tag{29}
\]

It follows that
\[
x(0) = - \left( \prod_{m=M-1}^{0} A_{i_m} \right)^{-1} \left( \sum_{m=0}^{M-2} \prod_{j=M-1}^{m+1} A_{i_j} B_{i_m} u(m) + B_{i_{M-1}} u(M-1) \right) = -B_{i_0} u(0) - \sum_{m=0}^{M-1-1} \prod_{j=0}^{m-1} A_{i_j} B_{i_m} u(m).
\]
Thus we can redefine the controllable state set as follows:

$$\mathcal{C}(\pi) = \left\{ x \mid x = B_{i_0} u(0) + \sum_{m=0}^{M-1} \prod_{j=0}^{m-1} A_{i_j} B_{i_m} u(m), \forall u(m) \right\}. \quad (30)$$

This implies (28). $\square$

It is easy to see that

$$\mathcal{C} = \bigcup_{\forall \pi} \mathcal{C}(\pi). \quad (31)$$

**Theorem 4.** Given switching sequence $\pi$,

$$T(\pi) = A_\pi \mathcal{C}(\pi), \quad \mathcal{C}(\pi) = A_{\pi}^{-1} T(\pi). \quad (32)$$

**Proof.** By (10) and (28), (32) is obvious. $\square$

Based on Theorem 4, we can establish the following corollaries directly.

**Corollary 2.** Consider the switching sequence $\pi = \{i, \ldots, i\}$; it follows that

$$\mathcal{C}(\pi) = \langle A_i | B_i \rangle. \quad (33)$$

**Corollary 3.** Given switching sequences $\pi_1$ and $\pi_2$, it follows that

$$\mathcal{C}(\pi_1 \wedge \pi_2) = \mathcal{C}(\pi_1) + A_{\pi_1}^{-1} \mathcal{C}(\pi_2). \quad (34)$$

**Corollary 4.** Given switching sequence $\pi$, it follows that

$$\mathcal{C}(\pi \wedge n) = \left\{ A_{\pi} \mathcal{C}(\pi) \right\} = T(\pi \wedge n). \quad (35)$$

**Corollary 5.** For any switching sequence $\pi$, it follows that

$$(A_{\pi \wedge n})^{-1} \mathcal{C}(\pi \wedge n) = \mathcal{C}(\pi \wedge n). \quad (36)$$

**Corollary 6.** For system (1), there exists a basic switching sequence $\pi_b$ such that $\mathcal{C}(\pi_b) = T(\pi_b) = \mathcal{W}_n$.

**Remark 5.** For system (1), since $\mathcal{C} \subseteq \mathcal{W}_n$ and $\mathcal{C}(\pi_b) \subseteq \mathcal{C}$, we have that $\mathcal{C}(\pi_b) = \mathcal{C} = \mathcal{W}_n$. This implies Proposition 1.

**Remark 6.** For system (1), we can use only one switching sequence to realize controllability.
3.4. Stabilizability

In this subsection we discuss stabilizability of linear switched systems. For notational simplicity, system (1) is represented with \(N\) pairs of matrices \((A_1, B_1), \ldots, (A_N, B_N)\). Namely, “system \((A_1, B_1), \ldots, (A_N, B_N)\)” means “system (1).”

**Definition 10** (Stabilizability). Given a switching path \(r(k): \infty \to \{1, \ldots, N\}\), system \((A_1, B_1), \ldots, (A_N, B_N)\) is said to be (asymptotically) stabilizable if for any nonzero state \(x_0\), there exists \(u(k)\), \(k \in \infty\) such that \(\lim_{k \to +\infty} x(k) = 0\).

**Proposition 5.** If the system \((A_1, B_1), \ldots, (A_N, B_N)\) is completely controllable, then there exists a switching path \(r(k): \infty \to \{1, \ldots, N\}\) such that system \((A_1, B_1), \ldots, (A_N, B_N)\) is stabilizable under \(r(k)\).

**Proof.** Since system \((A_1, B_1), \ldots, (A_N, B_N)\) is completely controllable, there exists a switching sequence \(\pi_b = \{i_m\}_{m=0}^{M-1}\), such that for any nonzero state \(x_0\), there exists \(u_0(k)\), \(k \in M\) such that \(x(M) = 0\). Thus, let

\[
gr(k) = \begin{cases} i_k, & k \in M, \\ 1, & k \geq M. \end{cases}
\]

\[
u(k) = \begin{cases} u_0(k), & k \in M, \\ 0, & k \geq M. \end{cases}
\]

It is obvious that \(\lim_{k \to +\infty} x(k) = 0\). \(\Box\)

Given a nonsingular matrix \(P\), under the transformation of state coordinate \(P = Px\), we get a new switched linear system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\) with state variable \(\overline{x}\), where \(\overline{A}_m = PA_m P^{-1}, \overline{B}_m = PB_m\) for \(m = 1, \ldots, N\).

Given any switching sequence \(\pi = \{i_m\}_{m=0}^{M-1}\), denote \(C(\pi)\) the controllable state set of \(\pi\) associated with system \((A_1, B_1), \ldots, (A_N, B_N)\) and denote \(\overline{C}(\pi)\) the controllable state set of \(\pi\) associated with system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\). We have

\[
C(\pi) = \mathcal{R}\left(\overline{B}_{i_0}, \overline{A}_{i_0}^{-1}\overline{B}_{i_1}, \ldots, \prod_{m=0}^{M-2} \overline{A}_{i_m}^{-1}\overline{B}_{i_{m+1}}\right)
\]

\[
= \mathcal{R}\left(PB_{i_0}, PA_{i_0}^{-1}P^{-1}PB_{i_1}, \ldots, \prod_{m=0}^{M-2} PA_{i_m}^{-1}P^{-1}PB_{i_{m+1}}\right)
\]

\[
= P\mathcal{R}\left(B_{i_0}, A_{i_0}^{-1}B_{i_1}, \ldots, \prod_{m=0}^{M-2} A_{i_m}^{-1}B_{i_{m+1}}\right)
\]

\[
= PC(\pi).
\]

Let \(C\) and \(\overline{C}\) be the controllable state sets of system \((A_1, B_1), \ldots, (A_N, B_N)\) and system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\), respectively. By Theorem 3, under the basic switching sequence \(\pi_b\), we have \(\overline{C} = \overline{C}(\pi_b) = PC(\pi_b) = PC\). This means that if any state \(x\) is controllable with system \((A_1, B_1), \ldots, (A_N, B_N)\), then \(\overline{x} = Px\) is controllable with system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\) and, conversely, if any state \(\overline{x}\) is controllable with
system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\), then \(x = P^{-1}\overline{x}\) is controllable with system \((A_1, B_1), \ldots, (A_N, B_N)\). Thus we get the following theorem.

**Theorem 5.** The controllability of system \((A_1, B_1), \ldots, (A_N, B_N)\) is equivalent to that of system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\).

Now we discuss the stabilizability condition when system \((A_1, B_1), \ldots, (A_N, B_N)\) is not controllable. Suppose \(\dim(\mathcal{V}_n) = d < n\), let \(\{q_1, \ldots, q_d\}\) be a basis for \(\mathcal{V}_n\), i.e., \(\mathcal{V}_n = \text{span}\{q_1, \ldots, q_d\}\). Then we can find \(n - d\) independent vectors \(q_{d+1}, \ldots, q_n \in \mathbb{R}^n \setminus \mathcal{V}_n\) such that \(\{q_1, \ldots, q_n\}\) is a basis for \(\mathbb{R}^n\). Denote \(Q = [q_1, \ldots, q_n]\) and \(P = Q^{-1}\). Under the transformation of state coordinate \(\overline{x} = Px\), we get a new system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\) with state variable \(\overline{x}\), where \(\overline{A}_m = PA_mP^{-1}\), \(\overline{B}_m = PB_m\) for \(m = 1, \ldots, N\). Denote \(P = [p_1^T, \ldots, p_n^T]^T\), where \(p_i^T\) is the \(i\)th row vector of \(P\), \(i = 1, \ldots, n\). Since \(p_i^Tq_j = 0\), for \(i \neq j\), and \(A_mq_j \in \mathcal{V}_n\), for \(j = 1, \ldots, d\), \(m = 1, \ldots, N\), we have \(p_i^T A_mq_j = 0\), for \(i = d + 1, \ldots, n\), \(j = 1, \ldots, d\), \(m = 1, \ldots, N\). Moreover, since \(\mathcal{R}(B_m) \subset \mathcal{V}_n\), we have \(p_i^T B_m = 0\), for \(i = d + 1, \ldots, n\), \(m = 1, \ldots, N\). Therefore, we have

\[\overline{A}_m = PA_mP^{-1} = \begin{bmatrix} \overline{A}_m^c & \overline{A}_m^n \\ 0 & \overline{A}_m^e \end{bmatrix}, \quad \overline{B}_m = PB_m = \begin{bmatrix} \overline{B}_m^c \\ 0 \end{bmatrix}\]

(37)

for \(m = 1, \ldots, N\), where \(\overline{A}_m^c, \overline{A}_m^e, \overline{A}_m^n\) and \(\overline{B}_m^c\) are \(d \times d\), \((n-d) \times (n-d)\) and \((d \times p)\)-dimensional matrix, respectively.

We separate state \(\overline{x}\) as \(\overline{x} = [\overline{x}_e^T, \overline{x}_n^T]^T\), where \(\overline{x}_e\) is a \(d\)-dimensional vector and \(\overline{x}_n\) is an \((n-d)\)-dimensional vector. Obviously, system \((\overline{A}_1^c, \overline{B}_1^c), \ldots, (\overline{A}_N^c, \overline{B}_N^c)\) with state \(\overline{x}_e\) is completely controllable.

**Theorem 6.** System \((A_1, B_1), \ldots, (A_N, B_N)\) is stabilizable if and only if system \(\overline{A}_1^c, \ldots, \overline{A}_N^c\) is stabilizable.

**Proof.** It is obvious that stabilizability of system \((A_1, B_1), \ldots, (A_N, B_N)\) is equivalent to stabilizability of system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\). Since system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\) is completely controllable, thus system \((\overline{A}_1, \overline{B}_1), \ldots, (\overline{A}_N, \overline{B}_N)\) is stabilizable if and only if system \((\overline{A}_1^c, 0), \ldots, (\overline{A}_N^c, 0)\) is stabilizable.

**Corollary 7.** System \((A_1, B_1), \ldots, (A_N, B_N)\) is stabilizable if system \(\overline{A}_1^c, \ldots, \overline{A}_N^c\) has a common Lyapunov function, i.e., there exists positive definite matrix \(P\) such that \((\overline{X}_i^c)^T \overline{X}_i^c - P < 0\), \(i = 1, \ldots, N\).

**Proof.** If \(P\) exists, then system \(\overline{A}_1^c, \ldots, \overline{A}_N^c\) is stable under arbitrary switching path.

4. Illustrating example

**Example 1.** Consider system (1) with \(n = 3\), \(N = 3\) and
Consider the switching sequence $\pi_b$. Obviously, it is long enough to realize reachability by $\pi_b$, and we can select $q \in Q$. Simple calculation gives

$$\mathcal{W}_3 = \text{span}\{B_1, A_1 B_1, A_2 A_1 B_1\} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}. \quad (39)$$

Consider the switching sequence $\pi_b = \{1, 2, 3, 1, 2, 1\}$; we have

$$T(\pi_b) = \text{span}\{A_1 A_2 A_3 A_2 B_1, A_1 A_2 A_1 A_3 B_2, A_1 A_2 A_1 B_3, A_1 A_2 B_1, A_1 B_2, B_1\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$= \mathcal{W}_3. \quad (40)$$

Obviously, it is long enough to realize reachability by $\pi_b$ completely.

**Example 2.** Consider system (1) with $n = 4$, $N = 3$ and

$$A_1 = \begin{bmatrix} 12.7 & 58.5 & -11.7 & -23.4 \\ -5.9 & -28.5 & 5.9 & 11.8 \\ 34.9 & 174.5 & -33.9 & -69.8 \\ -25.9 & -129.5 & 25.9 & 52.8 \end{bmatrix}, \quad B_1 = \begin{bmatrix} -2 \\ 1 \\ -5 \\ 4 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 8.4 & 25 & 8.6 & 9.2 \\ -2.8 & -7 & -5.2 & -6.4 \\ 13.8 & 33 & 35.2 & 44.4 \\ -9.8 & -22 & -26.2 & -33.4 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 8.6 & 40 & -9.6 & -18.2 \\ -4.2 & -20 & 4.2 & 8.4 \\ 26.2 & 125 & -19.2 & -43.4 \\ -19.2 & -92 & 15.2 & 33.4 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad (41)$$

Simple calculation gives

$$\mathcal{W}_4 = \text{span}\{B_1, A_1 B_1, A_2 A_1 B_1\}. \quad (42)$$

Then we can select $q_4 = [4 -9 -2 1]^T$ such that $\mathcal{W}_4 = \text{span}\{B_1, A_1 B_1, A_2 A_1 B_1, q_4\}$. Consider the transformation $x = [B_1, A_1 B_1, A_2 A_1 B_1, q_4] x$; we have

$$\overline{A}_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}, \quad \overline{B}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$\overline{B}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \quad (43)$$
From Example 1, Example 2 is stabilizable.

5. Conclusion

We have discussed the controllability of switched linear systems. The concept of controllable state set is introduced as the basic tools. Then we proved that there exist a basic switching sequence such that its controllable state set is exactly the controllable state set of the whole system. Then a sufficient condition for system stabilizability has been given. Finally, two examples are given to illustrate the results.

References